HIGH ORDER REGULARITY FOR SOLUTIONS OF THE INVISCID BURGERS EQUATION*

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Abstract. We discuss a recent Besov space regularity theory for discontinuous, entropy solutions of quasilinear, scalar hyperbolic conservation laws in one space dimension. This theory is very closely related to rates of approximation in L^1 by moving grid, finite element methods. In addition, we establish the Besov space regularity of solutions of the inviscid Burgers equation; the new aspect of this study is that no assumption is made about the local variation of the initial data.

Key words. regularity, moving grid finite elements, Besov spaces, conservation laws

AMS(MOS) subject classifications. 35L65, 35D10, 41A25, 46E35

1. Introduction

A regularity theory is developed in [2] and [8] for discontinuous, entropy solutions $u(x,t)$ of the scalar hyperbolic conservation law

$$\begin{aligned} &u_t + f(u)_x = 0, && x \in \mathbb{R}, \quad t > 0,\\ &u(x,0) = u_0(x), && x \in \mathbb{R}, \end{aligned} \tag{C}$$

under the assumption that f is uniformly convex and $u_0 \in \mathrm{BV}(\mathbb{R})$ has bounded support. In this theory one measures the regularity of $u(\,\cdot\,,t)$ in Besov spaces $B^\alpha_q(L^p(I))$; functions in these spaces have, roughly speaking, $\alpha > 0$ "derivatives" in $L^p(I)$, where I is a bounded interval, and q is a secondary index of regularity. (See §2 for precise definitions.)

Whereas the solutions of many evolution equations (such as the heat equation) have enough regularity in Sobolev spaces to be approximated to high order in $L^p(I)$ by piecewise polynomial splines defined on *uniform* grids with grid spacing $1/n$, discontinuous solutions of (C) can be approximated by splines on uniform grids to at most $O(n^{-1/p})$ in $L^p(I)$. Thus, if one would like high-order approximation by splines, one is led to consider approximations drawn from the class of piecewise polynomials defined on *arbitrary* grids with n intervals, i.e., *free knot* splines. Such approximations occur in moving grid finite element approximations to time-dependent partial differential equations, such as those used by Miller [9], Glimm et al. [5], and Lucier [7]. The following questions then arise: What regularity is needed to ensure high-order approximation in $L^p(I)$ by free knot splines, and do solutions of (C) maintain this regularity as time progresses? The answers are that regularity in certain Besov spaces is necessary and sufficient for certain orders

* Reprint from *Nonlinear Hyperbolic Problems,* C. Carasso et al., ed., Springer Lecture Notes in Mathematics, v. 1402, 1989, pp. 147–154.

of approximation by free knot splines, and that solutions of (C) retain this regularity if one considers approximation in L^1.

At this point, it is useful to contrast the approximation properties of functions in Sobolev spaces $W^{\alpha,p}(I)$, $\alpha > 0$, $p > 1$, with those of functions in the Besov spaces $B_q^\alpha(L^q(I))$, $\alpha > 0$, $q = 1/(\alpha + 1/p)$, $p > 0$. For $u \in L^p(I)$, I a finite interval, $p > 1$, define

$$s_n(u)_p := s_{n,r}(u)_p := \inf_{P \in S_n} \|u - P\|_{L^p(I)},$$

where $S_n := S_{n,r}$ is the set of all piecewise polynomials of degree less than r on a *uniform* grid of size $|I|/n$. (We use the notation := to mean "is defined as".) Then it can be shown that

$$u \in W^{\alpha,p}(I) \Longleftrightarrow \begin{cases} \sup\limits_{n>0} n^\alpha s_n(u)_p < \infty, & \alpha = r \in \mathbb{Z}, \\ \left(\sum\limits_{n=1}^{\infty} [n^\alpha s_n(u)_p]^p n^{-1} \right)^{1/p} < \infty, & r - 1 < \alpha < r \in \mathbb{Z}, \end{cases} \tag{1.1}$$

and that the quantities on the right of (1.1) are equivalent to the seminorm $|u|_{W^{\alpha,p}(I)}$. (For α not an integer, the Sobolev space $W^{\alpha,p}(I)$ is the same as the Besov space $B_p^\alpha(L^p(I))$; see [1, p. 223].)

