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On Sobolev Regularizations of Hyperbolic Conservation Laws

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1. INTRODUCTION

We study certain Sobolev-type regularizations of the hyperbolic conservation laws

$$u_t + f(u)_x = 0, \quad x \in \mathbf{R},\ t>0, \qquad \text{(C)}$$
$$u(x,0) = u_0(x), \quad x \in \mathbf{R},$$

that add terms simulating both dissipative and dispersive processes. These evolution equations have the form

$$u_t + f(u)_x - \nu g(u)_{xx} - \beta u_{xxt} = 0, \quad x \in \mathbf{R},\ t>0,\ \beta>0, \qquad \text{(S)}$$

with the auxiliary specification

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}.$$

The behaviour in $L^1(\mathbf{R})$ of the solutions of problem (S), as well as the value of (S) as an approximation to problem (C), is studied. Convergence results, with error estimates, are given as ν and β tend to zero. In a companion paper [19], finite-difference discretizations of (S) are studied as an approximation for (C).

As a tool to study nonlinear evolution equations posed in $L^1(\mathbf{R})$, it is shown that any nonlinear mapping from $L^1(\mathbf{R})$ to itself that preserves the integral, is a contraction, and commutes with translations satisfies a maximum principle. This lemma gives necessary and sufficient conditions that solutions of (S) satisfy a maximum principle, despite the dispersive nature of (S).

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We apply our Sobolev equation theory to the study of the singular perturbation problem,

$$u_t + f(u)_x + \eta u_{xt} = 0, \quad x \in \mathbf{R},\ t>0,\ \eta>0,$$
$$u(x,0) = u_0(x), \quad x \in \mathbf{R},$$

and show that solutions of the singularly perturbed problems converge to the solution of the conservation law if the flux f satisfies a certain "compatibility condition" discussed by Whitham in [25] for linear problems.

The plan of this paper is as follows. The remainder of this section introduces notation and discusses preliminary results. In Section 2, a theorem about maximum principles is proved. Other properties of solutions of evolution equations are reviewed. In Section 3 we give necessary and sufficient conditions for (S) to generate a contraction semigroup in $L^1(\mathbf{R})$. It is then shown that if these conditions are satisfied, the solution of (C) is recovered in the limit as ν and β tend to zero with ν^2/β held fixed. Error estimates are provided. In Section 4, the results of Section 3 are used to analyze second-order hyperbolic singular perturbations of (C).

Notation and Preliminary Results

Translations on $\mathbf{R}^n$ will be denoted by $\sigma : x \to x+y$. (Sometimes the notation σ_y will be used.) If u is a function whose domain is $\mathbf{R}$, then σu is defined by $\sigma u(x) = u(\sigma(x))$. An operator A that maps elements of some function class defined on $\mathbf{R}$ to elements of some other function class is said to commute with translations if $A(\sigma u) = \sigma A(u)$. For any set E, χ_E is the characteristic function for that set. The symbol C denotes various constants whose values need not be the same for each instance of its use. The Fréchet space $L^1_{loc}(\mathbf{R})$ is the space of all locally integrable functions on $\mathbf{R}$.

There is a natural partial ordering on the spaces $L^1(\mathbf{R})$ and $L^1_{loc}(\mathbf{R})$, with

$u \geq v$ if and only if $u(x) \geq v(x)$ a.e. $(d\mu)$. Endowed with the operations $(u \vee v)(x) = \max(u(x), v(x))$ and $(u \wedge v)(x) = \min(u(x), v(x))$, $L^1(\mathbf{R})$ is a Banach lattice, and $L^1_{loc}(\mathbf{R})$ is a complete vector lattice. Possibly nonlinear operators T of a vector lattice to itself that preserve the ordering, so that $u \geq v$ implies that $Tu \geq Tv$, are *order preserving*, or *monotone*. (If T is linear, it is called *positive*.)

The space $BV(\mathbf{R}^n)$ of functions of bounded variation is the set of all measurable functions u whose first distributional derivatives are finite measures. Two equivalent $BV(\mathbf{R})$ seminorms are given by

$$|u|_{BV(\mathbf{R}^n)} = \int_{\mathbf{R}^n} \sum_{i \leq n} \left| \frac{\delta u}{\delta x_i} \right|$$

and

$$\sup_y \frac{1}{y} \|\sigma_y u - u\|_{L^1(\mathbf{R})} \tag{1.1}$$

(See Volpert [24].) The definition of $BV(I)$, where I is a bounded interval, and $BV(\mathbf{R} \times I)$ is directly analogous to (1.1).

A mapping $A : X \to Y$, where X and Y are Banach spaces, is said to be *Lipschitz continuous* if there is a number C such that for all u, v in X,

$$\|A(u) - A(v)\|_Y \leq C \|u - v\|_X ; \tag{1.2}$$

$\|A\|^{Lip}$ is the least such C. If, instead, for every bounded subset E of X there is a number C such that (1.2) holds for u, v in E, then A is *locally Lipschitz continuous*.

We formulate our differential equations in terms of m-accretive operators and contraction semigroups. A good survey of the application of these topics to partial differential equations may be found in Evans [12]. If X is a Banach space, a *duality mapping* $J : X \to X^*$ has the properties that for all $x \in X$, $\|J(x)\|_{X^*} = \|x\|_X$, and $J(x)(x) = \|x\|_X^2$. A possibly multi-valued operator A, defined on some subset $D(A)$ of X is said to be *accretive* (or, equivalently, $-A$ is *dissipative*) if for every pair of elements $(x,A(x))$, $(y,A(y))$ in the graph of A, and for every duality mapping J on X,

$$J(x-y)(A(x)-A(y)) \geq 0.$$

If, in addition, for all positive λ, $I+\lambda A$ is a surjection, then A is *m-accretive*.

2. A MAXIMUM PRINCIPLE.

A number of time dependent partial differential equations, such as the heat equation, scalar hyperbolic conservation laws (see Crandall [5]), and the porous medium equation (see Evans [12]), may be formulated as

$$\begin{aligned} &u_t + A(u) = 0, \quad t>0, \\ &u(0) = u_0, \quad u_0 \in L^1(\Omega), \end{aligned} \tag{2.1}$$

where A is a (possibly multi-valued) m-accretive operator on $L^1(\Omega)$. The Crandall-Liggett theorem [6] states that, for any fixed $t \geq 0$, the mapping S_t that assigns to u_0 the value $u(t)$ is a nonexpansive mapping on $L^1(\Omega)$. Crandall and Tartar [8] proved the following useful lemma about nonexpansive mappings $T: L^1(\Omega) \to L^1(\Omega)$.

LEMMA 2.1 (Crandall-Tartar). *Let $T: L^1(\Omega) \to L^1(\Omega)$ be such that, for all u in $L^1(\Omega)$,*

$$\int_\Omega Tu \, d\mu = \int_\Omega u \, d\mu.$$

Then T is nonexpansive on $L^1(\Omega)$ if and only if T is order preserving on $L^1(\Omega)$.

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The next lemma extends Lemma 2.1.

LEMMA 2.2. (Maximum Principle). *Any contraction T from $L^1(\mathbf{R})$ or $L^1(\mathbf{Z})$ ($\mathbf{Z}$ denoting the integers) to itself that preserves the integral and commutes with translations satisfies a maximum (and minimum) principle. That is,*

$$\text{ess sup } Tu \leq \text{ess sup } u$$

$$(\text{ ess inf } Tu \geq \text{ess inf } u\ ).$$

Proof. A proof will be given for $L^1(\mathbf{R})$, the case of $L^1(\mathbf{Z})$ being slightly easier.

