A Short Reduction of Lucier's BV-Dual Noise Problem to the Ding–Wirth Theorem*

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Abstract

Lucier conjectured that independent Gaussian noise on an $N \times N$ pixel grid has expected dual BV norm of order $(\log N)/N$. We show that the Ding–Wirth greedy-lattice-animal theorem instead gives the sharp order $(\log N)^{3/4}/N$. The reduction becomes particularly short after replacing the usual BV norm by the equivalent anisotropic variation of the zero extension: pixel averaging is then contractive, and the resulting discrete coarea formula involves exactly the full lattice boundary used by Ding and Wirth.

The conjecture. Let $Q = (0,1)^2$, $h = N^{-1}$, and $V_N = \{0, \dots, N-1\}^2$. Let $Q_i = h(i + [0,1)^2)$, let $(g_i)_{i \in V_N}$ be independent standard Gaussians, and set

$$\varepsilon_N = \sum_{i \in V_N} g_i \mathbf{1}_{Q_i}, \qquad Z_N = \sup_{\|f\|_{L^1(Q)} + |Df|(Q) \le 1} \left| \int_Q f \varepsilon_N \right|.$$

Motivated by [2, Section 8.2], Lucier asks whether there are constants $c, C > 0$, independent of N, for which

Conjecture 1 (Lucier [4])**.**

$$c \frac{\log N}{N} \le \mathbb{E} Z_N \le C \frac{\log N}{N}.$$

The full BV norm is essential: using only $|Df|(Q)$ without a mean-zero constraint makes the supremum infinite almost surely.

The Ding–Wirth input. For a finite $A \subset \mathbb{Z}^2$, let ∂A be its full unoriented nearest-neighbor edge boundary in $\mathbb{Z}^2$. All connectivity is nearest-neighbor connectivity; a connected A is simply connected when $\mathbb{Z}^2 \setminus A$ has no finite connected components. Put $\Lambda_R = [-R, R]^2 \cap \mathbb{Z}^2$, let $\mathfrak{A}_R$ be the family of simply connected subsets of Λ_R containing 0, and let $\mathfrak{B}_R$ and $\mathcal{B}_R$ be, respectively, the families of all simply connected and all connected subsets of Λ_R, with no root requirement. Write $G(A) = \sum_{i \in A} G_i$ for an i.i.d. standard Gaussian field on $\mathbb{Z}^2$.

Theorem 2 (Ding–Wirth [3])**.** *There are universal constants $c, C > 0$ such that, for $R \ge 3$,*

$$\mathbb{E} \max_{A \in \mathfrak{A}_R} \frac{G(A)}{|\partial A|} \ge c (\log R)^{3/4}. \tag{1}$$

Moreover, for every $R > 1$ and $t > 0$,

$$\mathbb{P}\left(\max_{A \in \mathfrak{B}_R} \frac{G(A)}{|\partial A|} > C (\log R)^{3/4} + t \right) \le e^{-t^2/2}, \tag{2}$$

These are the lower bound in Theorem 1.6 and Proposition 2.2 of the third arXiv version of [3]. In particular, the upper estimate already concerns *unrooted* animals. Ding and Wirth trace the underlying lattice-animal estimate to Talagrand's matching theory [5, Section 4.4].

Lemma 3 (Hole filling)**.** *For every weight field $(w_i)_{i \in \Lambda_R}$, with $w(A) = \sum_{i \in A} w_i$,*

$$\max_{A \in \mathcal{B}_R} \frac{|w(A)|}{|\partial A|} = \max_{A \in \mathfrak{B}_R} \frac{|w(A)|}{|\partial A|}. \tag{3}$$

*Discovered by an internal OpenAI model, communicated by `jrl@openai.com`

Proof. If a connected $A \subseteq \Lambda_R$ is already simply connected, there is nothing to prove. Otherwise, let $H_1, \dots, H_k$ be the finite connected components of $\mathbb{Z}^2 \setminus A$ and put $\widehat{A} = A \cup H_1 \cup \cdots \cup H_k$. These sets remain in Λ_R, since its exterior lies in the infinite component of $\mathbb{Z}^2 \setminus A$. They are simply connected: the complement of $\widehat{A}$ is that infinite component, while the complement of each H_j is connected because every other component of $\mathbb{Z}^2 \setminus A$ has an edge to the connected set A. Moreover,

$$w(A) = w(\widehat{A}) - \sum_{j=1}^{k} w(H_j), \qquad |\partial A| = |\partial \widehat{A}| + \sum_{j=1}^{k} |\partial H_j|.$$

The triangle inequality and the weighted-average bound therefore give

$$\frac{|w(A)|}{|\partial A|} \le \max\left\{ \frac{|w(\widehat{A})|}{|\partial \widehat{A}|}, \max_{1\le j\le k} \frac{|w(H_j)|}{|\partial H_j|} \right\}.$$

The reverse inequality is immediate from $\mathfrak{B}_R \subseteq \mathcal{B}_R$. $\square$

An equivalent norm that removes boundary bookkeeping. For $f \in BV(Q)$, let $\widetilde{f}$ denote its extension by zero to $\mathbb{R}^2$, and define

$$J(f) = |D_1\widetilde{f}|(\mathbb{R}^2) + |D_2\widetilde{f}|(\mathbb{R}^2). \tag{4}$$

Lemma 4. *For every $f \in BV(Q)$,*

$$\frac{4}{5}\big(\|f\|_{L^1(Q)} + |Df|(Q)\big) \le J(f) \le 4\big(\|f\|_{L^1(Q)} + |Df|(Q)\big). \tag{5}$$