Recent results of Petrushev [10], [11] and DeVore and Popov [3], [4] provide a characterization of functions that can be approximated to high order by piecewise polynomials in $\Sigma_n := \Sigma_{n,r}$, the set of all piecewise polynomials of degree less than r on *arbitrary* grids with n intervals. Define for $p > 0$

$$\sigma_n(u)_p := \sigma_{n,r}(u)_p := \inf_{P \in \Sigma_n} \|u - P\|_{L^p(I)},$$

and let $q = 1/(\alpha + 1/p)$. Then for $\alpha < r$,

$$u \in B_q^\alpha(L^q(I)) \Longleftrightarrow \left(\sum_{n=1}^{\infty} [n^\alpha \sigma_n(u)_p]^q n^{-1} \right)^{1/q} < \infty, \tag{1.2}$$

where the right hand side of (1.2) is equivalent to the "seminorm" $|u|_{B_q^\alpha(L^q(I))}$. (This "seminorm" does not satisfy the triangle inequality if $q < 1$, in which case $B_q^\alpha(L^q(I))$ is not locally convex, but only locally quasiconvex.)

This suggests that perhaps there are certain spaces $X := B_q^\alpha(L^q(I))$ that are regularity spaces for (C); i.e., spaces X for which $u_0 \in X$ implies $u(\,\cdot\,, t) \in X$. In this direction, the following general theorem is proved in [2]:

THEOREM 1.1. *Assume that r is a positive integer and that $u_0 \in \mathrm{BV}(\mathbb{R})$ has support in $I := [0, 1]$. Then there exists a constant $C_1 := C_1(r)$ such that the following statements are valid. Let $\Omega = \{y \mid |y| < C_1 \|u_0\|_{L^\infty(\mathbb{R})}\}$. Assume that there is a constant C_2 such that for all $\xi \in \Omega$, $|f^{(r+1)}(\xi)| < C_2$ and $f''(\xi) \geq 1/C_2$. Then for any positive $\alpha < r$ and time $t > 0$ there exists a constant C such that if $u_0 \in B^\alpha(I) := B_q^\alpha(L^q(I))$, where $q = 1/(\alpha+1)$, then $u(\,\cdot\,, t)$, the solution of (C), has support in $I_t = [\inf_{\xi\in\Omega} f'(\xi)t, 1 + \sup_{\xi\in\Omega} f'(\xi)t]$ and $\|u(\,\cdot\,, t)\|_{B^\alpha(I_t)} \leq C(\|u_0\|_{B^\alpha(I)} + 1)$.*

In this paper we examine the special case of the inviscid Burgers equation,

$$\begin{aligned} &u_t + (u^2)_x = 0, \qquad && x \in \mathbb{R}, \quad t > 0, \\ &u(x,0) = u_0(x), \qquad && x \in \mathbb{R}. \end{aligned} \tag{B}$$

In this case, we are able to avoid the requirement that the total variation of u_0 be bounded, and we can show that the Besov space norm of u is bounded independently of time. Furthermore, the proof is simpler. Thus, in §3 we prove the following theorem:

Theorem 1.2. *Assume that $u_0 \in L^1(\mathbb{R})$ has support in $I := [0,1]$ and that $f(u) = u^2$. Then for any positive α there exists a constant C such that if $u_0 \in B^\alpha(I) := B^\alpha_q(L^q(I))$, where $q = 1/(\alpha+1)$, then $u(\,\cdot\,,t)$ has support in $I_t = [-(8\|u_0\|_{L^1(\mathbb{R})}t)^{1/2}, 1 + (8\|u_0\|_{L^1(\mathbb{R})}t)^{1/2}]$ and $\|u(\,\cdot\,,t)\|_{B^\alpha(I_t)} \le C\|u_0\|_{B^\alpha(I)}$.*

2. Preliminaries

In this section, we recall the definition of Besov spaces, present relevant properties of the solutions of (C), and, finally, restate lemmas found in [2] that will be useful here.

Let I be a finite interval. Fix $0 < \alpha < \infty$, $0 < q \le \infty$ and $0 < p < \infty$, and pick an integer $r > \alpha$. Define the $L^p(I)$ modulus of continuity $\omega_r(f,t)_p$ to be the supremum over all $0 < h < t$ of $\|\Delta^r_h f\|_{L^p(I_h)}$, where $I_h = \{x \in I \mid x + rh \in I\}$, and $\Delta^0_h f(x) := f(x)$ and $\Delta^r_h f(x) := \Delta^{r-1}_h f(x+h) - \Delta^{r-1}_h f(x)$. The Besov space $B^\alpha_q(L^p(I))$ is defined to be the set of all functions $f \in L^p(I)$ for which

$$|f|_{B^\alpha_q(L^p(I))} := \left(\int_0^\infty [t^{-\alpha}\omega_r(f,t)_p]^q \, dt/t\right)^{1/q}$$

is finite. Set $\|f\|_{B^\alpha_q(L^p(I))} := \|f\|_{L^p(I)} + |f|_{B^\alpha_q(L^p(I))}$.