From the previous lemma we know that T is order preserving, so our attention may be restricted to nonnegative functions u by considering $u \vee 0$. The lemma is obviously true if ess sup $u = \infty$.

If $u_n(x) = \chi_{[-n,n]}(x)\, u(x)$, then $u_n \to u$ in $L^1(\mathbf{R})$ as $n \to \infty$, and, since T is a contraction in $L^1(\mathbf{R})$, $Tu_n \to Tu$ in $L^1(\mathbf{R})$ as $n \to \infty$. Thus, some subsequence of Tu_n converges *a.e.* on $\mathbf{R}$, and it suffices to show that

$$\text{ess sup } Tu_n \leq \text{ess sup } u\ ,$$

to prove the lemma.

Since T commutes with translations,

$$\|\sigma Tu - Tu\|_{L^1(\mathbf{R})} = \|T(\sigma u) - Tu\|_{L^1(\mathbf{R})} \leq \|\sigma u - u\|_{L^1(\mathbf{R})}\ .$$

Thus, $\|Tu\|_{BV(\mathbf{R})} \leq \|u\|_{BV(\mathbf{R})}$.

If u is any function in $L^1(\mathbf{R})$, then

$$2\,\text{ess sup } u \leq \|u\|_{BV(\mathbf{R})}\ . \tag{2.2}$$

This inequality is clear if ess sup $u = 0$ or $\|u\|_{BV(\mathbf{R})} = \infty$. It is also easily seen for smooth functions, since if u is smooth,

$$\text{ess sup} u \leq \frac{1}{2}\int_{\mathbf{R}} |u'(x)|\, dx = \frac{1}{2}\|u\|_{BV(\mathbf{R})} \tag{2.3}$$

If u is not smooth, let $u_\varepsilon = \psi_\varepsilon * u$, where $\psi_\varepsilon(x) = \frac{1}{\varepsilon}\psi(\frac{x}{\varepsilon})$ and ψ is a nonnegative, smooth, integrable function whose integral is 1. Then u_ε is smooth, and (2.2) holds. Young's inequality implies that

$$\|\sigma u_\varepsilon - u_\varepsilon\|_{L^1(\mathbf{R})} \leq \|\sigma u - u\|_{L^1(\mathbf{R})}\ .$$

Therefore, $\|u_\varepsilon\|_{BV(\mathbf{R})} \leq \|u\|_{BV(\mathbf{R})}$. Also, $u_\varepsilon \to u$ *a.e.* as ε tends to zero, and, for any ε, sup $u_\varepsilon \leq$ ess sup u. Thus ess sup $u = \lim_{\varepsilon \to 0}$ ess sup u_ε, and

$$2 \lim_{\varepsilon \to 0} \operatorname{ess\,sup} u_\varepsilon \le \liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{BV(\mathbf{R})} \le \|u\|_{BV(\mathbf{R})} .$$

It is clear from (2.3) that equality in (2.2) is satisfied for smooth functions $\varphi \in L^1(\mathbf{R})$ if there is an x_0 with $\varphi'(x) \ge 0$ for $x \le x_0$, and $\varphi'(x) \le 0$ for $x \ge x_0$. For any n, pick such a φ with $\varphi \ge u_n$ *a.e.* and ess sup φ = ess sup u. Then $T\varphi \ge Tu_n$ *a.e.*, and

$$\begin{aligned}
\operatorname{ess\,sup} Tu_n &\le \operatorname{ess\,sup} T\varphi \\
&\le \frac{1}{2}\|T\varphi\|_{BV(\mathbf{R})} \\
&\le \frac{1}{2}\|\varphi\|_{BV(\mathbf{R})} \\
&= \operatorname{ess\,sup} \varphi \\
&= \operatorname{ess\,sup} u .
\end{aligned}$$

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The simple, but somewhat artificial, example $Tu = \chi_{[0,1]} \int_{\mathbf{R}} u(x)\,dx$ shows that the conclusion of the lemma does not hold if we do not assume that T commutes with translations.

In the preceding lemma, no smoothness properties are assumed of the mapping T, and T may map smooth functions to discontinuous ones. This occurs, for example, when the lemma is applied to the solution operator of a scalar conservation law (although the result is well known for these problems), or to solutions of (S) with nonsmooth initial data.

This pair of lemmas says a great deal about the solutions of evolution equations in $L^1(\mathbf{R})$. If T represents either S_t for some fixed t, or the solution of the backward difference equation

$$Tu + \Delta t A(Tu) = u , \quad \Delta t > 0 ,$$

then the implications indicated in Figure 1 hold.

It is sometimes simple to establish for an operator A that $\int A(u) = 0$ or

$$
\begin{tikzcd}
\begin{array}{c}\text{for all } u,\\ \int_{\mathbf{R}} Au = 0\end{array} \arrow[d, Rightarrow] & A \text{ is m-accretive in } L^1(\mathbf{R}) \arrow[d, Rightarrow] & \begin{array}{c}A \text{ commutes}\\ \text{with translations}\end{array} \arrow[d, Rightarrow] \\
\begin{array}{c}\text{for all } u,\\ \int_{\mathbf{R}} Tu = \int_{\mathbf{R}} u\end{array} \arrow[d, Rightarrow] & \begin{array}{c}\text{for all } u,v,\\ \|Tu - Tv\|_{L^1(\mathbf{R})} \le \|u - v\|_{L^1(\mathbf{R})}\end{array} & \begin{array}{c}T \text{ commutes}\\ \text{with translations}\end{array} \arrow[d, Rightarrow] \\
\begin{array}{c}u \ge v \text{ a.e.} \Rightarrow\\ Tu \ge Tv \text{ a.e.}\end{array} & & \begin{array}{c}\text{for all } u,\\ \|Tu\|_{BV(\mathbf{R})} \le \|u\|_{BV(\mathbf{R})}\end{array} \arrow[dl, Rightarrow] \\
& \begin{array}{c}\text{for all } u,\\ \text{ess sup } Tu \le \text{ess sup } u\end{array} &
\end{tikzcd}
$$

Figure 1. Properties of evolution equations in $L^1(\mathbf{R})$.

that A commutes with translations. If one can show that A is m-accretive on $L^1(\mathbf{R})$ in conjunction with these two properties, the other properties illustrated in Figure 1 then follow. These ideas will be exploited in the following section.

3. SOBOLEV EQUATIONS.

It is well known that solutions of (C) are not, in general, continuous; see Whitham [25] for a discussion of shocks. Weak solutions of (C), defined by the relation,

$$0 = \int_{\mathbf{R}} u_0(x)\,\varphi(x,0)\,dx + \iint_{\mathbf{R}\times[0,T]} (u\varphi_t + f(u)\varphi_x)\,dx\,dt\,, \qquad \text{for all } \varphi \in C_0^1(\mathbf{R}\times(-\infty,T])\,, \tag{3.1}$$

are in general not unique. The reader may refer to Le Roux [18] for a good description of this phenomenon.

Oleinik [20], Hopf [15], Volpert [24], and Kruzkov [16] provided existence and uniqueness results for certain classes of weak solutions of (C) through the prescription of an extra condition, known as an entropy condition. The theory for solutions of (C) used in this paper is expressed in the following theorem.