Proof. For $v \in BV(0,1)$, the one-sided traces satisfy

$$|v(0+)| + |v(1-)| \le |Dv|(0,1) + 2\,\|v\|_{L^1(0,1)}\,. \tag{6}$$

Indeed, for almost every $t \in (0,1)$, bound each endpoint value by $|v(t)|$ plus the variation between that endpoint and t, and then integrate in t. Slicing first horizontally and then vertically, and using the zero-extension trace formula, gives

$$J(f) \le 2\big(|D_1 f|(Q) + |D_2 f|(Q)\big) + 4\,\|f\|_{L^1(Q)} \le 4\big(|Df|(Q) + \|f\|_{L^1(Q)}\big).$$

Conversely, $|Df|(Q) \le J(f)$. Also, for the zero extension $\widetilde{v}$ of a one-dimensional BV function,

$$|D\widetilde{v}|(\mathbb{R}) \ge 2|v(t)| \quad \text{for almost every } t \in (0,1),$$

because the function starts and ends at zero. Integrating and slicing in both coordinate directions yields $J(f) \ge 4\,\|f\|_{L^1(Q)}$. Hence $|Df|(Q) + \|f\|_{L^1(Q)} \le \frac{5}{4} J(f)$, proving the other inequality. Standard BV slicing and trace facts are recorded in [1, Chapter 3]. $\square$

Exact discretization. Let $P_N f$ denote pixelwise averaging. Anisotropic variation is contractive under averaging on the uniform rectangular grid:

$$J(P_N f) \le J(f). \tag{7}$$

For completeness, if $v \in BV(\mathbb{R})$ and $a_k = h^{-1}\int_{kh}^{(k+1)h} v$, then

$$a_{k+1} - a_k = \int_{\mathbb{R}} \psi_k \, dDv, \qquad \psi_k(s) = h^{-1}\left|\{t \in (kh,(k+1)h) : t < s \le t+h\}\right|.$$

The triangular hat functions satisfy $\psi_k \ge 0$ and $\sum_{k\in\mathbb{Z}} \psi_k = 1$, whence

$$\sum_{k\in\mathbb{Z}} |a_{k+1} - a_k| \le |Dv|(\mathbb{R}).$$

Applying this estimate to horizontal and vertical slices, with Jensen for averaging in the transverse coordinate, proves (7). The grid aligns with ∂Q, so the grid average of $\widetilde{f}$ is precisely the zero extension of $P_N f$.

Since ε_N is pixelwise constant,

$$\int_Q f\varepsilon_N = \int_Q (P_N f)\varepsilon_N.$$

It follows from (7) that the J-dual norm is computed exactly by pixelwise-constant functions. For $f = \sum_{i \in V_N} u_i \mathbf{1}_{Q_i}$, extend u by zero outside V_N. Then

$$\int_Q f\varepsilon_N = h^2 \sum_{i \in V_N} g_i u_i, \qquad J(f) = h \sum_{\substack{\{i,j\} \subset \mathbb{Z}^2 \\ i \sim j}} |u_i - u_j|.$$

Here and below, edge sums count each unoriented edge once. Consequently,

$$\|\varepsilon_N\|_{J^*} = \frac{1}{N} \sup_{u \neq 0} \frac{|\sum_{i \in V_N} g_i u_i|}{\sum_{i \sim j} |u_i - u_j|}. \tag{8}$$

For $t > 0$, let $A_t^+ = \{i : u_i > t\}$ and $A_t^- = \{i : u_i < -t\}$. Discrete coarea gives

$$\sum_{i \sim j} |u_i - u_j| = \int_0^\infty \big(|\partial A_t^+| + |\partial A_t^-|\big)\, dt,$$

while layer cake gives

$$\sum_i g_i u_i = \int_0^\infty \big(g(A_t^+) - g(A_t^-)\big)\, dt, \qquad g(A) = \sum_{i \in A} g_i.$$

Therefore the supremum in (8) is bounded above by $\max_{\varnothing \neq A \subseteq V_N} |g(A)|/|\partial A|$; testing $u = \mathbf{1}_A$ gives equality. Decomposing A into connected components and using additivity of their full edge boundaries shows that the maximum may be restricted to connected sets. Hence

$$\boxed{\|\varepsilon_N\|_{J^*} = \frac{1}{N} \max_{\substack{\varnothing \neq A \subseteq V_N \\ A \text{ connected}}} \frac{|g(A)|}{|\partial A|}.} \tag{9}$$

Conclusion. Define the maximum on the right-hand side of (9), without the factor $1/N$, by T_N. Extend $(g_i)_{i \in V_N}$ to an i.i.d. field on $\mathbb{Z}^2$. Since $V_N \subseteq \Lambda_N$, equations (2) and (3), applied to both G and $-G$, imply

$$\mathbb{P}\big(T_N > C(\log N)^{3/4} + t\big) \leq 2e^{-t^2/2}, \qquad t > 0.$$

Integrating this inequality gives $\mathbb{E}T_N \leq C'(\log N)^{3/4}$. Conversely, for $N \geq 7$, a translate of Λ_R with $R = \lfloor (N-1)/2 \rfloor$ fits inside V_N; applying (1) to the simply connected animals containing its center gives $\mathbb{E}T_N \geq c'(\log N)^{3/4}$. For $3 \leq N < 7$, use $\mathbb{E}T_N \geq \mathbb{E}|g_0|/4$. Finally, (5) and (9) yield the sharp answer.

Theorem 5. *For universal constants $0 < c \leq C < \infty$ and every $N \geq 3$,*

$$c\frac{(\log N)^{3/4}}{N} \leq \mathbb{E}\, \|\varepsilon_N\|_{BV(Q)^*} \leq C\frac{(\log N)^{3/4}}{N}.$$

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