BEST APPROXIMATIONS IN L^1 ARE NEAR BEST IN L^p, $p < 1$

LAWRENCE G. BROWN AND BRADLEY J. LUCIER

(Communicated by J. Marshall Ash)

ABSTRACT. We show that any best L^1 polynomial approximation to a function f in L^p, $0 < p < 1$, is near best in L^p.

Let $I = [0,1]^d$ and let $\mathcal{P}_r$ be the set of all polynomials in d variables of total degree less than r. It is known that for each f in $L^1(I)$ a (not necessarily unique) best approximation $E_1 f$ to f exists in $\mathcal{P}_r$, and that $E_1 f = q$ if and only if

$$\left| \int_{E_+} s(x)\,dx - \int_{E_-} s(x)\,dx \right| \le \int_{E_0} |s(x)|\,dx \quad \text{for all } s \text{ in } \mathcal{P}_r, \tag{1}$$

where $E_+ = \{x \in I \mid q(x) > f(x)\}$, $E_- = \{x \in I \mid q(x) < f(x)\}$, and $E_0 = \{x \in I \mid q(x) = f(x)\}$. Condition (1) makes sense even if f is not in $L^1(I)$, and when (1) is satisfied, we call q a best $L^1(I)$ approximation to f and denote q by $E_1 f$. It is easy to show that for constant approximations ($r = 1$) the extended E_1, which is the median operator, is defined for all measurable f and bounded on $L^p(I)$ for any $p > 0$ (see [2] for a discussion of medians). It is also easy to show that the similarly extended L^2 best projection operator onto polynomials is bounded on L^p for $p \ge 1$ and any $r > 0$. These facts motivate us to prove the following theorem.

Theorem. *For each f in $L^p(I)$, $0 < p < 1$, and for all $r > 0$, a best $L^1(I)$ approximation $E_1 f$ exists in $\mathcal{P}_r$. Moreover, for all choices of $E_1 f$,*

$$\|f - E_1 f\|_p \le (1 + 2K)^{1/p} \inf_{q \in \mathcal{P}_r} \|f - q\|_p,$$

where

$$K = \sup_{q \in \mathcal{P}_r} \frac{\|q\|_\infty}{\|q\|_1}.$$

This theorem provides a method to find near-best polynomial approximations in $L^p(I)$ for $0 < p < 1$. Such approximations are useful in atomic decompositions of

Received by the editors October 29, 1990 and, in revised form, May 28, 1991.

1991 *Mathematics Subject Classification.* 41A50, 41A10, 46E30.

Key words and phrases. Best approximation, L^p spaces.

The second author was supported in part by the National Science Foundation (grants DMS-8802734 and DMS-9006219), the Office of Naval Research (contract N00014-91-J-1152), and by the Army High Performance Computing Research Center.

the Besov spaces $B_p^\alpha(L^p(I))$, $p<1$, which are the regularity spaces for nonlinear approximation in $L^q(I)$, $q^{-1}=p^{-1}-\alpha/d$, by wavelets and free-knot splines. Theory and applications to image and surface compression can be found in [2–6]. Of course, such near-best approximations are known to exist; the new thing here is that $L^1(I)$ projections provide them.

We make several remarks about the theorem. First, a more careful argument shows that

$$\|f-E_1f\|_p \le (2K)^{\frac{1}{p}-1} \inf_{q\in\mathcal{P}_r} \|f-q\|_p$$

for $0<p<1$; see [1]. Second, our proof extends to approximation by any finite-dimensional subpace of $L^1(\Omega)\cap L^\infty(\Omega)$ for any finite measure space $(\Omega,d\mu)$. (A measure space with atoms must first be embedded into a continuous measure space.) Generalizations to the behavior of the best $L^p(I)$ approximation in $L^{p-1}(I)$ for $p>1$, and the fact that the best $L^1(I)$ approximation is defined for any measurable f are contained in [1].

We first prove the following lemma, which contains the main argument.