THEOREM 3.1. *If f is locally Lipschitz continuous, then for any $u_0 \in BV(\mathbf{R})$ and for any $T>0$ there is a unique $u \in BV(\mathbf{R}\times[0,T])$ such that u satisfies* (3.1) *and, in addition, satisfies the entropy condition: for all $\varphi \in C_0^1(\mathbf{R}\times[0,T])$, with $\varphi \geq 0$, and for all $c \in \mathbf{R}$,*

$$\iint_{\mathbf{R}\times[0,T]} [\,|u-c\,|\varphi_t + \operatorname{sgn}(u-c)(f(u)-f(c))\varphi_x\,]\,dx\,dt \geq 0. \tag{3.2}$$

For a proof, see Volpert [24], Kruzkov [16], Crandall and Majda [7].

Kuznetsov [17] proposed a general theory of approximation for solutions of (3.2) in an arbitrary number of spatial dimensions. We formulate the one-dimensional version as follows.

THEOREM 3.2. *Let u be the entropy solution of* (C) *with $u_0 \in L^1(\mathbf{R})\cap BV(\mathbf{R})$, and let $v:\mathbf{R}^+ \to L^1(\mathbf{R})$ have left and right limits for every t and be right continuous. Pick a positive, symmetric function $\eta(\xi)$ with support in $[-1,1]$ and integral* 1, *positive numbers ε and ε_0, and let $\omega(x,t) = \frac{1}{\varepsilon_0}\eta(\frac{t}{\varepsilon_0})\frac{1}{\varepsilon}\eta(\frac{x}{\varepsilon})$. Define the "Kruzkov form"*

$$\begin{aligned}\Lambda_t^{\varepsilon_0,\varepsilon} = \iint_{S\times S} &|v(x'',t'')-u(x',t')|\,\frac{\partial}{\partial t''}\omega(x''-x',t''-t') \\ &+ \operatorname{sgn}(v(x'',t'')-u(x',t'))\,(f(v(x'',t''))-f(u(x',t')))\,\frac{\partial}{\partial x''}\omega(x''-x',t''-t')\,dx''\,dt''\,dx'\,dt' \\ + \int_{S\times\mathbf{R}} &[\omega(x''-x',0-t')\,|v_0(x'')-u(x',t')| - \omega(x''-x',t-t')\,|v(x'',t-0)-u(x',t')|\,]\,dx''\,dx'\,dt'\end{aligned}$$

where $S = \mathbf{R}\times[0,t]$. Then

$$\begin{aligned}\|u(t)-v(t)\|_{L^1(\mathbf{R})} \leq \|u(0)-v(0)\|_{L^1(\mathbf{R})} &+ (2\varepsilon + \|f\|_{Lip}\,\varepsilon_0)\,|u_0|_{BV(\mathbf{R})} \\ &+ \sup_{t',\,|\tau|<\varepsilon_0,\,-t'<\tau<t-t'} \|v(t'-\tau)-v(t')\|_{L^1(\mathbf{R})} - \Lambda_t^{\varepsilon_0,\varepsilon}.\end{aligned}$$

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We next investigate properties of the initial-value problem

$$\begin{gathered} u_t + f(u)_x - \nu g(u)_{xx} - \beta u_{xxt} = 0,\ \ x\in\mathbf{R},\ t>0,\ \nu,\ \beta>0, \\ u(x,0) = u_0(x),\ x\in\mathbf{R}. \end{gathered} \tag{3.3}$$

A particular instance of this equation has been used to model the propagation of small amplitude, shallow-water waves (see Benjamin et al. [1], and Bona et al. [2]). In this context, t is proportional to elapsed time and x is proportional to distance in the direction of propagation. The term $-\nu g(u)_{xx}$ models dissipative processes, and the term $-\beta u_{xxt}$ models the action of dispersion. (An equation is dispersive if waves of differing wavelengths move with different speeds.) The equation (3.3) may also be viewed as a regularization of the hyperbolic conservation law (C). It has been used in this way by Douglas et al. [10] to simulate a linear waterflood problem.

Existence, uniqueness and regularity of solutions of (3.3) have been studied by Showalter [23]. Ewing [13] has studied the numerical approximation of solutions of (3.3) and examined the behaviour of solutions of (3.3) as one of ν or β tends to zero. The present theory focuses on the behaviour of solutions of (3.3) in $L^1(\mathbf{R})$, and on the consequences of allowing ν and β to tend to zero simultaneously.

Using a different approach (compensated compactness), Schonbek [21] has investigated the behaviour of regularizations, like (3.3), that arise from the addition of small dissipative and dispersive terms. Her techniques apply, with some degree of success, to special cases of (3.3). Unlike the methods presented here, her methods also have application to the study of (3.3) with the dispersive term $-\beta u_{xxt}$ replaced by the Korteweg-de Vries dispersive term $+\beta u_{xxx}$.

Previously, Conley and Smoller [4] had proved the existence of travelling wave solutions of regularizations of hyperbolic systems incorporating both dissipation and a KdV type dispersive term. They also showed that these travelling wave solutions converged to weak shock solutions of the Riemann problem for the hyperbolic systems. Smoller and Shapiro [22] have considered which viscosity and dispersion matrices are ***admissible*** for systems, in that the regularized equations admit traveling wave solutions that converge to shock solutions of (C) as the levels of dissipation and dispersion go to zero.

We will assume throughout this section that the functions f and g are globally Lipschitz continuous and, without loss of generality, that $f(0)=g(0)=0$. The function space that contains u_0 will generally be taken as $L^1(\mathbf{R})$ or $BV(\mathbf{R})$ so that solutions of (3.3) are considered in the following weak sense. Following Benjamin et al. [1], we formally rewrite (3.3) as an integral equation. A scaling between the dissipative and dispersive terms is introduced that redefines g so that (3.3) becomes

$$u_t + f(u)_x - \alpha g(u)_{xx} - \alpha^2 u_{xxt} = 0, \quad x\in\mathbf{R},\ t>0,\ \alpha>0. \tag{3.4}$$

If ∂_x denotes differentiation with respect to x, (3.4) may be written as

$$\begin{aligned}(1-\alpha^2\partial_x^2)u_t &= -f(u)_x + \alpha g(u)_{xx} \\ &= -f(u)_x - \frac{1}{\alpha}((1-\alpha^2\partial_x^2)g(u) - g(u)).\end{aligned} \tag{3.5}$$

The elliptic operator $(1-\alpha^2\partial_x^2)$ can be inverted on the real line, subject to the condition that the solution be bounded at infinity, by convolution with the function

$$M_\alpha(x) = \frac{1}{2\alpha}e^{-\frac{|x|}{\alpha}}.$$

Thus, we may convert (3.5) to

$$\begin{aligned}u_t &= -M_\alpha * f(u)_x - \frac{1}{\alpha}(g(u) - M_\alpha * g(u)) \\ &= -M_\alpha * f(u)_x + \alpha K_\alpha * g(u)_x \\ &= K_\alpha * f(u) - \frac{1}{\alpha}(g(u) - M_\alpha * g(u)), \qquad x\in\mathbf{R},\ t>0,\ \alpha>0,\end{aligned} \tag{3.6}$$

where

$$K_\alpha(x) = -\frac{\operatorname{sgn}(x)}{2\alpha^2}e^{-\frac{|x|}{\alpha}}.$$

The operator on the right-hand side of (3.6), which we will write as $A(u)$, is a Lipschitz continuous map from $L^1(\mathbf{R})$ to itself. For, by Young's inequality,

$$\begin{aligned}\|A(u)-A(v)\|_{L^1(\mathbf{R})} &\le \|K_\alpha\|_{L^1(\mathbf{R})}\|f(u)-f(v)\|_{L^1(\mathbf{R})} \\ &\quad + \frac{1}{\alpha}(\|g(u)-g(v)\| + \|M_\alpha\|_{L^1(\mathbf{R})}\|g(u)-g(v)\|_{L^1(\mathbf{R})}) \\ &\le \frac{1}{\alpha}\|f\|_{Lip}\|u-v\|_{L^1(\mathbf{R})} + \frac{2}{\alpha}\|g\|_{Lip}\|u-v\|_{L^1(\mathbf{R})}.\end{aligned} \tag{3.7}$$

Thus, from the theory of ordinary differential equations in Banach spaces, Equation (3.6), posed as an initial-value problem in $L^1(\mathbf{R})$, has a unique solution in $C^1([0,\infty),L^1(\mathbf{R}))$ for any u_0 in $L^1(\mathbf{R})$ (see Hille and Phillips [14]).

