F-SINGULARITIES: A COMMUTATIVE ALGEBRA APPROACH

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Contents

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Introduction

The local theory of prime characteristic singularities is a beautiful and historied subject. Singularities which are defined in terms of the behavior of the Frobenius endomorphism have been labeled "F-singularities". We give an introduction on the four most prominent F-singularity classes; F-pure, F-injective, strongly F-regular, and F-rational singularities. Our approach is algebraic and we assume the reader is familiar with the basics of commutative algebra, see [Mat89] and Part I of [BH93].

Our treatment of prime characteristic singularities starts with Kunz's fundamental theorem from the 1960's, a point on a variety defined over a prime characteristic field is non-singular if and only if the Frobenius map is flat at that point, [Kun69]. We then begin our treatment of F-singularities with the first F-singularity class to be considered historically. The class of F-pure rings were born out of Hochster–Roberts's study of rings of invariants in the 1970's, [HR74, HR76]. Our initial presentation of F-pure rings in Chapter 2 is centered around Fedder's criterion, [Fed83], a containment test to determine if a homomorphic image of a regular ring is F-pure.

We deviate from the historical development of F-singularities in Chapter 3 and introduce the basic theory of strongly F-regular singularities, a singularity class that emerged from Hochster–Huneke's tight closure theory, [HH90, HH91, HH94a, HH94c]. Strongly F-regular rings are naturally studied in this text without the knowledge of tight closure theory.

We overlap the theory of F-injective and F-rational singularities in Chapter 4 through the study of Frobenius actions on local cohomology modules. In the 1980's, F-injective singularities came from the study of F-pure rings by Fedder, [Fed83], and the theory of F-rational singularities appeared alongside strongly F-regular singularities in tight closure theory. Similar to the theory of strongly F-regular singularities, the theory of F-rational singularities can be approached naturally without the knowledge of tight closure. Moreover, our study of F-rational rings through local cohomology gives valuable insight to more advanced topics treated in later chapters.

The problems of deforming the four fundamental F-singularity classes is presented in Chapter 5. We give self-contained treatments of the deformation problems as it pertains to F-rational singularities, $\mathbb{Q}$-Gorenstein strongly F-regular singularities, and $\mathbb{Q}$-Gorenstein F-pure singularities. We present some partial progress on the currently open problem of deforming F-injective singularities. Counterexamples to the deformation of strongly F-regular and F-pure singularities in non-$\mathbb{Q}$-Gorenstein rings are given in Chapter 8, among many other examples.

The study of F-singularities under local ring maps $R \to S$ given by Γ-constructions, completions, and other faithfully flat maps is the content of Chapter 6 and Chapter 7. Fundamentals of F-signature theory are presented in Chapter 9. We end our manuscript in Chapter 10, where we give self-contained and elementary proofs of: the Radu-André Theorem (a significant generalization of Kunz's theorem concerning the flatness of the Frobenius); another theorem of Kunz that F-finite rings are excellent, and Gabber's result that F-finite rings are homomorphic image of regular rings. At the end of every chapter we provide several supplemental exercises. We also provide several open problems throughout the text.

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Unless otherwise stated, all rings are assumed to be commutative, Noetherian and with multiplicative identity 1*. We will use the convention that* $(R, \mathfrak{m}, k)$ *is a (Noetherian) local ring with unique maximal ideal* $\mathfrak{m}$ *and residue field* $k = R/\mathfrak{m}$.

1. Kunz's theorem and F-finite rings

Rings of prime characteristic $p > 0$ come equipped with a special endomorphism, namely the Frobenius endomorphism $F : R \to R$ defined by $F(r) = r^p$. For each $e \in \mathbb{N}$ we can iterate the Frobenius endomorphism e times and obtain the e-th Frobenius endomorphism $F^e : R \to R$ defined by $F^e(r) = r^{p^e}$. Roughly speaking, the study of prime characteristic rings is the study of algebraic and geometric properties of the Frobenius endomorphism.

Throughout this text, we often need to distinguish the source and target of the Frobenius map. We adopt the commonly used notation F^e_*R to denote the target of the Frobenius as a module over the source, that is, $F^e\colon R \to F^e_*R$. Under this notation, elements in F^e_*R are denoted by F^e_*r where $r \in R$, and the R-module structure on F^e_*R is defined via $r_1 \cdot F^e_*r_2 = F^e_*(r_1^{p^e} r_2)$. On the other hand, $F^e_*R \cong R$ via $F^e_*r \leftrightarrow r$ as rings.

Suppose that R is reduced and let K be the total ring of fractions of R, thus $K = \prod K_i$ is a product of fields. Let $\overline{K} := \prod \overline{K_i}$. There are inclusions $R \subseteq K \subseteq \overline{K}$. We let

$$R^{1/p^e} := \{s \in \overline{K} \mid s^{p^e} \in R\}.$$

In other words, R^{1/p^e} is the collection of p^e-th roots of elements of R. Then R^{1/p^e} is unique up to non-unique isomorphism, and $R^{1/p^e} \cong R$ via $r^{1/p^e} \leftrightarrow r$ as rings. In this setup, we can view the Frobenius map as the natural inclusion $R \hookrightarrow R^{1/p^e}$, see Exercise 3.

As we already mentioned, the singularities of R are often studied via the behavior of the Frobenius map. A fundamental result in this direction is proved by Kunz [Kun69].

Theorem 1.1 (Kunz's Theorem)**.** *A ring R of prime characteristic $p > 0$ is regular if and only if the Frobenius map $F^e\colon R \to F^e_*R$ is flat for some (or equivalently, all) $e > 0$.*

Proof. First assume that R is regular, we want to show that F^e_*R is a flat R-module. Since flatness can be checked locally and we have $(F^e_*R)_P \cong F^e_*(R_P)$ as R_P-modules for all $P \in \operatorname{Spec}(R)$, we may assume $(R, \mathfrak{m}, k)$ is local. We next consider the commutative diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & F^e_*R \arrow[d] \\ \widehat{R} \arrow[r] & F^e_*\widehat{R} \end{tikzcd}$$

Since both vertical maps are faithfully flat, if we can show the bottom map is flat, then it will imply that the top map is flat. Therefore we may replace R by $\widehat{R}$ to assume $(R, \mathfrak{m}, k)$ is a complete regular local ring. By Cohen's structure theorem, $R \cong k[[x_1, \ldots, x_d]]$. In this case, it is straightforward to check that F^e_*R is a free R-module with basis

$$\{F^e_*(\lambda x_1^{i_1} \cdots x_d^{i_d}) \mid 0 \leq i_j < p^e, \text{ where } \{F^e_*\lambda\} \text{ is a free basis of } F^e_*k \text{ over } k\}.$$

Now we prove the converse. Note that if F^e_*R is flat over R for some $e > 0$, then after iterating we see that F^{ne}_*R is flat over R for all n. In particular, we can assume F^e_*R is flat over R for infinitely many $e > 0$. Since regularity and flatness are local conditions, we may again assume that $(R, \mathfrak{m}, k)$ is a local ring. Let $g = \operatorname{depth} R$. We pick a regular sequence in $\mathfrak{m}$ of maximal length: $x_1, \dots, x_g$. It follows that $R/(x_1, \dots, x_g)$ has depth 0 and thus $0 \neq N := \operatorname{Soc}(R/(x_1, \dots, x_g)) \cong \operatorname{Hom}_R(R/\mathfrak{m}, R/(x_1, \dots, x_g))$. Hence there exists n such that $N \nsubseteq \mathfrak{m}^n(R/(x_1, \dots, x_g))$.

Claim 1.2. *For any finitely generated R-module M of infinite projective dimension with minimal free resolution*

$$\cdots \to R^{n_{g+2}} \xrightarrow{\phi_{g+2}} R^{n_{g+1}} \xrightarrow{\phi_{g+1}} R^{n_g} \to \cdots \to R^{n_1} \to R^{n_0} \to M \to 0,$$

the entries in the matrix representing ϕ_{g+2} are not all contained in $\mathfrak{m}^n$.