Lemma. *If there exists a best $L^1(I)$ approximation $E_1f\in\mathcal{P}_r$ to $f\in L^p(I)$, then*

$$\|E_1f\|_p \le \left(2\sup_{q\in\mathcal{P}_r}\frac{\|q\|_\infty}{\|q\|_1}\right)^{1/p}\|f\|_p.$$

Proof. Condition (1) implies, in particular, that for $q=E_1f$,

$$\left|\int_{E_+} q(x)\,dx - \int_{E_-} q(x)\,dx\right| \le \int_{E_0} |q(x)|\,dx. \tag{2}$$

Our approach is to bound from below $\|f\|_p$ among all f satisfying (2) for a particular q.

We introduce the function

$$g(x)=\begin{cases} 0, & x\in E_+,\ q(x)>0,\\ q(x), & x\in E_+,\ q(x)\le 0,\\ 0, & x\in E_-,\ q(x)<0,\\ q(x), & x\in E_-,\ q(x)\ge 0,\\ q(x), & x\in E_0.\end{cases}$$

Clearly, $\|g\|_p\le\|f\|_p$ and the sets $\tilde E_+\subset E_+$, $\tilde E_-\subset E_-$, and $\tilde E_0\supset E_0$ for g and q satisfy

$$\left|\int_{\tilde E_+} q(x)\,dx - \int_{\tilde E_-} q(x)\,dx\right| \le \int_{\tilde E_0} |q(x)|\,dx. \tag{3}$$

Now $\int_I |g|^p = \int_{\tilde E_0}|q|^p$, and to further bound $\|f\|_p$ from below we minimize $\int_{\tilde E_0}|q(x)|^p\,dx$ among all partitions $(\tilde E_+,\tilde E_-,\tilde E_0)$ with $q>0$ on $\tilde E_+$, $q<0$ on $\tilde E_-$, and (3) holding. These conditions imply that

$$\int_{\tilde E_+\cup\tilde E_-}|q(x)|\,dx \le \int_{\tilde E_0}|q(x)|\,dx;$$

i.e.,

$$\int_{\tilde{E}_0} |q(x)|\,dx \geq \frac{1}{2}\|q\|_1.$$

We claim that the best choice of $\tilde{E}_0$ satisfies

$$\inf_{x\in\tilde{E}_0} |q(x)| \geq \sup_{x\notin\tilde{E}_0} |q(x)| \quad \text{and} \quad \int_{\tilde{E}_0} |q(x)|\,dx = \frac{1}{2}\|q\|_1.$$

Suppose $\tilde{E}_0'$ is any other choice. We can assume $\int_{\tilde{E}_0'} |q(x)|\,dx = \frac{1}{2}\|q\|_1$, because otherwise we could make $\tilde{E}_0'$, and, *a fortiori*, $\int_{\tilde{E}_0'} |q(x)|^p\,dx$, smaller.

Let a be any number between $\sup_{x\notin\tilde{E}_0} |q(x)|$ and $\inf_{x\in\tilde{E}_0} |q(x)|$, and let $A = \tilde{E}_0 \setminus \tilde{E}_0'$ and $B = \tilde{E}_0' \setminus \tilde{E}_0$. Then $\int_A |q(x)|\,dx = \int_B |q(x)|\,dx$ and

$$\int_A |q(x)|\,dx = \int_A |q(x)|^p |q(x)|^{1-p}\,dx \geq a^{1-p} \int_A |q(x)|^p\,dx,$$
$$\int_B |q(x)|\,dx = \int_B |q(x)|^p |q(x)|^{1-p}\,dx \leq a^{1-p} \int_B |q(x)|^p\,dx.$$

Therefore,

$$\int_A |q(x)|^p\,dx \leq \int_B |q(x)|^p\,dx,$$

and

$$\int_{\tilde{E}_0} |q(x)|^p\,dx \leq \int_{\tilde{E}_0'} |q(x)|^p\,dx.$$

So

$$\frac{\int_I |q(x)|^p\,dx}{\int_I |f(x)|^p\,dx} \leq \frac{\int_I |q(x)|^p\,dx}{\int_{\tilde{E}_0} |q(x)|^p\,dx}.$$