We will consider solutions of the integral equation (3.6) to be weak solutions of (3.4). If u is a solution of the integral equation (3.6), then u is a solution of (3.4) in the sense of distributions. Also, if u is a solution of (3.6) possessing continuous, bounded derivatives of up to second order, and f and g have two continuous bounded derivatives, one may differentiate (3.6) to see that u is a classical solution of (3.4). Thus, such an interpretation of solutions of (3.4) is reasonable.

Qualitatively, the solutions of (3.3) behave differently for various ranges of the parameters ν and β. When ν and β are 0, the solutions of (3.3) are the solutions of the conservation law (C). Here, shocks, or discontinuities, develop in the solution u (cf. Whitham [25]), and solutions of (C) must be considered in the weak sense of Theorem 3.1. If $g(u) = u$, ν is positive, and β is still 0, the solution u of (3.3) is smooth for all positive time. When ν is small, however, large gradients arise in the function u near the time when the first shock occurs in the solution of the conservation law (C) with the same initial data. For each problem, the solution operator S_t is a nonexpansive mapping on $L^1(\mathbf{R})$, it satisfy a maximum principle, and it is order preserving on $L^1(\mathbf{R})$.

Solutions of (3.3) when ν is 0 and β is positive behave quite differently. When β is small, near the time when shocks occur in the solution of the conservation law (C), oscillations occur in the solution of (3.3) in about the same position as those shocks. These solutions of (3.3) are not contraction semigroups on $L^1(\mathbf{R})$, they are not order preserving, and they do not satisfy a maximum principle. Thus, it is interesting to see how the dissipative and dispersive terms interact.

The following theorem plays a central role in our analysis.

THEOREM 3.3. *The mapping* $A : L^1(\mathbf{R}) \to L^1(\mathbf{R})$, *defined by*

$$A(u) = K_\alpha * f(u) - \frac{1}{\alpha}(g(u) - M_\alpha * g(u)), \tag{3.8}$$

is dissipative on $L^1(\mathbf{R})$ *if and only if the functions* $g(\xi) \pm f(\xi)$ *are nondecreasing on* $\mathbf{R}$.

The proof of Theorem 4.2 contains the proof that if A is dissipative on $L^1(\mathbf{R})$ then the functions $g \pm f$ are nondecreasing on $\mathbf{R}$.

Proof. Since the dual of $L^1(\mathbf{R})$ is $L^\infty(\mathbf{R})$, any duality mapping J for $L^1(\mathbf{R})$ is of the form $J(u)(v) = \int_{\mathbf{R}} \hat{J}(u)(x)v(x)\,dx$, where

$$\hat{J}(u)(x) = \|u\|_{L^1(\mathbf{R})} \cdot \begin{cases} 1 & u(x)>0, \\ -1 & u(x)<0, \\ a(x) & u(x)=0, \end{cases}$$

where $a(x)$ is any measurable function with $|a(x)| \le 1$ *a.e.*.

Writing out the integrals in $A(u)$ explicitly gives

$$\begin{aligned} &\hat{J}(u-v)(x)(A(u)-A(v))(x) \\ &\quad = \hat{J}(u-v)(x)\left[\int_x^\infty \frac{e^{\frac{-|x-y|}{\alpha}}}{2\alpha}\left\{-(f(u(y))-f(v(y)))+g(u(y))-g(v(y))\right\}dy\right. \\ &\qquad \left. + \int_{-\infty}^x \frac{e^{\frac{-|x-y|}{\alpha}}}{2\alpha}\left\{(f(u(y))-f(v(y)))+g(u(y))-g(v(y))\right\}dy\right] \\ &\qquad - \frac{1}{\alpha}\,|g(u(x))-g(v(x))|\;\|u-v\|_{L^1(\mathbf{R})}. \end{aligned} \tag{3.9}$$

Note that $\operatorname{sgn}(u-v)(g(u)-g(v)) = |g(u)-g(v)|$, since g is nondecreasing by hypothesis.

Integrate (3.9) with respect to x, replace the quantities in braces with their absolute values, and change the order of integration to obtain

$$J(u-v)(A(u)-A(v)) \le \frac{\|u-v\|_{L^1(\mathbf{R})}}{\alpha} \times$$
$$\left\{ \int_{\mathbf{R}} \frac{|-(f(u(y))-f(v(y)))+g(u(y))-g(v(y))|}{2} + \frac{|f(u(y))-f(v(y))+g(u(y))-g(v(y))|}{2} dy - \int_{\mathbf{R}} |g(u(x))-g(v(x))|\, dx \right\}.$$

If $g \pm f$ are nondecreasing on $\mathbf{R}$, then

$$2|g(\xi)-g(\eta)| = |-(f(\xi)-f(\eta))+g(\xi)-g(\eta)| + |f(\xi)-f(\eta)+g(\xi)-g(\eta)|$$

for any ξ, η in $\mathbf{R}$, so that

$$J(u-v)(A(u)-A(v)) \le 0.$$

Thus, A is dissipative.

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THEOREM 3.4. *Let the mapping A be given by (3.8). Then the following conditions are equivalent:*

(1) *The functions $g(\xi) \pm f(\xi)$ are nondecreasing on* $\mathbf{R}$.

(2) *The mapping $-A$ is accretive on* $L^1(\mathbf{R})$.

(3) *The mappings $S_t : u_0 \to u(\cdot, t)$, where $u(x,t)$ is the solution of Equation (3.6), are contractions in $L^1(\mathbf{R})$. Since S_t commutes with translations,* $|S_t(u)|_{BV(\mathbf{R})} \le |u|_{BV(\mathbf{R})}$.

(4) *The mappings S_t are order preserving on* $L^1(\mathbf{R})$.

(5) *The mappings S_t satisfy a maximum and a minimum principle.*

Proof. Theorem 4.1 has a proof that (1) implies (2).

Because the operator A is a Lipschitz continuous map in $L^1(\mathbf{R})$, the mapping $-A$ is not only accretive on $L^1(\mathbf{R})$ but also m-accretive. Thus the Crandall-Liggett theorem [6] shows that there exists a semigroup S_t for which

$$S_t(u) = \lim_{n\to\infty}(I + \frac{t}{n}A)^{-n}(u), \text{ and} \tag{3.10a}$$

$$\|S_t(u) - S_t(v)\|_{L^1(\mathbf{R})} \le \|u - v\|_{L^1(\mathbf{R})}. \tag{3.10b}$$

Since the solution of the differential equation (3.6) is in $C^1([0,\infty),L^1(\mathbf{R}))$, Brezis and Pazy [3] show that the function $S_t(u)$ defined by (3.10a) is the same as $u(t)$ defined by (3.6). Thus property (3) follows.