We are particularly interested in the spaces $B^\alpha(I) := B^\alpha_q(L^q(I))$, $\alpha > 0$, where $q := 1/(\alpha+1)$. These spaces have the property that if $\alpha' > \alpha$ then $B^{\alpha'}(I)$ is continuously embedded in $B^\alpha(I)$, which in turn is continuously embedded in $L^1(I)$. We define $B^0(I) := L^1(I)$.

The spaces $B^\alpha(I)$, $\alpha > 0$, form a real interpolation family. The real method of interpolation using K-functionals can be described as follows: For any two linear, complete, quasi-normed spaces X_0 and X_1 continuously embedded in a linear Hausdorff topological space X, define the following functional for all f in $X_0 + X_1$:

$$K(f,t,X_0,X_1) := \inf_{f=f_0+f_1} \{\|f_0\|_{X_0} + t\|f_1\|_{X_1}\},$$

where $f_0 \in X_0$ and $f_1 \in X_1$. The new space $X_{\theta,q} := (X_0,X_1)_{\theta,q}$ ($0 < \theta < 1$, $0 < q \le \infty$) consists of functions f for which

$$\|f\|_{X_{\theta,q}} := \|f\|_{X_0+X_1} + \left(\int_0^\infty [t^{-\theta}K(f,t,X_0,X_1)]^q \, dt/t\right)^{1/q} < \infty,$$

where $\|f\|_{X_0+X_1} := K(f,1,X_0,X_1)$. DeVore and Popov [3] showed that if $\beta > \gamma > \alpha \ge 0$, $q = 1/(\gamma+1)$, and θ is defined by $\gamma = (1-\theta)\alpha + \theta\beta$, then $(B^\alpha(I), B^\beta(I))_{\theta,q} = B^\gamma(I)$. In particular, $(L^1(I), B^\beta(I))_{\alpha/\beta, 1/(\alpha+1)} = B^\alpha(I)$.

As we have noted earlier, the Besov spaces $B^\alpha(I)$ are intimately related to approximation by piecewise polynomials with free knots. For each pair of positive integers n

and r, let $\Sigma_n := \Sigma_{n,r}$ denote the collection of all piecewise polynomials on I of degree less than r with at most 2^n pieces. (This is slightly different than in §1.) If f is in $L^1(I)$ and $n \geq 0$, we let

$$\sigma_n(f)_1 := \sigma_{n,r}(f)_1 := \inf_{v \in \Sigma_n} \|f - v\|_{L^1(I)}$$

be the error in approximating f in the $L^1(I)$ norm by the elements of Σ_n; $s_{-1}(f)_1 := \|f\|_{L^1(I)}$. As a special case of (1.2) we have that a function f is in $B^\alpha(I)$ with $\alpha > 0$ if and only if

$$\|f\|_{\mathcal{A}_q^\alpha(L^1(I))} := \left(\sum_{n=-1}^{\infty} (2^{n\alpha} \sigma_n(f)_1)^q \right)^{1/q} < \infty, \tag{2.1}$$

and $\|f\|_{\mathcal{A}_q^\alpha(L^1(I))}$ is equivalent to $\|f\|_{B^\alpha(I)}$. More generally (see [4]), if $\beta > \alpha$ and $0 < q \leq \infty$ then $\mathcal{A}_q^\alpha(L^1(I)) = (L^1(I), B^\beta(I))_{\alpha/\beta,q}$. The characterization given here of the equivalence between approximation and regularity is more suited to our present purposes than the one given in §1.

We will now relate certain properties of conservation laws, all of which can be found in the monograph by Lax [6]. When $f(u) = u^2$, (C) is the inviscid Burgers equation, (B). Given x and t, Lax shows that $u(x,t) = u_0(y)$, where $y := y(x,t)$ is a solution (there may be many) of the implicit equation $y = x - 2tu_0(y)$, and furthermore $y(x,t)$ is an increasing function of x for each fixed $t > 0$. It follows that when u_0 has support in $[0,1]$ and is piecewise polynomial of degree less than r with 2^n pieces then $u(x,t)$ is piecewise an algebraic curve in x for each t. On each piece, u satisfies the equation

$$u = P_i(x - 2ut), \tag{2.2}$$

where P_i is one of the polynomial pieces of u_0. Furthermore, u can only have jump discontinuities that decrease. It follows that there are no more than $r2^n$ pieces in the definition of $u(\,\cdot\,,t)$ for all $t > 0$.