Proof of Claim. Since $\operatorname{pd}_R R/(x_1, \dots, x_g) = g$, we have $\operatorname{Tor}^R_{g+1}(M, R/(x_1, \dots, x_g)) = 0$. Therefore tensoring the above minimal free resolution with $R/(x_1, \dots, x_g)$, we know that

$$(R/(x_1, \dots, x_g))^{n_{g+2}} \xrightarrow{\phi_{g+2}} (R/(x_1, \dots, x_g))^{n_{g+1}} \xrightarrow{\phi_{g+1}} (R/(x_1, \dots, x_g))^{n_g}$$

is exact in the middle. Since the resolution is minimal, the socle $N^{n_{g+1}} \subseteq (R/(x_1, \dots, x_g))^{n_{g+1}}$ is contained in $\operatorname{Ker} \phi_{g+1} = \operatorname{Im} \phi_{g+2}$. If all entries in the matrix representing ϕ_{g+2} are contained in $\mathfrak{m}^n$, then $N^{n_{g+1}} \subseteq \mathfrak{m}^n(R/(x_1, \dots, x_g))^{n_{g+1}}$ and thus $N \subseteq \mathfrak{m}^n(R/(x_1, \dots, x_g))$. This is a contradiction. $\square$

We now continue the proof of the theorem. Suppose $\operatorname{pd}_R R/\mathfrak{m} = \infty$. Since the Frobenius map is flat, tensoring a minimal free resolution of $R/\mathfrak{m}$ with F^e_*R and identifying F^e_*R with R, we obtain a minimal free resolution of $R/\mathfrak{m}^{[p^e]}$ such that the entries in the matrix representing each differential (in particular the $(g+2)$-th differential) are all contained in $\mathfrak{m}^{[p^e]}$, the ideal generated by p^e-th powers of elements of $\mathfrak{m}$. But for $e \gg 0$ this contradicts Claim 1.2 because n is independent of e. Therefore $\operatorname{pd}_R R/\mathfrak{m} < \infty$ and thus R is regular. $\square$

Remark 1.3. Since the Frobenius map F^e induces a bijection on $\operatorname{Spec}(R)$, F^e is flat if and only if it is faithfully flat. Hence Theorem 1.1 implies that a ring R of prime characteristic $p > 0$ is regular if and only if F^e is faithfully flat for some (or equivalently, all) $e > 0$.

Remark 1.4. Our proof of the converse direction in Theorem 1.1 follows from [KL98] (which originates from ideas in [Her74]).

We next introduce a rather "mild" condition on the Frobenius map.

Definition 1.5. A ring R of prime characteristic $p > 0$ is called *F-finite* if for some (or equivalently, all) $e > 0$, the Frobenius map $F^e\colon R \to R$ is a finite morphism, i.e., $F^e_* R$ is a finitely generated R-module.

For example, a field k of prime characteristic $p > 0$ is F-finite if and only if $[k^{1/p} : k] < \infty$. More generally, it follows from Exercise 5 below (and Cohen's structure theorem) that rings essentially finite type over F-finite fields are F-finite, and complete local rings of prime characteristic $p > 0$ with F-finite residue fields are F-finite.

The F-finite property turns out to imply that the rings are not pathological. We will sometimes implicitly use the following two results, due to Gabber [Gab04] and Kunz [Kun76] respectively, throughout. We will give proofs of these results in Chapter 10.

Theorem 1.6. *Let R be an F-finite ring of prime characteristic $p > 0$. Then R is a homomorphic image of an F-finite regular ring. In particular, F-finite rings admit canonical modules.*

Theorem 1.7. *If R is an F-finite ring of prime characteristic $p > 0$, then R is excellent. Moreover, if $(R, \mathfrak{m}, k)$ is a local ring of prime characteristic $p > 0$, then R is F-finite if and only if R is excellent and $R/\mathfrak{m}$ is F-finite.*

Recall that a ring R is called *excellent* if R satisfies the following:

(1) R is universally catenary.
(2) If S is an R-algebra of finite type, then the regular locus of S is open in $\operatorname{Spec}(S)$.
(3) For all $P \in \operatorname{Spec}(R)$, the map $R_P \to \widehat{R_P}$ has geometrically regular fibers. That is, for all $Q \in \operatorname{Spec}(R)$ such that $Q \subseteq P$, $\kappa(Q)' \otimes_{R_P} \widehat{R_P}$ is regular for all finite (or equivalently, finite and purely inseparable) field extensions $\kappa(Q)'$ of $\kappa(Q)$.

Excellent rings include most examples arising from algebraic geometry. For example, all rings essentially finite type over a field and all complete local rings are excellent.

Exercise 1. Let R be a ring of prime characteristic $p > 0$. Verify that if $F^e_* R$ is finitely generated for one $e > 0$, then $F^e_* R$ is finitely generated for all $e > 0$.

Exercise 2. Let R be a ring of prime characteristic $p > 0$. Prove that R is reduced if and only if $R \to F^e_* R$ is injective for one (or equivalently, all) $e > 0$.

Exercise 3. Let R be a reduced ring of prime characteristic $p > 0$. Show that the eth iterate of the Frobenius map $F^e : R \to F^e_* R$ is isomorphic to the inclusion of algebras $R \subseteq R^{1/p^e}$.

Exercise 4. Let R be a ring of prime characteristic $p > 0$. Prove that R is F-finite if and only if $R_{\mathrm{red}} := R/\sqrt{0}$ is F-finite. (Hint: First show that $R \to F^e_* R$ factors through R_{red} for $e \gg 0$. Then consider a filtration $0 = J^n \subseteq J^{n-1} \subseteq \cdots \subseteq J = \sqrt{0} \subseteq R$ and show that each $F^e_*(J^i/J^{i+1})$ is finitely generated over R_{red}.)

Exercise 5. Let R be an F-finite ring of prime characteristic $p > 0$. Prove the following:

(1) If $I \subseteq R$ an ideal then R/I is F-finite.
(2) If W a multiplicative subset of R then $W^{-1}R$ is F-finite.
(3) If x an indeterminate then $R[x]$ and $R[[x]]$ are F-finite.

Conclude that rings essentially of finite type over F-finite rings are F-finite.

2. F-pure rings and Fedder's criterion

An F-singularity is a class of prime characteristic singularities defined in terms of the behavior of Frobenius endomorphism. Theorem 1.1 equates flatness of the Frobenius endomorphism with non-singularity of the ambient ring. Therefore non-singularity is an F-singularity and thus it motivates the study of other F-singularities. Our first class of F-singularities are F-pure and F-split singularities.

Definition 2.1. A map of R-modules $M_1 \to M_2$ is *pure* if $M_1 \otimes_R N \to M_2 \otimes_R N$ is injective for every R-module N. A ring R of prime characteristic $p > 0$ is called *F-pure* (resp., *F-split*) if the Frobenius map $F^e \colon R \to F^e_* R$ is pure (resp., split) for some (or equivalently, all) $e > 0$.

Clearly, a split map is always pure, hence F-split implies F-pure. Moreover, if R is F-pure then the Frobenius map is injective and thus R is reduced, see Exercise 2. So in this case we can always view the Frobenius map as the natural inclusion $R \hookrightarrow R^{1/p^e}$. Therefore R is F-pure if and only if R is reduced and the natural map $R \to R^{1/p^e}$ is pure for some (or equivalently, all) $e > 0$. Similarly, R is F-split if R is reduced and $R \to R^{1/p^e}$ is split for some (or equivalently, all) $e > 0$.

We will prove that F-singularity classes of F-pure and F-split singularities are equivalent for F-finite rings and complete local rings. To establish this we prove a general criterion for purity of maps.