We use Lemma 2.1 to show that properties (3) and (4) are equivalent. This requires

$$\int_{\mathbf{R}} u(x,t)\,dx = \int_{\mathbf{R}} u_0(x)\,dx\,,\ t>0.$$

For any $u\in L^1(\mathbf{R})$, $\int_{\mathbf{R}} A(u)\,dx = 0$, since $\int_{\mathbf{R}} K_\alpha\,dx = 0$, and $\int_{\mathbf{R}} M_\alpha\,dx = 1$. Thus, if v is the solution of $v - \frac{t}{n}A(v) = u$, $\int_{\mathbf{R}} v\,dx = \int_{\mathbf{R}} u\,dx$. Because the function $S_t(u)$ is the limit as $n\to\infty$ of functions v in $L^1(\mathbf{R})$ all having the same integral, the mappings S_t satisfy the hypotheses of Lemma 2.1.

Lemma 2.2 may be used to show that property (4) implies property (5). This requires that the mappings S_t commute with translations. Since the operator A commutes with translations, an argument similar to the one above shows that S_t commutes with translations.

Finally, we show that if property (1) is false, then property (5) does not hold.

Assume that (1) does not hold. Without loss of generality we may assume that there is some $\eta>\xi$ with $g(\eta)+f(\eta)<g(\xi)+f(\xi)$. It must be true that $\eta>0$ or $\xi<0$; assume, again for definiteness, that $\eta>0$.

Let

$$u_0(x) = \begin{cases} 0 & x<-R-1 \\ \xi & -R<x<\frac{-1}{R} \\ \eta & 0<x<R \\ 0 & R+1<x \end{cases}$$

and define u_0 linearly on the intervals where it is not defined above so that it is continuous on $\mathbf{R}$. The positive parameter R is to be determined. For this initial datum, one determines from (3.6) that

$$u_t(0,0) = \frac{1}{2\alpha}(-f(\eta) - g(\eta)) + \frac{1}{2\alpha}(f(\xi) + g(\xi)) + \varepsilon(R), \tag{3.11}$$

where $\varepsilon(R) \to 0$ as $R \to \infty$. Since the sum of the other terms on the right-hand side of (3.11) is positive, for some value of R, $u_t(0,0) > 0$. Thus for some t, S_t does not satisfy a maximum principle, so that (5) implies (1).

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Property (1) of Theorem 3.4 may be stated in terms of the original parameters ν and β of equation (3.3) as Property (1)′.

(1)′ The functions $\nu g(\xi) \pm \beta^{1/2} f(\xi)$ are nondecreasing on $\mathbf{R}$.

If f and g are in $C^1(\mathbf{R})$, then Property (1)′ is equivalent to

$$\text{for all } \xi \in \mathbf{R}, \quad \nu g'(\xi) \geq \beta^{1/2} |f'(\xi)| . \tag{3.12}$$

The quantity on the right side of this relation measures the interaction between the dispersive term ($\beta^{1/2}$), and the nonlinear transport term ($|f'(\xi)|$), while the expression on the left is a measure of the strength of the dissipative term. This condition defines the sense in which the dissipative term must dominate the interaction between the nonlinear and dispersive terms so that the properties of Theorem 4.2 hold. Condition (3.12) does *not* say that dissipation must be large with respect to the initial data. Rather, it is similar to certain stability conditions for finite-difference equations (see [19]).

The next theorem proves that, under certain conditions, solutions of (3.4) converge to the entropy solution of the conservation law (3.1).

THEOREM 3.5. *Assume that the functions $g \pm f$ are nondecreasing on $\mathbf{R}$ and that $T>0$ is given. Then for any $u_0 \in BV(\mathbf{R})$ there exists a unique solution $u(x,t)$ of (3.4) with $u \in C^1([0,T], L^1_{loc}(\mathbf{R}))$ and $|u(t)|_{BV(\mathbf{R})}$ uniformly bounded for*

$0 \le t \le T$. *Also, as* $\alpha \to 0$, *the solutions* u *of* (3.4) *converge in* $C^0([0,T], L^1_{loc}(\mathbf{R}))$ *to the entropy solution of* (C) *for the same* u_0.

Proof The first part of the proof constructs solutions of (3.4) in the space $BV(\mathbf{R})$, and shows that these solutions satisfy properties analogous to properties (b) to (f) of Theorem 3.4.

For any positive integer k, let $u^k = \chi_{[-k,k]} \cdot u_0$. Then u^k is in $L^1(\mathbf{R}) \cap BV(\mathbf{R})$, $|u^k|_{BV(\mathbf{R})} \le |u_0|_{BV(\mathbf{R})} + 2\|u_0\|_{L^\infty(\mathbf{R})}$, and $\|u^k\|_{L^\infty(\mathbf{R})} \le \|u_0\|_{L^\infty(\mathbf{R})}$. Thus, by Theorem 4.2, for every k there is a unique mapping $\rho_k \in C^1([0,T], L^1(\mathbf{R}))$ such that $\rho_k(t) = S_t(u^k)$. A fortiori, ρ_k is in $C^1([0,T], L^1_{loc}(\mathbf{R}))$. Also, for any positive time t,

$$|\rho_k(t)|_{BV(\mathbf{R})} \le |u_0|_{BV(\mathbf{R})} + 2\|u_0\|_{L^\infty(\mathbf{R})}, \tag{3.13a}$$

and

$$\|\rho_k(t)\|_{L^\infty(\mathbf{R})} \le \|u_0\|_{L^\infty(\mathbf{R})}. \tag{3.13b}$$

Since $\|M_\alpha\|_{L^1(\mathbf{R})} = \|\alpha K_\alpha\|_{L^1(\mathbf{R})} = 1$, and $|f(u)|_{BV(\mathbf{R})} \le \|f\|_{Lip} |u|_{BV(\mathbf{R})}$, Equation (3.6) yields

$$\|\frac{\partial \rho_k}{\partial t}(t)\|_{L^1(\mathbf{R})} = \|A(\rho_k)\|_{L^1(\mathbf{R})} \le (\|f\|_{Lip} + \|g\|_{Lip})\,|\rho_k|_{BV(\mathbf{R})} \tag{3.14a}$$

$$\le (\|f\|_{Lip} + \|g\|_{Lip})(|u_0|_{BV(\mathbf{R})} + 2\|u_0\|_{L^\infty(\mathbf{R})}). \tag{3.14b}$$

Hence, the functions ρ_k are equicontinuous in $L^1_{loc}(\mathbf{R})$.

For every k, the range of ρ_k is contained in the set

$$S = \{u \in L^1_{loc}(\mathbf{R}) \mid |u|_{BV(\mathbf{R})} \le |u_0|_{BV(\mathbf{R})} + 2\|u_0\|_{L^\infty(\mathbf{R})}, \text{ and } \|u\|_{L^\infty(\mathbf{R})} \le \|u_0\|_{L^\infty(\mathbf{R})}\} \tag{3.15}$$

which is precompact in $L^1_{loc}(\mathbf{R})$ (see [11]). Thus, by the Arzela-Ascoli theorem, some subsequence of the ρ_k, renamed ρ_k, converges in $C^0([0,T], L^1_{loc}(\mathbf{R}))$ to a function $\rho(t)$.

The functions $\rho_k(t) \to \rho(t)$ and $u^k \to u_0$ (where u_k and u_0 are extended to $\mathbf{R} \times [0,T]$ as constant functions on $[0,T]$) in $C^0([0,T], L^1_{loc}(\mathbf{R}))$. We now show that $A(\rho_k(t)) \to A(\rho(t))$ in the same space.

