A SHORT PROOF OF THE DIRECT SUMMAND THEOREM VIA THE FLATNESS LEMMA

LINQUAN MA

The goal is to explain a short proof of the direct summand theorem which relies only on André's flatness lemma [And18], the idea follows from [And20] and we use the same notation as in op. cit.. I thank Bhargav Bhatt and Kestutis Cesnavicius for discussions on the proof at MSRI Spring 2019.

Theorem 0.1. *Every complete local domain R of mixed characteristic $(0,p)$ with perfect residue field admits a big Cohen-Macaulay algebra. In particular, the direct summand theorem holds.*

Proof. Write $R = S/P$ where $(S, \mathfrak{m}) = W(k)[[z_1, \ldots, z_n]]$. Since $p \notin P$, by standard prime avoidance, we can pick $f_1, \ldots, f_c \in P$ such that $p, f_1, \ldots, f_c$ is a regular sequence in S with $c = \operatorname{ht} P$. Extend $f_1, \ldots, f_c, p$ to a full system of parameters on S: $f_1, \ldots, f_c, p, x_2, \ldots, x_d$. Then $p, x_2, \ldots, x_d$ is a system of parameters on S/P and $S/(f_1, \ldots, f_c)$. Since P is a minimal prime of $(f_1, \ldots, f_c)$, there exists $g \notin P$ such that $gP \in \sqrt{(f_1, \ldots, f_c)}$.

Let $S_{\infty,0} = S[\widehat{p^{1/p^\infty}, z_1^{1/p^\infty}, \ldots, z_n^{1/p^\infty}}]$ and let $S_\infty = S_{\infty,0}\langle f_1^{1/p^\infty}, \ldots, f_c^{1/p^\infty}, g^{1/p^\infty}\rangle[\frac{1}{p}]^\circ$. Then both $S_{\infty,0}$ and S_∞ are perfectoid algebras. Moreover, let $T = S_\infty/(f_1^{1/p^\infty}, f_2^{1/p^\infty}, \ldots, f_c^{1/p^\infty})^-$, where $(-)^-$ denotes closure in the p-adic topology. Then T is also a perfectoid algebra.

Claim 0.2. *$p, x_2, \ldots, x_d$ is a p^{1/p^∞}-almost regular sequence on T.*

Proof. By André's flatness lemma (see [Bha18, Theorem 2.3]), S_∞ is p^{1/p^∞}-almost faithfully flat over $S_{\infty,0}$ mod p and thus $p, f_1^{1/p^e}, \ldots, f_c^{1/p^e}, x_2, \ldots, x_d$ is a p^{1/p^∞}-almost regular sequence on S_∞ for every e. This implies $p, x_2, \ldots, x_d$ is a p^{1/p^∞}-almost regular sequence on $S_\infty/(f_1^{1/p^e}, \ldots, f_c^{1/p^e})$ for every e.[1] Hence after taking direct limit for e and p-adic completion, we know that $p, x_2, \ldots, x_d$ is a p^{1/p^∞}-almost regular sequence on T. □

Claim 0.3. *$p, x_2, \ldots, x_d$ is a $(pg)^{1/p^\infty}$-almost regular sequence on $g^{-1/p^\infty}T := \operatorname{Hom}((g^{1/p^\infty}), T)$.*

Proof. By Claim 0.2, we only need to check that $(g^{-1/p^\infty}T)/(p, x_2, \ldots, x_d)$ is not $(pg)^{1/p^\infty}$-almost zero. This comes down to show that $(pg)^{1/p^\infty}$ is not contained in $(p, x_2, \ldots, x_d)T$. Therefore it is enough to show that $S_\infty/(\mathfrak{m}S_\infty + \sqrt{PS_\infty})$ is not $(pg)^{1/p^\infty}$-almost zero. Now S_∞ is a p^{1/p^∞}-almost Cohen-Macaulay S-algebra by André's flatness lemma again, so we can use Hochster's algebra modification or Gabber's method (see [Gab18]) to map S_∞ to a big Cohen-Macaulay S-algebra B. If $S_\infty/(\mathfrak{m}S_\infty + \sqrt{PS_\infty})$ is $(pg)^{1/p^\infty}$-almost zero, then so is $B/(\mathfrak{m}B + \sqrt{PB})$. But the latter one is not $(pg)^{1/p^\infty}$-almost zero thanks to [And20, Proposition 2.5.1 (2)] (applied to $R = S$ and $\pi = pg$, note that we are using $pg \notin P$ to guarantee the conditions are satisfied). □

[1] We are using the fact that if y_1, y_2 is an almost regular sequence, then y_1 is always an almost nonzerodivisor mod y_2 (but y_2 may not be an almost nonzerodivisor itself).

Finally, since T is reduced, T is an $S/\sqrt{(f_1,\dots,f_c)}$-algebra, and thus $gP = 0$ in T. But then $g^{1/p^\infty}P = 0$ in T and thus $g^{-1/p^\infty}T$ is an $R = S/P$-algebra. By Claim 0.3 above, $g^{-1/p^\infty}T$ is a $(pg)^{1/p^\infty}$-almost Cohen-Macalay R-algebra. So we can use Hochster's algebra modification or Gabber's method to map $g^{-1/p^\infty}T$ to a big Cohen-Macaulay algebra. □

Remark 0.4. (a) By Gabber's refinement of the flatness lemma [GR04, 16.9.17], if we instead let S_∞ be the p-root closure of $S_{\infty,0}[f_1^{1/p^\infty},\dots,f_c^{1/p^\infty},g^{1/p^\infty}]$, then S_∞ is honestly faithfully flat over $S_{\infty,0}$. Therefore in Claim 0.2, $p, x_2, \dots, x_d$ is an honest regular sequence in T, and Claim 0.3 follows directly from [And20, Proposition 2.5.1 (2)].

(b) There is an elementary and self-contained proof of (a generalization of) André's flatness lemma in [CS19, Section 2.3], and recently Bhatt–Scholze proved a general form of the purity theorem using the flatness lemma, see [BS19, Theorem 10.9].

References

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Department of Mathematics, Purdue University, West Lafayette, IN 47907
Email address: ma326@purdue.edu