Proposition 2.2. *Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-module. Then a map $R \to M$ is pure if and only if the induced map $E \to E \otimes_R M$ is injective where $E := E_R(k)$ denotes the injective hull of the residue field.*

Proof. One direction is obvious. So suppose $R \to M$ is not pure, then there exists an R-module N such that $N \to N \otimes_R M$ is not injective. Since N is a directed union of its finitely generated submodules and injectivity is preserved under direct limit, we may assume N is finitely generated. Now we pick $u \in \operatorname{Ker}(N \to N \otimes_R M)$, there exists n such that $u \notin \mathfrak{m}^n N$. Consider the commutative diagram:

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N \otimes_R M \arrow[d] \\ N/\mathfrak{m}^n N \arrow[r] & (N/\mathfrak{m}^n N) \otimes_R M \end{tikzcd}$$

Since the image of $u \in N/\mathfrak{m}^n N$ is nonzero, the bottom map is not injective. Now $N/\mathfrak{m}^n N$ has finite length, so it embeds in $E^{\oplus r}$ for some r. The commutative diagram

$$\begin{tikzcd} N/\mathfrak{m}^n N \arrow[r] \arrow[d, hook] & (N/\mathfrak{m}^n N) \otimes_R M \arrow[d] \\ E^{\oplus r} \arrow[r] & E^{\oplus r} \otimes_R M \end{tikzcd}$$

then shows that the bottom map is not injective. Thus $E \to E \otimes_R M$ is not injective. $\square$

Corollary 2.3. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p > 0$. Then R is F-pure if and only if $\widehat{R}$ is F-pure.*

Proof. We have canonical isomorphisms $E := E_R(k) \cong E_R(k) \otimes_R \widehat{R} \cong E_{\widehat{R}}(k)$. Thus we have

$$E \to E \otimes_R F^e_* R \to E \otimes_R F^e_* \widehat{R} \cong E \otimes_{\widehat{R}} F^e_* \widehat{R}.$$

Since $F^e_* R \to F^e_* \widehat{R}$ is faithfully flat and hence pure (see Exercise 10) and thus also pure as an R-module map, the second map is injective. Hence the composition is injective if and only if the first map is injective. Therefore the conclusion follows from Proposition 2.2. $\square$

Corollary 2.4. *Let $R \to M$ be a pure map. If either R is complete local or M is finitely generated, then $R \to M$ is split. In particular, if R is a ring of prime characteristic $p > 0$, F-pure, and is either complete local or F-finite, then R is F-split.*

Proof. If $(R, \mathfrak{m}, k)$ is complete local, then taking the Matlis dual of the injection $E \hookrightarrow E \otimes_R M$ yields a surjection $\operatorname{Hom}_R(E \otimes_R M, E) \twoheadrightarrow \operatorname{Hom}_R(E, E) \cong R$. By adjunction we have

$$\operatorname{Hom}_R(E \otimes_R M, E) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(E, E)) \cong \operatorname{Hom}_R(M, R).$$

Thus we have a surjection $\operatorname{Hom}_R(M, R) \twoheadrightarrow R$, one can check that this is precisely the natural map induced by applying $\operatorname{Hom}_R(-, R)$ to $R \to M$. Thus $R \to M$ is split.

Next we assume M is finitely generated. We want to show that the map $\operatorname{Hom}_R(M, R) \to R$ is surjective. It is enough to check this locally on $\operatorname{Spec}(R)$. Since M is finitely genrated, we have

$$R_P \otimes_R \operatorname{Hom}_R(M, R) \cong \operatorname{Hom}_{R_P}(M_P, R_P).$$

Since $R \to M$ is pure, $R_P \to M_P$ is pure for all $P \in \operatorname{Spec}(R)$, we may thus assume that R is local. But then the surjectivity of $\operatorname{Hom}_R(M, R) \to R$ can be checked after base change to $\widehat{R}$. Since M is finitely generated, we know that

$$\widehat{R} \otimes_R \operatorname{Hom}_R(M, R) \cong \operatorname{Hom}_{\widehat{R}}(M \otimes_R \widehat{R}, \widehat{R}).$$

Therefore it remains to show that $\operatorname{Hom}_{\widehat{R}}(M\otimes_R \widehat{R}, \widehat{R}) \to \widehat{R}$ is surjective. But since $R \to M$ is pure, $\widehat{R} \to M \otimes_R \widehat{R}$ is pure, and hence split by the first conclusion. So $\operatorname{Hom}_{\widehat{R}}(M\otimes_R \widehat{R}, \widehat{R}) \to \widehat{R}$ is surjective as wanted. $\square$

Since faithfully flat maps are always pure (see Exercise 10 below), regular rings of prime characteristic $p > 0$ are F-pure by Theorem 1.1, and thus by Corollary 2.4, complete regular local rings and F-finite regular rings are F-split. However, we warn the reader that there are examples of regular local rings (even DVRs) of prime characteristic $p > 0$ that are *not* F-split. The first such example was discovered by Datta–Smith [DS16b] who constructed a non-excellent DVR of prime characteristic $p > 0$ that is not F-split. Very recently, Datta–Murayama [DM20] constructed an excellent, local, henselian DVR of prime characteristic $p > 0$ that is not F-split. Thus without the assumptions of Corollary 2.4, it frequently happens that F-pure rings fail to be F-split. We will not treat these examples in this text though: for most questions that we will study, one can first localize and then complete (one can further pass to F-finite rings, see Chapter 6) so Corollary 2.4 can be applied to tell us that we do not need to distinguish between F-pure and F-split.

We next state and prove a fundamental result of Fedder [Fed83].

Theorem 2.5 (Fedder's criterion)**.** *Let $(S, \mathfrak{m}, k)$ be a regular local ring of prime characteristic $p > 0$ and let $I \subseteq S$ be an ideal. Then $R := S/I$ is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$ where $I^{[p]}$ is the ideal generated by p-th powers of elements of I.*

Proof. We first assume $(S, \mathfrak{m}, k)$ is a complete regular local ring with perfect residue field. By Cohen's structure theorem, $S \cong k[[x_1, \dots, x_d]]$ and we know that F_*S is a finite free S-module with basis $\{F_*(x_1^{i_1} \cdots x_d^{i_d}) \mid 0 \le i_j < p\}$.

Claim 2.6. *For each tuple $(i_1, \dots, i_d)$ with $0 \le i_1, \dots, i_d < p$ there is an S-linear map $\varphi_{(i_1,\dots,i_d)} \colon F_*S \to S$ which is defined on basis elements as follows:*

$$\varphi_{(i_1,\dots,i_d)}(F_*(x_1^{j_1} \cdots x_d^{j_d})) = \begin{cases} 1 & (j_1, \dots, j_d) = (i_1, \dots, i_d) \\ 0 & (j_1, \dots, j_d) \neq (i_1, \dots, i_d) \end{cases}.$$

Moreover, $\operatorname{Hom}_S(F_*S, S) \cong (F_*S) \cdot \Phi$ *where* $\Phi = \varphi_{(p-1,\dots,p-1)}$.

Proof of Claim. The first assertion is clear and we only prove the second assertion. Since all the $\varphi_{(i_1,\dots,i_d)}$s generate $\operatorname{Hom}_S(F_*S, S)$ as an S-module, it is enough to observe that

$$\varphi_{(i_1,\dots,i_d)}(F_* \cdot -) = \Phi(F_*(x_1^{p-1-i_1} \cdots x_d^{p-1-i_d} \cdot -)) = F_*(x_1^{p-1-i_1} \cdots x_d^{p-1-i_d}) \cdot \Phi.$$

Therefore Φ generates $\operatorname{Hom}_S(F_*S, S)$ as an F_*S-module as wanted. $\square$

Since F_*S is a finite free S-module, every map $F_*(S/I) \to S/I$ can be lifted to a map $F_*S \to S$, and thus can be written as $\Phi(F_*(s \cdot -))$ for some $s \in S$ by Claim 2.6.

Claim 2.7. *$\Phi(F_*(s \cdot -))$ induces a map $F_*(S/I) \to S/I$ if and only if $s \in (I^{[p]} : I)$.*

Proof of Claim. If $s \in (I^{[p]} : I)$, then $\Phi(F_*(s \cdot -))$ sends F_*I to I hence it induces a map $F_*(S/I) \to S/I$. To prove the converse, suppose $r = sr' \in sI$ such that $r \notin I^{[p]}$. Since $\{F_*(x_1^{i_1} \cdots x_d^{i_d}) \mid 0 \le i_j < p\}$ is a free basis of F_*S over S, F_*r can be written uniquely as $\sum r_{i_1 i_2 \dots i_d} F_*(x_1^{i_1} \cdots x_d^{i_d})$ where $r_{i_1 i_2 \dots i_d} \in S$. Since $F_*r \notin F_*I^{[p]}$ by our choice, there exists $r_{i_1 i_2 \dots i_d} \notin I$ and by Claim 2.6 $\varphi_{(i_1,\dots,i_d)}(F_*r) \notin I$. But then $\Phi(F_*(r x_1^{p-1-i_1} \cdots x_d^{p-1-i_d})) \notin I$ and thus $\Phi(F_*(s \cdot r' x_1^{p-1-i_1} \cdots x_d^{p-1-i_d})) \notin I$. Therefore $\Phi(F_*(s \cdot -))$ does not send F_*I to I so it does not induce a map $F_*(S/I) \to S/I$. $\square$

By Claim 2.7, S/I is F-pure (equivalently, F-split in this case by Corollary 2.4) if and only if there exists $s \in (I^{[p]} : I)$ such that $\Phi(F_*(s \cdot -))$ is surjective. But it is easy to see that $\Phi(F_*(s \cdot -))$ is surjective if and only if $s \notin \mathfrak{m}^{[p]}$: if $s \in \mathfrak{m}^{[p]}$ then the image of $\Phi(F_*(s \cdot -))$ is contained in $\mathfrak{m}$ so it cannot be surjective, while if $s \notin \mathfrak{m}^{[p]}$ then s contains a monomial $x_1^{i_1} \cdots x_d^{i_d}$ with nonzero coefficient for some $0 \le i_1, \dots, i_d < p$, so $\Phi(F_*(s \cdot x_1^{p-1-i_1} \cdots x_d^{p-1-i_d}))$ is a unit and thus $\Phi(F_*(s \cdot -))$ is surjective. Putting all these together, we see that S/I is F-pure if and only if $(I^{[p]} : I) \nsubseteq \mathfrak{m}^{[p]}$.