Clearly, $f(\rho_k(t)) \to f(\rho(t))$ in $C^0([0,T], L^1_{loc}(\mathbf{R}))$ if f is Lipschitz continuous. If I is any bounded interval in $\mathbf{R}$, $I_R = \{x \mid |x-y| \le R \text{ for some } y \text{ in } I\}$, and $K^+(x) = \chi_{[0,\infty)} e^{-|x|/\alpha}/2\alpha$ then

$$\begin{aligned}
\|K^+ * u - K^+ * v\|_{L^1(I)} &\le \int_I \int_{x-R}^{x+R} K^+(x-y)\,|u(y)-v(y)|\,dy\,dx \\
&\quad + \int_I \int_{|y-x|>R} K^+(x-y)\,|u(y)-v(y)|\,dy\,dx \\
&\le \frac{1}{2}\int_{I_R} |u(y)-v(y)|\,dy \qquad (3.16)\\
&\quad + \|u-v\|_{L^\infty(\mathbf{R})}\,|I| \int_{|x|>R} K^+(x)\,dx\,.
\end{aligned}$$

In our case, $u = f(\rho(t))$, $v = f(\rho_k(t))$ are uniformly bounded in $L^\infty(\mathbf{R})$, and $f(\rho_k) \to f(\rho)$ in $C^0([0,T], L^1_{loc}(\mathbf{R}))$. Therefore, the second term of the last expression in (3.16) may be made arbitrarily small by letting R be sufficiently large. For any fixed R the first term tends to zero uniformly for $t \in [0,T]$. Thus, for any bounded interval $I \subset \mathbf{R}$, and $\varepsilon > 0$, if k is large enough,

$$\|K^+ * f(\rho_k) - f(\rho)\|_{L^1(I)} \le \varepsilon\,.$$

Because the other terms of $A(\rho_k)$ may be treated in the same way, $A(\rho_k) \to A(\rho)$. Since $\rho(t)$ is continuous in $L^1_{loc}(\mathbf{R})$, the same argument shows that $A(\rho(t))$ is also continuous. Thus $\rho(t)$ satisfies

$$\rho(t) = u_0 + \int_0^t A(\rho(\tau))\,d\tau\,. \qquad (3.17)$$

Differentiating (3.17) with respect to t shows that $\rho(t)$ is in $C^1([0,T], L^1_{loc}(\mathbf{R}))$, and that $\rho(t)$ satisfies (3.6).

Because of (3.14), $|\rho(t)|_{BV(\mathbf{R})} \le |u_0| + 2\|u_0\|_{L^\infty(\mathbf{R})}$ independently of t.

We now show uniqueness. If ω is any solution of (3.6) with initial data v_0 and $\omega \in C^1([0,T], L^1_{loc}(\mathbf{R}))$ with $|\omega|_{BV(\mathbf{R})}$ uniformly bounded for $t \in [0,T]$, (3.14a) shows that $\|\frac{\partial \omega}{\partial t}(t)\|_{L^1(\mathbf{R})} \le C$ for some number C independent of t, and that $\|\omega(t) - v_0\|_{L^1(\mathbf{R})} \le Ct$.

Assume now that there are two solutions ρ and ω of (3.6) corresponding to the same initial data u_0. Then, for any t, $\rho(t)-\omega(t)\in L^1(\mathbf{R})$, and

$$\begin{aligned}\frac{d}{dt}\|\rho(t)-\omega(t)\|_{L^1(\mathbf{R})} &= J(\rho(t)-\omega(t))\left(\frac{\partial\rho(t)}{\partial t}-\frac{\partial\omega(t)}{\partial t}\right) \\ &= J(\rho(t)-\omega(t))\,(A(\rho(t))-A(\omega(t)))\end{aligned} \tag{3.18}$$

where $J: L^1(\mathbf{R})\to L^\infty(\mathbf{R})$ is the duality mapping of Theorem 4.1. The argument used in Theorem 4.1 shows that the right hand side of (3.18) is nonpositive. Therefore, $\rho(t)=\omega(t)=S_t(u_0)$. More generally, if $u-v$ is in $L^1(\mathbf{R})$, then $\|S_t(u)-S_t(v)\|_{L^1(\mathbf{R})}\le\|u-v\|_{L^1(\mathbf{R})}$.

The other properties are as follows. By setting $v=\sigma u$ in the last inequality, we see that $|u(t)|_{BV(\mathbf{R})}\le|u_0|_{BV(\mathbf{R})}$. Hence, if u_0 is constant, $u(t)=u_0$. Because of part (4) of Theorem 4.2, the mapping $u_0\to u(t)$ is order preserving by construction. Thus S_t satisfies a maximum and minimum principle. The inequality (3.14a) also applies to u, since u satisfies (3.6).

The second part of this proof shows that the solutions of (3.4) converge to the entropy solution of (C) as α tends to zero. The dependence on α of the solutions u of (3.4) will be made explicit by writing u^α.

Note that, since (3.14a) and $|u^\alpha(t)|_{BV(\mathbf{R})}\le|u_0|_{BV(\mathbf{R})}$ hold independently of α, the functions u^α mapping $[0,T]$ into $L^1_{loc}(\mathbf{R})$ are equicontinuous. The range of each mapping u^α is again contained in the set S of (3.15). Thus there exists a sequence of the numbers α tending to zero, such that the corresponding solutions of (3.4) converge in $C^0([0,T],L^1_{loc}(\mathbf{R}))$ to a function u. *A fortiori*, the functions u^α converge to u in the weaker topology of $L^1_{loc}(\mathbf{R}\times[0,T])$.

This function $u\in BV(\mathbf{R}\times[0,T])$ since

$$\|u(\cdot+y,\cdot)-u(\cdot,\cdot)\|_{L^1(\mathbf{R}\times[0,T])}\le T\,|y|\,|u_0|_{BV(\mathbf{R})},$$

and similarly,

$$\|u(\cdot,\cdot+\Delta t)-u(\cdot,\cdot)\|_{L^1(I\times[0,T])}\le T\,|\Delta t|\,(\|f\|_{Lip}+\|g\|_{Lip})\,|u_0|_{BV(\mathbf{R})}.$$

We now show that the functions u^α satisfy an approximate entropy condition and that u satisfies the entropy condition of Theorem 3.1.

Fix $\alpha > 0$. For any $t_0 \in [0, T)$ and $c \in \mathbf{R}$, solve the initial value problem

$$\begin{gathered} v_t + f(v)_x - \alpha g(v)_{xx} - \alpha^2 v_{xxt} = 0, \quad t > t_0, \ x \in \mathbf{R}, \\ v(x, t_0) = u^\alpha(x, t_0) \vee c, \ x \in \mathbf{R}. \end{gathered} \tag{3.19}$$

Since the operators S_t are order preserving, and u and c are solutions of (3.19) with initial data $u^\alpha(x, t_0)$ and c, respectively, it is true that

$$v(x,t) \geq u^\alpha(x,t) \vee c \quad \text{for all } x \in \mathbf{R}, \ t \geq t_0.$$