Lax also shows that for any $u_0 \in L^1([0,1])$ the support of $u(\,\cdot\,,t)$ is contained in $I_t := [-(8\|u_0\|_{L^1(\mathbb{R})}t)^{1/2}, 1 + (8\|u_0\|_{L^1(\mathbb{R})}t)^{1/2}]$. Also, if $u(x,t)$ and $v(x,t)$ are solutions of (B) with initial data u_0 and v_0 respectively, then

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})}. \tag{2.3}$$

Thus, if v_0 is a best piecewise polynomial approximation in $L^1([0,1])$ from Σ_n to u_0 (i.e., $\|u_0 - v_0\|_{L^1(I)} = \sigma_n(u_0)_1$) and $U_n(x,t) := v(x,t)$ is the solution of (B) with initial data v_0, then

$$\|u(\,\cdot\,,t) - U_n(\,\cdot\,,t)\|_{L^1(I_t)} \leq \|u_0 - U_n(\,\cdot\,,0)\|_{L^1(I)} = \sigma_n(u_0)_1. \tag{2.4}$$

It will be useful to redefine the values of $U_n(x,t)$ for $x \notin I_t$ to be zero. Then (2.4) remains valid because $u(x,t) = 0$ for $x \notin I_t$.

We will need the following lemmas, which are proved in [2]. Let

$$\|g\|_p^*(I) := \left(\frac{1}{|I|} \int_I |g|^p \right)^{1/p}.$$

Lemma 2.1 (Equivalence of Norms). *Let ϕ and ψ be defined on an interval I as the functional inverses of polynomials P and Q of degree $\le d$; assume that ϕ and ψ are monotone on I. Then for all $1 \le p < d/(d-1)$*

$$\|\phi - \psi\|_p^*(I) \le C(p,d)\|\phi - \psi\|_1^*(I). \tag{2.5}$$

Lemma 2.2 (Bounded Oscillation). *Assume that P and Q are polynomials with real coefficients in two variables of total degree less than r. Let ϕ and ψ be functions that are real analytic in the interior of an interval I and satisfy $P(x,\phi)=0$ and $Q(x,\psi)=0$ for $x \in I$. Let $A = \phi - \psi$. Then for $k = 0, 1, \ldots, r+1$ either $A^{(k)}$ is identically zero on I or $A^{(k)}(x) = 0$ has finitely many solutions x in I. The number of solutions depends only on r.*

Lemma 2.3 (Inverse Inequality). *Let v be twice continuously differentiable on an open interval I and assume that v, v', and v'' each have one sign on I. If numbers p and q are given such that $0 < p \le 1$ and $qp < q - p$, then there exists a constant C such that whenever $v \in L^q(I)$ then $v' \in L^p(I)$ and*

$$\|v'\|_p^*(I) \le C|I|^{-1}\|v\|_q^*(I). \tag{2.6}$$

3. Proof of Theorem 1.2

If $u_0 \in B^\alpha(I) := B_q^\alpha(L^q(I))$, $q = 1/(\alpha+1)$, then by (2.1) u_0 can be approximated well in $L^1(I)$ by piecewise polynomial functions of degree less than r; inequality (2.4) then shows that $u(\cdot,t)$ can be approximated well by piecewise algebraic curves of a certain degree. The proof of Theorem 1.2 consists of showing that good approximation by algebraic curves of the form (2.2) implies $u(\cdot,t) \in B^\alpha(I_t)$.

Assume first that α is less than, but close to, an integer r, and $u_0 \in B^\alpha(I)$. Then by the characterization (2.1), $\sum[2^{n\alpha}\sigma_n(u_0)_1]^q < \infty$. From (2.4) we obtain that $U_n(\cdot,t)$ converges to $u(\cdot,t)$ in $L^1(I_t)$ and therefore

$$u = U_0 + \sum_{n=0}^{\infty}(U_{n+1} - U_n) = \sum_{n=-1}^{\infty} T_n,$$

where $T_{-1} := U_0$ and for later use we define $U_{-1} := 0$.