We next treat the general case. Consider the following commutative diagram:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \widehat{S} \cong k[[x_1, \dots, x_d]] \arrow[r] \arrow[d] & \widetilde{S} := \overline{k}[[x_1, \dots, x_d]] \arrow[d] \\ R = S/I \arrow[r] & \widehat{R} = \widehat{S}/I\widehat{S} \arrow[r] & \widetilde{R} := \widetilde{S}/I\widetilde{S} \end{tikzcd}$$

It is clear that all the maps in the horizontal rows are faithfully flat. Moreover, since $E_S(k) \cong k[x_1^{-1}, \dots, x_d^{-1}]$ and similarly for $\widetilde{S}$, we have $E_{\widetilde{S}}(\overline{k}) \cong E_S(k) \otimes_S \widetilde{S}$. It follows that

$$E_R(k) \otimes_R \widetilde{R} \cong (\operatorname{Ann}_{E_S(k)} I) \otimes_S \widetilde{S} \cong \operatorname{Ann}_{E_{\widetilde{S}}(\overline{k})} I\widetilde{S} \cong E_{\widetilde{R}}(\overline{k}).$$

Therefore we have the following commutative diagram:

$$\begin{tikzcd} E_R(k) \arrow[r] \arrow[d] & E_R(k) \otimes_R F_*R \arrow[d] \\ E_{\widetilde{R}}(\overline{k}) \cong E_R(k) \otimes_R \widetilde{R} \arrow[r] & E_{\widetilde{R}}(\overline{k}) \otimes_{\widetilde{R}} F_*\widetilde{R} \cong E_R(k) \otimes_R F_*\widetilde{R} \end{tikzcd}$$

Note that a socle representative $u \in E_R(k)$ maps to a socle representative $u \otimes 1 \in E_{\widetilde{R}}(\overline{k})$. Thus u maps to zero in $E_R(k) \otimes_R F_*R$ if and only if $u \otimes 1$ maps to zero in $E_{\widetilde{R}}(\overline{k}) \otimes_{\widetilde{R}} F_*\widetilde{R}$ (the right vertical map is injective as $F_*R \to F_*\widetilde{R}$ is faithfully flat and hence pure, see Exercise 10). Thus the top map is injective if and only if the bottom map is injective. By Proposition 2.2, R is F-pure if and only if $\widetilde{R}$ is F-pure. Now $\widetilde{R} = \widetilde{S}/I\widetilde{S}$ and $\widetilde{S}$ is complete local with perfect residue field, so by what we have proved, $\widetilde{R}$ is F-pure if and only if $(I^{[p]}\widetilde{S} :_{\widetilde{S}} I\widetilde{S}) \not\subseteq \mathfrak{m}^{[p]}\widetilde{S}$. But since $S \to \widetilde{S}$ is faithfully flat, the latter holds if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$. □

Remark 2.8. There is a graded version of Fedder's criterion: let $S = k[x_1, \dots, x_d]$ be a polynomial ring over a field k and let $I \subseteq S$ be a homogeneous ideal. Then $R := S/I$ is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$ where $\mathfrak{m} = (x_1, \dots, x_d)$. The proof follows from the same line as in Theorem 2.5: the key point is that, when k is perfect, $\mathrm{Hom}_S(F_*S, S) \cong F_*S$ still holds and we have a graded version of Proposition 2.2 (with graded injective hull of k in place of the injective hull of k). We leave the details to the interested reader.

Remark 2.9. With the same setup as in Theorem 2.5 or Remark 2.8, it follows from the same argument that R is F-pure if and only if $(I^{[p^e]} : I) \not\subseteq \mathfrak{m}^{[p^e]}$ for some (or equivalently, all) $e > 0$. We leave the details to the interested reader.

Fedder's criterion is *extremely* useful as it allows us to determine if a particular ring is F-pure.

Example 2.10. Let k be a field of prime characteristic $p > 0$.

(1) Let S be $k[[x_1, \dots, x_d]]$ or $k[x_1, \dots, x_d]$ and let $R = S/I$ be a Stanley-Reisner ring (i.e., I is generated by square free monomials). Then R is F-pure. The point is that $x_1x_2 \cdots x_d$ is a multiple of every square free monomial, thus $(x_1 \cdots x_d)^{p-1} \cdot f \in (f^p)$ for any square free monomial f. Hence $(x_1 \cdots x_d)^{p-1} \in (I^{[p]} : I)$ since I is generated by square free monomials, but $(x_1 \cdots x_d)^{p-1} \notin \mathfrak{m}^{[p]}$.

(2) Let R denote either $k[[x, y, z]]/(x^3 + y^3 + z^3)$ or $k[x, y, z]/(x^3 + y^3 + z^3)$. Then $(I^{[p]} : I) = (x^3 + y^3 + z^3)^{p-1}$. If $p \equiv 1 \bmod 3$, then there is a term $(xyz)^{p-1}$ in the monomial expansion of $(x^3 + y^3 + z^3)^{p-1}$ with nonzero coefficient thus R is F-pure. On the other hand, if $p \equiv 2 \bmod 3$, then one checks that $(x^3 + y^3 + z^3)^{p-1} \in \mathfrak{m}^{[p]} = (x^p, y^p, z^p)$ so R is not F-pure.

Exercise 6. Let R be a ring of prime characteristic $p > 0$. Verify that $R \to F^e_*R$ is pure (resp., split) for one $e > 0$, then $R \to F^e_*R$ is pure (resp., split) for all $e > 0$.

Exercise 7. Suppose that R is an F-finite ring of prime characteristic $p > 0$ and F^e_*R admits a free summand. Show that the Frobenius map $R \to F^e_*R$ is split. (Assuming F^e_*R admits a free summand is equivalent to assuming that there exists $F^e_* r \in F^e_* R$ and $\varphi : F^e_* R \to R$ so that $\varphi(F^e_* r) = 1$. We are asking you to show the existence of a map $\psi : F^e_* R \to R$ so that $\psi(F^e_* 1) = 1$.)

Exercise 8. Let k be a field of prime characteristic $p > 0$. Use Fedder's criterion to show that $R = k[[x, y, z]]/(x^2 + y^3 + z^7)$ is not F-pure.

Exercise 9. Prove that if $R \to S$ is pure (resp., split) map of rings of prime characteristic $p > 0$ and S is F-pure (resp., F-split), then R is F-pure (resp., F-split).

Exercise 10. Prove that if $R \to S$ is faithfully flat, then $R \to S$ is pure. Give an example of a faithfully flat ring extension that is not split.

Exercise 11. Show that a map of R-modules $N \to M$ is pure if and only if $N_P \to M_P$ is pure for all $P \in \operatorname{Spec}(R)$. In particular, if R is a ring of prime characteristic $p > 0$, then R is F-pure if and only if R_P is F-pure for all $P \in \operatorname{Spec}(R)$, also prove that if R is F-split, then R_P is F-split for all $P \in \operatorname{Spec}(R)$.[1]

[1]The authors do not know whether R_P is F-split for all $P \in \operatorname{Spec}(R)$ implies R is F-split in general, i.e., outside the case that F-pure and F-split are known to be equivalent.

3. F-regular rings: splitting finite extensions

In this chapter, we introduce and study the arguably most important class of F-singularities: strongly F-regular rings, [HH90, HH94a].