Therefore, for all $t_0 \in [0, t)$,

$$v_t(x, t_0) \geq (u^\alpha(x, t_0) \vee c)_t,$$

where the right-hand side may be interpreted as a measure, since $u^\alpha \vee c$ is also in $BV(\mathbf{R} \times [0, T])$. But

$$v_t(x, t_0) = A(u^\alpha(t_0) \vee c)(x),$$

so that, as measures,

$$(u^\alpha \vee c)_t \leq A(u^\alpha \vee c),$$

and similarly,

$$(u^\alpha \wedge c)_t \geq A(u^\alpha \wedge c).$$

For any nonnegative $\varphi \in C_0^2(\mathbf{R} \times [0, T])$, we may compute

$$\begin{aligned} 0 &\geq \iint_{\mathbf{R} \times [0,T]} [(u^\alpha \vee c)_t - (u^\alpha \wedge c)_t - (A(u^\alpha \vee c) - A(u^\alpha \wedge c))]\, \varphi \, dx\, dt \\ &= \iint_{\mathbf{R} \times [0,T]} -((u^\alpha \vee c) - (u^\alpha \wedge c))\varphi_t - (f(u^\alpha \vee c) - f(u^\alpha \wedge c))\, M_\alpha * \varphi_x \\ &\qquad - (g(u^\alpha \vee c) - g(u^\alpha \wedge c))\alpha M_\alpha * \varphi_{xx}\, dx\, dt \\ &= -\iint_{\mathbf{R} \times [0,T]} |u^\alpha - c|\, \varphi_t + \operatorname{sgn}(u^\alpha - c)(f(u^\alpha) - f(c))\, M_\alpha * \varphi_x \\ &\qquad + \operatorname{sgn}(u^\alpha - c)(g(u^\alpha) - g(c))\, \alpha M_\alpha * \varphi_{xx}\, dx\, dt\,. \end{aligned} \tag{3.20}$$

As α tend to zero, the functions $u^\alpha \to u$, $|u^\alpha - c| \to |u - c|$, and $\operatorname{sgn}(u^\alpha - c)(f(u^\alpha) - f(c)) \to \operatorname{sgn}(u - c)(f(u) - f(c))$ in $L^1_{loc}(\mathbf{R} \times [0, T])$ and boundedly $a.e.$ (passing to a further subsequence, if necessary). Also, $M_\alpha * \varphi_x \to \varphi_x$ and

$\alpha M_a * \varphi_{xx} \to 0$ in $L^1(\mathbf{R}\times[0,T])$. The Lebesgue dominated convergence theorem implies that

$$0 \le \iint_{\mathbf{R}\times[0,T]} |u-c|\varphi_t + \mathrm{sgn}(u-c)(f(u)-f(c))\varphi_x \, dx\, dt .$$

This is the required entropy condition.

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The next theorem uses Kuznetsov's theory to obtain an error estimate.

THEOREM 3.6. *If the assumptions of Theorem 4.3 are valid, u^α is the solution of (3.3), and u is the solution of (C), then there exist a C such that*

$$\|u^\alpha(t) - u(t)\|_{L^1(\mathbf{R})} \le C(\alpha t)^{1/2} |u_0|_{BV(\mathbf{R})} .$$

Proof. It is shown in Theorem 3.4 that $\sup_{t',|\tau|<\varepsilon_0,-t'<\tau<t-t'} \|v(t'-\tau)-v(t')\|_{L^1(\mathbf{R})}$ is bounded by $\varepsilon_0(\|f\|_{Lip}+\|g\|_{Lip})$. The argument of Theorem 3.5 shows that, as a measure,

$$|u^\alpha - c|_t + A(u^\alpha \vee c)_x - A(u^\alpha \wedge c)_x \le 0,$$

If one multiplies this inequality by $\omega = \omega(x''-x', t''-t')$, with $u^\alpha = u^\alpha(x'',t'')$, and substitutes $u = u(x',t')$ for c, an integration by parts shows that

$$-\Lambda_t^{\varepsilon_0,\varepsilon} \le -\iint_{S\times S} \alpha M_\alpha * [\mathrm{sgn}(u^\alpha - u)(g(u^\alpha)-g(u))]_{x''}\omega_{x''}$$
$$+ (M_\alpha - \delta_I) * [\mathrm{sgn}(u^\alpha-u)(f(u^\alpha)-f(u))]\,\omega_{x''}\, dx''\, dt''\, dx'\, dt' .$$

Here δ_I is the Dirac delta measure. Since $|u^\alpha(\cdot)|_{BV(\mathbf{R})} \le |u_0|_{BV(\mathbf{R})}$ uniformly for $t\in[0,T]$ and f and g are Lipschitz, the first and second terms on the right can be bounded by $\alpha t\, \|g\|_{Lip}\, |u_0|_{BV(\mathbf{R})} \|\omega_{x''}\|_{L^1(S)}$ and $\alpha t\, \|f\|_{Lip}\, |u_0|_{BV(\mathbf{R})} \|\omega_{x''}\|_{L^1(S)}$ respectively. Because $\|\omega_{x''}\|_{L^1(S)} \le C/\varepsilon$, $-\Lambda_t^{\varepsilon_0,\varepsilon}$ is bounded by $Ct(\|f\|_{Lip}+\|g\|_{Lip})|u_0|_{BV(\mathbf{R})}\alpha/\varepsilon$. By letting ε_0 tend to zero, we see from Theorem 3.2 that

$$\|u(t) - u^\alpha(t)\|_{L^1(\mathbf{R})} \le 2\varepsilon |u_0|_{BV(\mathbf{R})} + Ct(\|f\|_{Lip}+\|g\|_{Lip})|u_0|_{BV(\mathbf{R})}\alpha/\varepsilon .$$

Minimizing this expression with respect to ε gives

$$\|u(t)-u^{\alpha}(t)\|_{L^1(\mathbf{R})} \le C\,|u_0|_{BV(\mathbf{R})} t^{1/2}\alpha^{1/2}(\|f\|_{Lip}+\|g\|_{Lip})^{1/2}.$$

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None of the above theorems hold if the dispersive term $-\beta u_{xxt}$ is replaced by the KdV dispersive term $+\beta u_{xxx}$. In particular, it is simple enough to show that, no matter how small the coefficient β, for certain initial data the initial value problem

$$\begin{aligned} u_t + f(u)_x - \nu g(u)_{xx} + \beta u_{xxx} &= 0, \quad x\in\mathbf{R}, t>0, \\ u(x,0) &= u_0(x), \quad x\in\mathbf{R}, \end{aligned} \tag{3.21}$$

will *not* satisfy a maximum principle. This initial data can be chosen to be a third degree polynomial in the neighborhood of the maximum of u_0. However, Schonbek [21] has used the theory of compensated compactness to show that certain regularizations of the form (3.21) converge to the entropy solution of the conservation law (C).

The initial data u_0 for which (3.21) does not satisfy a maximum principle has a significant high frequency component at its maximum. The KdV equation is sensitive to high frequency waves: the dispersion relation for the linearized equation shows that as the wavelength gets shorter, waves travel with an increasing, unbounded speed in the direction of $-\infty$. The model equation (S) does not have this property. Instead, the phase and group velocities of the Fourier components of the initial waveform are bounded independently of the wavelength. This difference may provide some intuition as to why the KdV model does not satisfy a maximum principle.

4. ON WAVE HIERARCHIES

In a chapter in [25] entitled "Wave Hierarchies," Whitham studies a class of singular perturbation problems of the form

$$\eta\left(\frac{\partial}{\partial t}+c_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+c_2\frac{\partial}{\partial x}\right)\varphi+\left(\frac{\partial}{\partial t}+a\frac{\partial}{\partial x}\right)\varphi=0, \tag{4.1}$$

posed on the quarter-plane $x>0$, $t>0$; the positive parameter $\eta>0$ is small. Equation (4.1) can be considered a second-order hyperbolic regularization of the first-order equation

$$\frac{\partial\varphi}{\partial t}+a\frac{\partial\varphi}{\partial x}=0, \tag{4.2}$$

whereas Equation (4.2) is the *reduced equation* for Equation (4.1).