From the form of the function $U_n(x,t)$ discussed in §2, we can write for $n = -1, 0, \ldots$

$$T_n = \sum_{j=1}^{N} A_j, \qquad N \le C2^n,$$

where C depends on r. Here $A_j = (\phi_j - \psi_j)\chi_j$ with ϕ_j and ψ_j algebraic functions and χ_j the characteristic function of an interval I_j. The intervals I_j, $j = 1, \ldots, N$ are piecewise disjoint. We can further assume by Lemma 2.2 that $A_j^{(k)}$ has one sign on I_j for each $k = 0, \ldots, r+1$ and $1 \le j \le N$.

We fix j and measure the smoothness of $A := A_j$. For this, fix h and consider the sets Γ of all x such that $\{x, x+h, \ldots, x+rh\} \subset I := I_j$, Γ' of all $x \notin \Gamma$ for which $\{x, x+h, \ldots, x+rh\} \cap I \ne \phi$, and Γ'' of all remaining $x \in \mathbb{R}$.

For $x \in \Gamma''$, $\Delta_h^r(A,x) = 0$, so

$$\int_{\Gamma''} |\Delta_h^r(A,x)|^q \, dx = 0. \tag{3.1}$$

For $x \in \Gamma'$, $\Delta_h^r(A,x) \le 2^r(|A(x)| + \cdots + |A(x+rh)|)$. Since Γ' has measure no greater than $2r\min(h,|I|)$, we have for a fixed $p > 1$ with $p < 1 + 1/r$, by Hölder's inequality

$$\int_{\Gamma'} |\Delta_h^r(A,x)|^q \, dx \le C[\min(h,|I|)]^{1-q/p} \left(\int_I |A(x)|^p \right)^{q/p}. \tag{3.2}$$

We can write $A = \phi - \psi$ where ϕ is a piece of U_{n+1} and ψ is a piece of U_n. From (2.2), we can write ϕ as

$$\phi = \frac{x - (I + 2tP_1)^{-1}(x)}{2t}, \tag{3.3}$$

where P_1 is one of the polynomial pieces in the definition of $U_{n+1}(0)$; similarly for ψ. Therefore,

$$\begin{aligned} \|\phi - \psi\|_p^*(I) &= \frac{1}{2t} \|(I + 2tP_1)^{-1} - (I + 2tP_2)^{-1}\|_p^*(I) \\ &\le \frac{C}{2t} \|(I + 2tP_1)^{-1} - (I + 2tP_2)^{-1}\|_1^*(I) \\ &= C\|\phi - \psi\|_1^*(I). \end{aligned} \tag{3.4}$$

Here the first equality is (3.3) and the inequality that follows is by Lemma 2.1. Therefore, from (3.2) and (3.4) we can conclude that

$$\int_{\Gamma'} |\Delta_h^r(A,x)|^q \, dx \le C[\min(h,|I|)]^{1-q/p} |I|^{-q+q/p} \left(\int_I |A(x)| \, dx \right)^q. \tag{3.5}$$

We next consider $x \in \Gamma$. Because $A^{(r)}$ is monotone on I, we know that for each x there is a ξ such that

$$|\Delta_h^r(A,x)| = C(r)h^r |A^{(r)}(\xi)| \le Ch^r \max(|A^{(r)}(x)|, |A^{(r)}(x+rh)|).$$

Without loss of generality assume that the maximum is attained by the first term. For a number $\epsilon > 0$ to be specified in a moment, let $\alpha_r := \alpha$ and $\alpha_k := \alpha_{k+1} - 1 - \epsilon$, $k = r-1, \ldots, 0$, and let $q_k := 1/(\alpha_k + 1)$. Then by choosing ϵ appropriately, we will have $q_0 = p$, where p is as in (3.4). (Here we must assume that α is close enough to r.) We also have that $0 < q_k \le 1$ for $k = r, \ldots, 1$, and that $q_k q_{k-1} < q_{k-1} - q_k$; therefore, Lemma 2.3 implies that

$$\|A^{(r)}\|_{q_r}^*(I) \le C|I|^{-1} \|A^{(r-1)}\|_{q_{r-1}}^*(I) \le \cdots \le C|I|^{-r} \|A\|_{q_0}^*(I).$$

We then apply (3.4) to find that

$$\begin{aligned}\int_\Gamma |\Delta_h^r(A,x)|^q\,dx &\le Ch^{rq}\int_I |A^{(r)}(x)|^q\,dx \\ &\le Ch^{rq}|I|^{-rq+1}\left(\frac{1}{|I|}\int_I |A(x)|^p\,dx\right)^{q/p} \\ &\le Ch^{rq}|I|^{-rq-q+1}\left(\int_I |A(x)|\,dx\right)^q.\end{aligned}\tag{3.6}$$