Definition 3.1. An F-finite ring R of prime characteristic $p > 0$ is called *strongly F-regular* if for every $c \in R$ that is not in any minimal prime of R, there exists $e > 0$ such that the map $R \to F^e_* R$ sending 1 to $F^e_* c$ splits as a map of R-modules.

Clearly, strongly F-regular rings are F-split and in particular reduced. For local rings, we can say more.

Lemma 3.2. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring of prime characteristic $p > 0$. Then R is a domain.*

Proof. Since R is reduced, it is enough to show that R has only one minimal prime. Let $P_1, \ldots, P_n$ be the minimal primes of R. Suppose $n \geq 2$, we pick $f_i \in \cap_{j \neq i} P_j - P_i$. Then we have $\sum_{i=1}^n f_i$ is not contained in any minimal prime of R. Thus as R is strongly F-regular, there exists $e > 0$ and an R-linear map $\phi\colon F^e_* R \to R$ such that $\phi(F^e_*(\sum_{i=1}^n f_i)) = 1$ and thus $\sum_{i=1}^n \phi(F^e_* f_i) = 1$. Since $(R, \mathfrak{m}, k)$ is local, at least one of $\phi(F^e_* f_i)$ is a unit. Without loss of generality, we may assume $\phi(F^e_* f_1) = u \in R$ is a unit. But then as $f_1 f_2 = 0$ (since $f_1 f_2$ is contained in all minimal primes of R and R is reduced), we have

$$u f_2 = \phi(f_2 \cdot F^e_* f_1) = \phi(F^e_*(f_2^{p^e} f_1)) = \phi(F^e_* 0) = 0$$

which is a contradiction. $\square$

Like F-purity, strong F-regularity is a local property.

Lemma 3.3. *Let R be an F-finite ring of prime characteristic $p > 0$. Then R is strongly F-regular if and only if R_P is strongly F-regular for every $P \in \operatorname{Spec}(R)$.*

Proof. First suppose R is strongly F-regular. Let $P_1, \ldots, P_n$ be the minimal primes of R. It is enough to show that for any $c \in R$ whose image in R_P is not contained in any minimal prime of R_P, we can find $e > 0$ and an R_P-linear map $F^e_* R_P \to R_P$ sending $F^e_* c$ to 1. We may assume c is not in any minimal prime of R: for suppose c is contained in $P_1, \ldots, P_i$ but not in the other minimal primes of R, then we can pick $c' \in \cap_{j=i+1}^n P_j - \cup_{j=1}^i P_i$ and replace c by $c + c'$ (the image of c' in R_P is 0 since $P_j \nsubseteq P$ for each $j = 1, \ldots, i$). But then since R is strongly F-regular, there exists $e > 0$ such that the map $R \to F^e_* R$ sending 1 to $F^e_* c$ splits as a map of R-modules. So after localizing the splitting we get the desired R_P-linear map $F^e_* R_P \to R_P$ sending $F^e_* c$ to 1.

We next prove the converse. We fix $c \in R$ not in any minimal prime of R. We know that for every $P \in \operatorname{Spec}(R)$, there exists e (which may depend on P) such that $R_P \to F^e_* R_P$ sending 1 to $F^e_* c$ splits. Since R is F-finite, $\operatorname{Hom}_{R_P}(F^e_* R_P, R_P) \cong R_P \otimes_R \operatorname{Hom}_R(F^e_* R, R)$ hence there exists a map $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ sending $F^e_* c$ to $f \notin P$. But then $R_f \to F^e_* R_f$ sending 1 to $F^e_* c$ splits. Now for every $P \in \operatorname{Spec} R$ we can find such f thus $\cup D(f) = \operatorname{Spec}(R)$. Hence there exists $f_1, \dots, f_n$ such that $\cup_{i=1}^n D(f_i) = \operatorname{Spec}(R)$ and for each f_i there exists $e_i > 0$ such that $R_{f_i} \to F^{e_i}_* R_{f_i}$ sending 1 to $F^{e_i}_* c$ splits. It is then easy to check that, for $e_0 = \max\{e_1, \dots, e_n\}$, the map $R \to F^{e_0}_* R$ sending 1 to $F^{e_0}_* c$ splits. □

The following is a consequence of Kunz's theorem, Theorem 1.1.

Theorem 3.4. *An F-finite regular ring of prime characteristic $p > 0$ is strongly F-regular.*

Proof. By Lemma 3.3, we may assume that $(R, \mathfrak{m}, k)$ is an F-finite regular local ring. By Theorem 1.1, $F^e_* R$ is a finite free R-module. For any $0 \neq c \in R$, there exists $e > 0$ such that $F^e_* c \in F^e_* R$ is part of a minimal basis of $F^e_* R$ over R: otherwise $F^e_* c \in \mathfrak{m} \cdot F^e_* R = F^e_*(\mathfrak{m}^{[p^e]})$ for all e which implies that $c \in \cap_e \mathfrak{m}^{[p^e]} = 0$ which is a contradiction. Since $F^e_* c \in F^e_* R$ is part of a minimal basis of $F^e_* R$ over R, the map $R \to F^e_* R$ sending 1 to $F^e_* c$ splits as a map of R-modules. □

We next prove that every strongly F-regular ring R splits out of every finite extension of R, a crucial property of strongly F-regular rings.

Theorem 3.5. *Let R be an F-finite and strongly F-regular ring of prime characteristic $p > 0$. Then $R \to S$ splits for any module-finite extension S of R.*

Proof. Since S is module-finite over R, it is enough to show $R_P \to (R - P)^{-1} S$ is split for every prime $P \in \operatorname{Spec}(R)$. Thus by Lemma 3.3, we may assume $(R, \mathfrak{m}, k)$ is a strongly F-regular local ring and hence a domain by Lemma 3.2. By killing a minimal prime of S, we may further assume that S is also a domain. Now S is a torsion-free R-module, thus there exists an R-linear map $\theta\colon S \to R$ such that $\theta(1) = c \neq 0$. Since R is strongly F-regular, we can find e such that $R \to F^e_* R$ sending 1 to $F^e_* c$ splits, call the splitting ϕ. Now we consider the following commutative diagram with natural maps:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ F^e_* R \arrow[r] & F^e_* S. \end{tikzcd}$$

We know that $F^e_*\theta\colon\ F^e_*S \to F^e_*R$ sends F^e_*1 to F^e_*c, thus $\phi \circ F^e_*\theta$ sends $F^e_*1 \in F^e_*S$ to $1 \in R$. Therefore, $R \to F^e_*S$ splits, this clearly implies $R \to S$ splits by the commutative diagram. ☐

Combining the results we have so far, we obtain:

Corollary 3.6. *If R is a regular ring of characteristic $p > 0$, then $R \to S$ splits for any module-finite extension S of R.*

Proof. Since S is module-finite over R, it is enough to show $R_P \to (R-P)^{-1}S$ is split for every prime $P \in \operatorname{Spec}(R)$ thus we may assume R is a regular local ring. We then consider the faithfully flat extensions $R \to \widehat{R} \cong k[[x_1, \ldots, x_d]] \to \widetilde{R} \cong \overline{k}[[x_1, \ldots, x_d]]$. Again since S is module-finite over R, it is enough to show $\widetilde{R} \to \widetilde{R} \otimes_R S$ is split. Now $\widetilde{R}$ is F-finite and regular thus strongly F-regular by Theorem 3.4. So $\widetilde{R} \to \widetilde{R} \otimes_R S$ splits by Theorem 3.5. ☐

Remark 3.7. Corollary 3.6 holds without assuming R has characteristic $p > 0$, see [And18]. We will not discuss this result beyond the characteristic $p > 0$ setting.