Since $\int_{\tilde{E}_0} |q(x)|\,dx = \frac{1}{2}\|q\|_1$, we have

$$\|q\|_\infty |\tilde{E}_0| \geq \frac{1}{2}\|q\|_1, \quad \text{or} \quad |\tilde{E}_0| \geq \frac{1}{2}\frac{\|q\|_1}{\|q\|_\infty}.$$

Now

$$\int_{I\setminus\tilde{E}_0} |q(x)|^p\,dx \leq a^p(1-|\tilde{E}_0|)$$

and

$$\int_{\tilde{E}_0} |q(x)|^p\,dx \geq a^p |\tilde{E}_0|.$$

So

$$\frac{\int_I |q(x)|^p\,dx}{\int_{\tilde{E}_0} |q(x)|^p\,dx} = 1 + \frac{\int_{I\setminus\tilde{E}_0} |q(x)|^p\,dx}{\int_{\tilde{E}_0} |q(x)|^p\,dx} \leq 1 + \frac{1-|\tilde{E}_0|}{|\tilde{E}_0|} = \frac{1}{|\tilde{E}_0|} \leq 2\frac{\|q\|_\infty}{\|q\|_1}.$$

Therefore, $\|q\|_p \leq (2K)^{1/p}\|f\|_p$, where $K = \sup_{q\in\mathcal{P}_r}(\|q\|_\infty/\|q\|_1)$. $\square$

Corollary. *For each f in $L^p(I)$, $0<p<1$, a best $L^1(I)$ polynomial approximation $E_1 f$ exists.*

Proof. For each positive integer n, define f_n by

$$f_n(x)=\begin{cases} n, & f(x)>n,\\ f(x), & |f(x)|\le n,\\ -n, & f(x)<-n.\end{cases}$$

Then $f_n\in L^\infty(I)$ and $\|f_n\|_p\le\|f\|_p$. Best $L^1(I)$ approximations q_n to f_n exist for all n, and

$$\|q_n\|_p\le C\|f_n\|_p\le C\|f\|_p.$$

However, for each p and r there is a constant C_1 such that for all $q\in\mathcal{P}_r$, $\|q\|_\infty\le C_1\|q\|_p$. Therefore for all n we have $\|q_n\|_\infty\le CC_1\|f\|_p$, so that for some n we have $\|q_n\|_\infty<n$. This implies that we can choose $E_1 f=q_n$ because the sets E_+, E_-, and E_0 are the same for f as for f_n. $\square$

Proof of the main theorem. The corollary shows that $E_1 f$ exists. Now E_1 is linear with respect to addition of polynomials in $\mathcal{P}_r$: If we let $g=f+q$, $q\in\mathcal{P}_r$, and $E_1 g=E_1 f+q$, then clearly condition (1) is satisfied because E_+, E_-, and E_0 are the same for g and $E_1 g$ as for f and $E_1 f$. So $E_1 f+q$ is a best $L^1(I)$ approximation to $f+q$ according to our definition.

Finally, because $\|f+g\|_p^p\le\|f\|_p^p+\|g\|_p^p$, we have for all $q\in\mathcal{P}_r$,

$$\begin{aligned}\|f-E_1 f\|_p^p&\le\|f-q\|_p^p+\|q-E_1 f\|_p^p\\ &=\|f-q\|_p^p+\|E_1(q-f)\|_p^p\\ &\le(1+2K)\|f-q\|_p^p\end{aligned}$$

by the lemma. $\square$

References

1. L. G. Brown, *L^p best approximation operators are bounded on L^{p-1}*, in preparation.
2. R. A. DeVore, B. Jawerth, and B. J. Lucier, *Image compression through wavelet transform coding*, IEEE Trans. Information Theory **38** (1992), 719–746.
3. ______, *Surface compression*, Computer Aided Geometric Design **9** (1992), 219–239.
4. R. A. DeVore, B. Jawerth, and V. A. Popov, *Compression of wavelet decompositions*, Amer. J. Math. **114** (1992), 737–785.
5. R. A. DeVore and V. A. Popov, *Free multivariate splines*, Constr. Approx. **3** (1987), 239–248.
6. ______, *Interpolation of Besov spaces*, Trans. Amer. Math. Soc. **305** (1988), 397–414.

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address: lgb@math.purdue.edu, lucier@math.purdue.edu