Because $\Gamma=\phi$ if $h>|I|/r$, (3.6), (3.5), and (3.1) imply that

$$\begin{aligned}&\int_{\mathbb{R}} |\Delta_h^r(A,x)|^q\,dx \\ &\qquad \le C\left([\min(h,|I|)]^{1-q/p}|I|^{-q+q/p}+|I|^{-rq-q+1}h^{rq}\chi(h)\right)\left(\int_I |A(x)|\,dx\right)^q,\end{aligned}$$

where χ is the characteristic function of $[0,|I|/r]$. It follows that $\omega_r(A,h)_q^q$ is also less than the right hand side of our latest inequality. Therefore,

$$\begin{aligned}&\int_0^\infty h^{-\alpha q}\omega_r(A,h)_q^q\,dh/h \\ &\qquad\le C\left(|I|^{-q+q/p}\int_0^{|I|} h^{-\alpha q-q/p}\,dh+|I|^{1-q}\int_{|I|}^\infty h^{-\alpha q-1}\,dh\right. \\ &\qquad\qquad\left.+|I|^{-rq-q+1}\int_0^{|I|} h^{(r-\alpha)q-1}\,dh\right)\left(\int_I |A(x)|\,dx\right)^q \\ &\qquad\le C|I|^{-\alpha q-q+1}\left(\int_I |A(x)|\,dx\right)^q \\ &\qquad= C\left(\int_I |A(x)|\,dx\right)^q,\end{aligned}\tag{3.7}$$

because $-\alpha q-q+1=0$.

We can now estimate the smoothness of $T_n=T_n(\,\cdot\,,t)$. Because $q<1$, we know that

$$\omega_r(T_n,h)_q^q\le\sum_{j=1}^N \omega_r(A_j,h)_q^q.\tag{3.8}$$

Hence, (3.7) and Hölder's inequality imply that

$$\begin{aligned}\int_0^\infty h^{-\alpha q}\omega_r(T_n,h)_q^q\,dh/h &\le C\sum_{j=1}^N\left(\int_{I_j}|A(x)|\,dx\right)^q \\ &\le CN^{1-q}\|T_n\|_{L^1(I_t)}^q \\ &\le CN^{\alpha q}\|T_n\|_{L^1(I_t)}^q.\end{aligned}\tag{3.9}$$

Consider now the expression for u, $u(\,\cdot\,,t)=\sum_{n=-1}^\infty T_n$. Using (3.8) and the continuous embedding of $B^\alpha([0,1])$ into $L^1([0,1])$, we obtain

$$
\begin{aligned}
\int_0^\infty \omega_r(u,h)_q^q h^{-\alpha q-1}\,dh &\le \sum_{n=-1}^{\infty}\int_0^\infty \omega_r(T_n,h)_q^q h^{-\alpha q-1}\,dh \\
&\le C\sum_{n=-1}^{\infty} 2^{n\alpha q}\|T_n\|_{L^1(I_t)}^q \\
&\le C\sum_{n=-1}^{\infty} 2^{n\alpha}\sigma_n(u_0)_1^q \\
&\le C\|u_0\|_{B^\alpha([0,1])}^q + C\|u_0\|_{L^1([0,1])}^q \\
&\le C\|u_0\|_{B^\alpha([0,1])}^q,
\end{aligned}
\tag{3.10}
$$

because from (2.3), for $n=-1,0\dots,$

$$
\begin{aligned}
\|T_n(t)\|_{L^1(I_t)} &= \|U_{n+1}(t)-U_n(t)\|_{L^1(I_t)} \\
&\le \|U_{n+1}(t)-u(t)\|_{L^1(I_t)} + \|u(t)-U_n(t)\|_{L^1(I_t)} \\
&\le \|U_{n+1}(0)-u_0\|_{L^1(I_t)} + \|u_0-U_n(0)\|_{L^1(I_t)} \\
&\le 2\sigma_n(u_0)_1.
\end{aligned}
$$

By (3.10), $\|u(\,\cdot\,,t)\|_{B^\alpha(I_t)} \le C\|u_0\|_{B^\alpha([0,1])}$.

This proves the theorem for α close to r. The proof for other values of $\alpha < r$ can be completed using interpolation; see [2]. ∎

Acknowledgments

The first author was supported in part by the National Science Foundation under grant DMS–8620108. The second author was supported in part by the National Science Foundation under grants DMS–8403219 and DMS–8802734.

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