Another consequence of Theorem 3.5 is the following:

Corollary 3.8. *Let R be an F-finite and strongly F-regular ring of prime characteristic $p > 0$. Then R is normal. In particular, one-dimensional strongly F-regular rings are regular.*

Proof. Suppose R is not normal, then there exists $\frac{a}{b}$ integral over R (with b a nonzerodivisor in R) but $\frac{a}{b} \notin R$. Let $R' = R[\frac{a}{b}]$. Since $R \to R'$ is a finite extension, by Theorem 3.5, there exists an R-linear map $\theta\colon\ R' \to R$ such that $\theta(1) = 1$. Thus

$$b \cdot \theta(\frac{a}{b}) = \theta(a) = a.$$

But then $\frac{a}{b} = \theta(\frac{a}{b}) \in R$, which is a contradiction. ☐

Another important property of strongly F-regular rings is the following:

Theorem 3.9. *Let R and S be F-finite rings of prime characteristic $p > 0$. If R is a direct summand of S and S is strongly F-regular (e.g., S is regular), then R is strongly F-regular.*

Proof. By Lemma 3.3, it is enough to show R_P is strongly F-regular for each $P \in \operatorname{Spec}(R)$. Now R_P is a direct summand of $(R-P)^{-1}S$ and the latter is strongly F-regular by Lemma 3.3 again. Thus we may assume $(R, \mathfrak{m}, k)$ is local. Since S is strongly F-regular, it is normal by Corollary 3.8 and hence a product of normal domains $S \cong S_1 \times S_2 \times \cdots \times S_n = \prod S_i e_i$ where e_i is the i-th idempotent corresponding to S_i (e.g., $e_1 = (1, 0, \ldots, 0)$). Now a splitting

ϕ: $S \to R$ sends $1 = (1, \ldots, 1) = \sum e_i$ to 1. Since $(R, \mathfrak{m}, k)$ is local, there exists i such that $\phi(e_i)$ is a unit in R. But then the induced map $\widetilde{\phi}$: $S_i \to R$ defined via $\widetilde{\phi}(s_i) := \phi(s_i e_i)$ for all $s_i \in S_i$ is an R-linear surjection $S_i \twoheadrightarrow R$. Therefore $R \to S_i$ is split (i.e., R is a direct summand of S_i). Note that, as S_i can be viewed as a localization of S, S_i is still strongly F-regular by Lemma 3.3.

Thus replacing S by S_i, we may assume that both R and S are domains. Let $0 \neq c \in R$ be given. Since S is strongly F-regular, there exists $e > 0$ and an S-linear map ϕ: $F^e_* S \to S$ such that $\phi(F^e_* c) = 1$. Let θ: $S \to R$ be a splitting. Then $\theta \circ \phi$: $F^e_* S \to R$ is an R-linear map sending $F^e_* c$ to 1. Restricting this map to $F^e_* R$ then yields an R-linear map $F^e_* R \to R$ sending $F^e_* c$ to 1. $\square$

Theorem 3.9 allows us to write many examples of strongly F-regular rings:

Example 3.10. Let k be an F-finite field of prime characteristic $p > 0$.

(1) Let $R = k[x, y, z]/(xy - z^2)$. Then $R \cong k[s^2, st, t^2]$ is a direct summand of $S = k[s, t]$. Hence R is strongly F-regular. More generally, Veronese subrings of polynomial rings (over F-finite fields) are strongly F-regular.

(2) Let $R = k[x, y, u, v]/(xy - uv)$. Then $R \cong k[a, b]\#k[c, d] \cong k[ac, ad, bc, bd]$ is a direct summand of $S = k[a, b, c, d]$. Hence R is strongly F-regular. More generally, Segre product of polynomial rings (over F-finite fields) are strongly F-regular.

Finally, we point out that to check strong F-regularity, one actually only needs to check the splitting condition in the definition for one single c. This will be very useful in later chapters.

Theorem 3.11. *Let R be an F-finite ring of prime characteristic $p > 0$. Suppose there exists c not in any minimal prime of R such that R_c is strongly F-regular (e.g., R_c is regular). Then R is strongly F-regular if and only if there exists $e > 0$ such that the map $R \to F^e_* R$ sending 1 to $F^e_* c$ splits as a map of R-modules.*

Proof. Given any $d \in R$ that is not in any minimal prime of R, the image of d is not in any minimal prime of R_c. Therefore, since R_c is strongly F-regular, there exists $e_0 > 0$ and a map $\phi \in \operatorname{Hom}_{R_c}(F^{e_0}_* R_c, R_c)$ such that $\phi(F^{e_0}_* d) = 1$. Since R is F-finite, we have $\operatorname{Hom}_{R_c}(F^{e_0}_* R_c, R_c) \cong R_c \otimes_R \operatorname{Hom}_R(F^{e_0}_* R, R)$ and thus $\phi = \frac{\varphi}{c^n}$ for some $n > 0$ and some $\varphi \in \operatorname{Hom}_R(F^{e_0}_* R, R)$. It follows that $\varphi(F^{e_0}_* d) = c^n$. Next we pick $e_1 > 0$ such that $n < p^{e_1 - e}$, so (the image of) $F^e_* c$ in $F^{e_1}_* R$ is a multiple of $F^{e_1}_* c^n$. Since $R \to F^e_* R$ sending 1 to $F^e_* c$ splits, it follows that $R \to F^{e_1}_* R$ sending 1 to $F^{e_1}_* c^n$ splits (since R is F-pure). We pick such a splitting θ and consider the map $\theta \circ (F^{e_1}_* \varphi)$: $F^{e_1 + e_0}_* R \to R$. It is straighforward to check that this map sends $F^{e_1 + e_0}_* d$ to 1. $\square$

Corollary 3.12. *An F-finite local ring $(R, \mathfrak{m}, k)$ of prime characteristic $p > 0$ is strongly F-regular if and only if $\widehat{R}$ is strongly F-regular.*

Proof. We may assume R is a domain by Lemma 3.2. Since R is excellent, there exists $0 \neq c \in R$ such that R_c is regular and then $\widehat{R}_c$ is also regular. Consider the following commutative diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\cong"] & E \otimes_R F^e_* R \arrow[d] \\ E \arrow[r] & E \otimes_{\widehat{R}} F^e_* \widehat{R} \end{tikzcd}$$

where $E = E_R(k) = E_{\widehat{R}}(k)$ and the horizontal maps are induced by $R \to F^e_* R$ (resp., $\widehat{R} \to F^e_* \widehat{R}$) sending 1 to $F^e_* c$. It is easy to see that the first row is injective if and only if the second row is injective. Since R and $\widehat{R}$ are F-finite, by Corollary 2.4 and Proposition 2.2, $R \to F^e_* R$ sending $1 \to F^e_* c$ splits if and only if $\widehat{R} \to F^e_* \widehat{R}$ sending $1 \to F^e_* c$ splits. By Theorem 3.11, R is strongly F-regular if and only if $\widehat{R}$ is strongly F-regular. $\square$

Exercise 12. Let R be an F-finite ring of prime characteristic $p > 0$. Suppose that M is a finitely generated module, $m \in M$, and that there exists $e_0 \in \mathbb{N}$ and $\varphi \in \operatorname{Hom}_R(F^{e_0}_* M, R)$ such that $\varphi(F^{e_0}_* m) = 1$. Show that R is F-pure and that for all $e \geq e_0$ there exists a $\psi \in \operatorname{Hom}_R(F^e_* M, R)$ such that $\psi(F^e_* m) = 1$.

Exercise 13. Let $R \to S$ be a faithfully flat extension of F-finite rings of prime characteristic $p > 0$. Prove that if S is strongly F-regular, then R is strongly F-regular.

Exercise 14. Let R be an F-finite and strongly F-regular ring of prime characteristic $p > 0$. Show that for each nonzero element $g \in R$ that there exists an $e \in \mathbb{N}$ so that $R \to F^e_* R \subseteq F^e_* R[1/g] = F^e_* R(\operatorname{Div}(g))$ splits. (Hint: Show that $R \to F^e_* R \subseteq F^e_* R[1/g]$ is isomorphic to $R \xrightarrow{\cdot F^e_* g} F^e_* R$.)

Exercise 15. Let R be an F-finite and strongly F-regular ring of prime characteristic $p > 0$. Prove that for all effective divisors D (see Appendix A), there exists e_0 (depending on D) such that for all $e \geq e_0$, the composition $R \to F^e_* R \to F^e_* R(D)$ splits. (Hint: Show that there exists a nonzero element $g \in R$ such that $D \leq \operatorname{Div}(g)$ and use Exercise 14.)

Exercise 16 (Glassbrenner [Gla96]). Let $(S, \mathfrak{m}, k)$ be an F-finite regular local ring of prime characteristic $p > 0$ (resp., a polynomial ring over an F-finite field of prime characteristic $p > 0$) and let $I \subseteq S$ be an ideal (resp., a homogeneous ideal). Then the following are equivalent for $R = S/I$:

(1) R is strongly F-regular.
(2) For every $c \in S$ not in any minimal prime of I, there exists $e > 0$ such that $c(I^{[p^e]} : I) \not\subseteq \mathfrak{m}^{[p^e]}$.
(3) For some $c \in S$ not in any minimal prime of I such that R_c is strongly F-regular, there exists $e > 0$ such that $c(I^{[p^e]} : I) \not\subseteq \mathfrak{m}^{[p^e]}$.