Using Laplace transform techniques, Whitham shows that Equation (4.1) is linearly stable if the characteristics of the reduced equation (4.2) are compatible with the characteristics of Equation (4.1), in the sense that $c_1<a<c_2$. This occurs when, at any point (x,y), the first-order characteristics of (4.2) point into the cone whose boundary consists of the second-order characteristics of (4.1). In this case, the fastest and slowest signals in (4.1) travel along the characteristics with speed c_2 and c_1, respectively, while decaying like $Ce^{-\xi/\eta}$, where ξ is the characteristic variable. The bulk of the signal travels at speed a, the wave speed of the reduced equation. Finally, any incongruities in the boundary conditions are resolved in a boundary layer near $x=0$ of width proportional to η. Whitham's analysis can also be used to cover the pure initial-value problem in which (4.1) is posed on the half-space $x\in\mathbf{R}$, $t>0$, and initial data $u_0(x)$ is provided on the line $t=0$.

We use a different approach to provide a thorough analysis for a similar nonlinear problem. Consider the conservation law (C) and its second-order hyperbolic regularization

$$\begin{aligned} u_t+f(u)_x+\eta u_{xt}&=0, \qquad x\in\mathbf{R},\ t>0,\\ u(x,0)&=u_0(x), \qquad x\in\mathbf{R}. \end{aligned} \tag{4.3}$$

The hyperbolic regularization (4.3), which results in a singular form of (4.1), introduces both dissipative and dispersive effects, and we will show that it is a special case of the Sobolev equations introduced and analyzed in the previous section, for which results similar to those in Section 3 are true. We must first show that Equation (4.3) is well posed with the given initial conditions.

Because the characteristics of Equation (4.3) are the lines $x = c$ and $t = c$ for any constant c, we are specifying initial data along a characteristic, and the solution of (4.3) is not unique. Transform (4.3) by taking $(1-\eta\partial_x)$ of each side to obtain

$$u_t + f(u)_x - \eta f(u)_{xx} - \eta^2 u_{xxt} = 0. \tag{4.4}$$

By this device, we have separated the effects of the regularization ηu_{xt} into its purely dissipative part $(-\eta f(u)_{xx})$ and its purely dispersive part $(-\eta^2 u_{xxt})$. Because the operator $(1-\eta\partial_x)$ has a non-zero kernel, Equation (4.4) will have more solutions than (4.3).

It is immediately obvious that Equation (4.4) is a special case of Equation (3.3). Our analysis of (3.3) relies on the fact that the operator $1-\eta^2\partial_x^2$ can be inverted, *subject to boundedness at infinity*, by convolution with the function $M_\eta = \frac{1}{2\eta} e^{\frac{-|x|}{\eta}}$. Equation (4.4) can therefore be transformed into the integral equation,

$$u_t = K_\eta * f(u) - \frac{1}{\eta}(f(u) - M_\eta * f(u)), \tag{4.5}$$

($K_\eta = \frac{-\mathrm{sgn}(x)}{2\eta^2} e^{\frac{-|x|}{\eta}}$), which may be viewed as an ordinary differential equation on $L^p(\mathbf{R})$ for any $1 \le p \le \infty$. Posed as an initial-value problem on $L^1(\mathbf{R})$, (4.5) has a unique solution for any Lipschitz continuous f. Thus, we choose the "correct" solution of (4.3) to be the unique weak solution of (4.3) with $\|u_t\|_{L^\infty(\mathbf{R}\times[0,T])} < \infty$ for any $T>0$.

With this definition, Theorem 3.4 yields the following stability results.

THEOREM 4.1. *The following properties are equivalent.*

1. *The function f is nondecreasing on* $\mathbf{R}$.

2. *The mappings $S_t: u_0 \to u(\cdot,t)$, where $u(x,t)$ is the solution of* (4), *are contractions on $L^1(\mathbf{R})$. Since S_t commutes with translations, this implies that $|S_t(u_0)|_{BV(\mathbf{R})} \le |u_0|_{BV(\mathbf{R})}$.*

3. *The mappings S_t are order preserving on $L^1(\mathbf{R})$: if $u_0(x) \ge v_0(x)$ for all $x \in \mathbf{R}$, then $S_t(u_0)(x) \ge S_t(v_0)(x)$ for all $x \in \mathbf{R}$, $t > 0$.*

4. *The mappings S_t satisfy maximum and minimum principles:* $\operatorname{ess\,sup} S_t(u_0) \le \operatorname{ess\,sup} u_0$ *and* $\operatorname{ess\,inf} S_t(u_0) \ge \operatorname{ess\,inf} u_0$.

In light of Theorem 4.1, we say that (4.3) is *stable* if f is nondecreasing.

It seems natural to require that f be nondecreasing for stability; otherwise the "dissipative" term $-\eta f(u)_{xx}$ in (5) is not dissipative on all of $L^1(\mathbf{R})$! However, the proof of the preceding theorem relies on the delicate balance between the dissipative and dispersive effects of the regularization in (4.3).

Once Theorem 4.1 is in hand, the solution operator S_t can be extended to $u_0 \in BV(\mathbf{R})$ as a continuous map on the space of locally integrable functions $L^1_{loc}(\mathbf{R})$. Besides being total variation diminishing, the map S_t has the following properties.

THEOREM 4.2. *Assume that the function f is Lipschitz continuous and nondecreasing. Let $S_t(u_0)$ be the solution of* (4.3) *with $u_0 \in BV(\mathbf{R})$ and let $u(x,t)$ be the entropy solution of* (C). *Then there exists a constant C depending on the Lipschitz constant for f such that for any $t > 0$,* $\|S_t(u_0) - u(\cdot,t)\|_{L^1(\mathbf{R})} \le C(\eta t)^{1/2} |u_0|_{BV(\mathbf{R})}$.

Theorem 4.2 is a direct consequence of Theorem 3.6.

The results of our analysis can be compared with the linear analysis of Whitham. In Equation (4.3), information travels in the positive x and t directions.

Whitham's compatibility condition for the regularized equation requires that the first-order characteristics point into the first quadrant, ie. that f be nonnegative, so that f is nondecreasing. This is exactly the stability condition of Part 1 of Theorem 4.1. If this compatibility condition holds, then Theorem 4.2 provides strict error estimates for the full nonlinear problem in terms of the regularization parameter η.

The problem

$$\begin{aligned} u_t + f(u)_x - \eta u_{xt} &= 0, \quad x \in \mathbf{R},\ t > 0, \\ u(x,0) &= u_0(x), \quad x \in \mathbf{R}. \end{aligned} \tag{4.6}$$

can be analvzed similarly. The following theorem applies.

THEOREM 4.3. *Assume that the function f is Lipschitz continuous and nonincreasing. Let $S_t(u_0)$ be the solution of* (4.6) *with $u_0 \in BV(\mathbf{R})$ and let $u(x,t)$ be the entropy solution of* (C). *Then there exists a constant C depending on the Lipschitz constant for f such that for any $t>0$, $\|S_t(u_0) - u(\cdot,t)\|_{L^1(\mathbf{R})} \le C(\eta t)^{1/2} |u_0|_{BV(\mathbf{R})}$.*

Theorem 4.3's stability condition is that f be nonincreasing on $\mathbf{R}$, and the characteristics of (4.6) can be interpreted as pointing in the negative x direction and the positive t direction. Whitham's compatibility condition is now that the first-order characteristics of (4.6) point into the second quadrant; this is equivalent to requiring that f be nonincreasing on $\mathbf{R}$.

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