(Hint: Mimic the strategy of the proof of Theorem 2.5.)

Exercise 17. Let k be an F-finite field of prime characteristic $p > 0$ and $R = k[x_1, \dots, x_d]/(x_1^n + \cdots + x_d^n)$. Use Exercise 16 to show that R is strongly F-regular if $n < d$ and $p \gg 0$, and R is not strongly F-regular if $n \geq d$.

Exercise 18. Let R be an $\mathbb{N}$-graded ring over a field k of prime characteristic $p > 0$ with homogenous maximal ideal $\mathfrak{m}$. Use Theorem 2.5 and Exercise 16 to prove that R is F-pure (resp., F-finite and strongly F-regular) if and only if so is $R_{\mathfrak{m}}$.

A *very big* open question in F-singularity theory, and tight closure theory, is whether the converse of Theorem 3.5 holds.

Open Problem 1. *Let R be an F-finite domain of prime characteristic $p > 0$. If $R \to S$ splits for any module-finite extension S of R, then is R strongly F-regular?*

This has an affirmative answer in the following cases:

(1) If R is Gorenstein by [HH94c].
(2) If R is $\mathbb{Q}$-Gorenstein by [Sin99a].
(3) If the anti-canonical cover of R is a Noetherian ring by an unpublished result of Singh, see also [CEMS18] for more general results. (Recall that the condition means, with K_X a choice of the canonical divisor of $X = \operatorname{Spec}(R)$, $S := \oplus_{n \geq 0} R(-nK_X)$ is a finitely generated R-algebra).

We refer the readers to Appendix A for basics on divisors and $\mathbb{Q}$-Gorenstein rings. Here we just point out that there are (obvious) implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) since every Gorenstein ring is $\mathbb{Q}$-Gorenstein and every $\mathbb{Q}$-Gorenstein ring has Noetherian anti-canonical cover.

Discussion 3.13. In Hochster–Huneke's foundational work [HH90, HH94a], there are three notions of F-regularity: weakly F-regular, F-regular, and strongly F-regular. The former two are defined using tight closure. Conjecturally all these notions are equivalent (for F-finite rings), but to this date this is still not clear. It turns out that even weakly F-regular rings split from all their module-finite extensions [HH90]. Thus an affirmative answer to Open Problem 1 will imply that all these notions are equivalent. For related results on the equivalence of different notions of F-regularity, see [HH94a, Wil95, Mac96, LS99, LS01, AP19]. On the other hand, it becomes apparent in recent years that strong F-regularity is the most useful concept and has most applications to algebraic geometry.

Discussion 3.14. We can define strongly F-regular rings beyond the F-finite setting, there are actually several ways to extend the definition, for example see [HH94a] or [DS16b]. For technical reasons, and also because it will be quite technical to define F-signature without F-finite assumptions, we decide to keep the F-finite assumption in the definition of strong F-regularity in this text.

4. F-rational and F-injective rings

In this chapter we discuss F-rational and F-injective rings [Fed83, HH94a, HH94c]. We begin by collecting some basic facts about Frobenius structure on local cohomology modules. Let $I = (f_1, \ldots, f_n)$ be an ideal of R, then we have the Čech complex:

$$C^\bullet(f_1, \ldots, f_n; R) := 0 \to R \to \oplus_i R_{f_i} \to \cdots \to R_{f_1 f_2 \cdots f_n} \to 0.$$

The i-th local cohomology module $H^i_I(R)$ is the i-th cohomology of $C^\bullet(f_1, \ldots, f_n; R)$. The local cohomology modules $H^i_I(R)$ only depends on the radical of I. Since the Frobenius endomorphism on R naturally induces the Frobenius endomorphism on all localizations of R, it induces a natural Frobenius action on $C^\bullet(f_1, \ldots, f_n; R)$, and hence it induces a natural Frobenius action on each $H^i_I(R)$.

We know from the definition that a ring homomorphism $R \to S$ induces a map $H^i_I(R) \to H^i_{IS}(S)$. The natural Frobenius action on $H^i_I(R)$ discussed above can be alternatively described as $H^i_I(R) \to H^i_{I \cdot F_* R}(F_* R) = H^i_{F_* I^{[p]}}(F_* R)$ and then identify $H^i_{F_* I^{[p]}}(F_* R)$ with $H^i_I(R)$, where the last identification is induced by $F_* R \cong R$ as rings (note that $H^i_{I^{[p]}}(R) = H^i_I(R)$). We will be mostly interested in the case that $(R, \mathfrak{m}, k)$ is local and $I = \mathfrak{m}$. In this case, we can compute $H^i_\mathfrak{m}(R)$ using the Čech complex on a system of parameters $x_1, \ldots, x_d$ of R. For example, the top local cohomology module $H^d_\mathfrak{m}(R)$ is isomorphic to

$$\frac{R_{x_1 \cdots x_d}}{\sum_i \operatorname{Im}(R_{x_1 \cdots \widehat{x_i} \cdots x_d})},$$

and with this description, the natural Frobenius action on $H^d_\mathfrak{m}(R)$ is given by

$$\frac{r}{x_1^n \cdots x_d^n} \to \frac{r^p}{x_1^{np} \cdots x_d^{np}}.$$

Definition 4.1. A local ring $(R, \mathfrak{m}, k)$ of dimension d and of prime characteristic $p > 0$ is called *F-rational* if R is Cohen-Macaulay and for every $c \in R$ that is not in any minimal prime of R, there exists $e > 0$ such that the composition

$$H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(F^e_* R) \xrightarrow{\cdot F^e_* c} H^d_\mathfrak{m}(F^e_* R)$$

is injective. Equivalently, using $F^e\colon H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(R)$ to denote the e-th Frobenius action, this is saying that $c \cdot F^e(-)$ is injective on $H^d_\mathfrak{m}(R)$. An arbitrary ring R of prime characteristic $p > 0$ is called F-rational if and only if $R_\mathfrak{m}$ is F-rational for all maximal ideals $\mathfrak{m} \subseteq R$.

Remark 4.2. Our definition of F-rational rings is not the original one as in [HH94a, HH94c], but it is an equivalent definition for all rings that are homomorphic images of Cohen-Macaulay rings. This is a very mild assumption: for example, all excellent local rings satisfy this condition [Kaw02]. In fact, in Hochster's more recent lecture notes on tight closure [Hoc07], being a homomorphic image of a Cohen-Macaulay ring is built into the definition of F-rationality, thus almost nothing is lost.

Example 4.3. Suppose $(R, \mathfrak{m}, k)$ is a regular local ring of dimension d and of prime characteristic $p > 0$. We will show that R is F-rational. Note that a socle representative of $H^d_\mathfrak{m}(R)$ is $\eta = \frac{1}{x_1 \cdots x_d}$ where $x_1, \dots, x_d$ is a regular system of parameters of R. If $c \neq 0$ such that $c \cdot F^e(\eta) = 0$ for all $e > 0$, then $\frac{c}{x_1^{p^e} \cdots x_d^{p^e}} = 0$ in $H^d_\mathfrak{m}(R)$ for all $e > 0$. But then $c \in \bigcap_e (x_1^{p^e}, \dots, x_d^{p^e}) = 0$, a contradiction.

Proposition 4.4. *Suppose R is an F-rational ring of prime characteristic $p > 0$, then R is normal. In particular, one-dimensional F-rational rings are regular.*

Proof. We may assume $(R, \mathfrak{m}, k)$ is local. In order to show R is normal, it is enough to prove that every principal ideal of height one is integrally closed by [HS06, Proposition 1.5.2] (if $\dim(R) = 0$, then the condition implies R is a field so R is trivially normal). Suppose $y \in \overline{(x)}$ where x is not in any minimal prime of R, then there exists $m > 0$ such that $(y, x)^n = (y, x)^m (x)^{n-m}$ for all $n > m$. Thus $x^m y^n \in (x)^n$ for every n. We can extend x to a full system of parameters $x, x_2^t, \dots, x_d^t$ of R. Then the Čech class $\eta = \frac{y}{x x_2^t \cdots x_d^t}$ satisfies

$$x^m \cdot F^e(\eta) = x^m \cdot \frac{y^{p^e}}{x^{p^e} x_2^{tp^e} \cdots x_d^{tp^e}} = 0$$

for all $e > 0$ since $x^m y^{p^e} \in (x^{p^e})$ by construction. So by the definition of F-rationality, $\eta = 0$ in $H^d_\mathfrak{m}(R)$. But since R is Cohen-Macaulay, we know that $y \in (x, x_2^t, \dots, x_d^t)$. As this is true for every $t > 0$, $y \in \bigcap_t (x, x_2^t, \dots, x_d^t) = (x)$. Thus (x) is integrally closed. □

An important result we want to prove next is that strongly F-regular rings are F-rational. We need a well-known lemma.

Lemma 4.5. *Let $(R, \mathfrak{m}, k)$ be a complete and equidimensional local ring of dimension d. Suppose R_P is Cohen-Macaulay for all $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$. Then $H^i_\mathfrak{m}(R)$ has finite length for all $i < d$.*

Proof. By Cohen's structure theorem, we can write $R = S/I$ where S is a complete regular local ring. By local duality, $H^i_\mathfrak{m}(R)^\vee \cong \operatorname{Ext}_S^{n-i}(R, S)$ where $n = \dim(S)$. It follows that

$$\operatorname{Ext}_S^{n-i}(R, S)_P \cong \operatorname{Ext}_{S_P}^{n-i}(R_P, S_P) = \operatorname{Ext}_{S_P}^{\dim(S_P) - (i - \dim(R/P))}(R_P, S_P),$$

where we abuse notation and also use P to denote the pre-image of P in S. Now by local duality over S_P,

$$\operatorname{Ext}_{S_P}^{\dim(S_P)-(i-\dim(R/P))}(R_P, S_P)^\vee \cong H_{PR_P}^{i-\dim(R/P)}(R_P).$$

Since R is equidimensional, $\dim(R/P) + \dim(R_P) = d$ hence if $i < d$ then $i - \dim(R/P) < \dim(R_P)$. Thus if $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ and $i < d$, then $H_{PR_P}^{i-\dim(R/P)}(R_P) = 0$ since R_P is Cohen-Macaulay, which gives $\operatorname{Ext}_S^{n-i}(R, S)_P = 0$. Thus $\operatorname{Ext}_S^{n-i}(R, S)$ is supported only at $\{\mathfrak{m}\}$ when $i < d$. By local duality, $H_\mathfrak{m}^i(R)$ has finite length whenever $i < d$. $\square$

We can now prove the following result.

Theorem 4.6. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring of prime characteristic $p > 0$. Then R is F-rational (and hence Cohen-Macaulay).*

Proof. Note that $H_\mathfrak{m}^d(R) = H_\mathfrak{m}^d(\widehat{R})$ and if $c \in R$ is not in any minimal prime of R, then c is not in any minimal prime of $\widehat{R}$. Thus it is clear that $\widehat{R}$ is F-rational implies R is F-rational. Therefore we may assume R is a complete local domain by Corollary 3.12 and Lemma 3.2. Since strong F-regularity is preserved under localization by Lemma 3.3, by induction on $\dim(R)$ we may further assume R_P is Cohen-Macaulay for all $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$. Thus by Lemma 4.5, $H_\mathfrak{m}^i(R)$ has finite length whenever $i < d = \dim(R)$.

Let $0 \neq c \in \mathfrak{m}$. Since $H_\mathfrak{m}^i(R)$ has finite length for $i < d$, there exists n such that $c^n H_\mathfrak{m}^i(R) = 0$. Replacing c with c^n we may assume $cH_\mathfrak{m}^i(R) = 0$. Thus $(F_*^e c) \cdot H_\mathfrak{m}^i(F_*^e R) = 0$. Since R is strongly F-regular, there exists $e > 0$ and an R-linear map $F_*^e R \to R$ such that the composition of the following maps is the identity map on R:

$$R \to F_*^e R \xrightarrow{\cdot F_*^e c} F_*^e R \to R.$$

Applying the i-th local cohomology functor $H_\mathfrak{m}^i(-)$ to the above composition of maps we see that the identity map on $H_\mathfrak{m}^i(R)$ factors through the zero map on $H_\mathfrak{m}^i(F_*^e R)$ and thus $H_\mathfrak{m}^i(R) = 0$ whenever $i < d$. This proves that R is Cohen-Macaulay. Finally, applying the d-th local cohomology functor $H_\mathfrak{m}^d(-)$ to the same composition of maps, we see that the identity map on $H_\mathfrak{m}^d(R)$ factors through

$$H_\mathfrak{m}^d(R) \to H_\mathfrak{m}^d(F_*^e R) \xrightarrow{\cdot F_*^e c} H_\mathfrak{m}^d(F_*^e R).$$

In particular, the above map is injective and thus R is F-rational. $\square$

As a consequence of the results we proved so far, we can prove the following.

Corollary 4.7. *Let $R \to S$ be a pure map of rings of prime characteristic $p > 0$. If S is regular, then R is Cohen-Macaulay.*

Proof. We first observe that if R and S are both F-finite and the map $R \to S$ is split (this includes most cases of interest). Then the conclusion follows by combining Theorem 3.4, Theorem 3.9, and Theorem 4.6.

But with a careful examination of the methods we used in proving these results, we can prove the general case of the corollary. We now give the details. First of all we may assume $(R, \mathfrak{m}, k)$ is a local ring. Since $R \to S$ is pure, $E \to E \otimes_R S$ is injective so $u \otimes 1 \neq 0$ in $E \otimes_R S$, where $E = E_R(k)$ and u is a socle representative of E. But then $u \otimes 1 \neq 0$ in $E \otimes_R S_Q$ for some $Q \in \operatorname{Spec}(S)$, and thus $E \to E \otimes_R S_Q$ is injective. This implies $R \to S_Q$ is pure by Proposition 2.2. So we may assume S is also a local ring. We may then replace R by $\widehat{R}$ and S by $\widehat{S} \cong k[[x_1, \ldots, x_n]]$ and further replace $\widehat{S}$ by $\widetilde{S} := \overline{k}[[x_1, \ldots, x_n]]$. Therefore we may assume $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ is pure where S is a complete and F-finite regular local ring and $(R, \mathfrak{m}, k)$ is a complete local domain. Furthemore, by induction on $\dim(R)$ we may assume R_P is Cohen-Macaulay for all $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$. By Lemma 4.5, $H^i_\mathfrak{m}(R)$ has finite length for all $i < \dim(R)$.

For each $i < \dim(R)$, let $0 \neq c \in R$ that annihilates $H^i_\mathfrak{m}(R)$. By Theorem 3.4, S is strongly F-regular so there exists $e > 0$ such that $S \to F^e_* S$ sending 1 to $F^e_* c$ splits. We consider the following commutative diagram:

$$\begin{tikzcd}
H^i_\mathfrak{m}(R) \arrow[r] \arrow[d] & H^i_\mathfrak{m}(F^e_* R) \arrow[r, "\cdot F^e_* c"] \arrow[d] & H^i_\mathfrak{m}(F^e_* R) \arrow[d] \\
H^i_\mathfrak{m}(S) \arrow[r] & H^i_\mathfrak{m}(F^e_* S) \arrow[r, "\cdot F^e_* c"] & H^i_\mathfrak{m}(F^e_* S)
\end{tikzcd}$$

From the bottom row, we see that the map from top left $H^i_\mathfrak{m}(R)$ to bottom right $H^i_\mathfrak{m}(F^e_* S)$ is injective, while from the first row, we see that the same map is the zero map from $H^i_\mathfrak{m}(R)$ to $H^i_\mathfrak{m}(F^e_* S)$ as c annihilates $H^i_\mathfrak{m}(R)$. This shows that $H^i_\mathfrak{m}(R) = 0$ and hence R is Cohen-Macaulay. $\square$

Remark 4.8. Corollary 4.7 holds without assuming the rings have characteristic $p > 0$, see [HH95] and [HM18]. We will not discuss the result beyond the characteristic $p > 0$ setting.

The converse of Theorem 4.6 holds if R is Gorenstein.

Proposition 4.9. *Suppose R is an F-finite ring of prime characteristic $p > 0$ which is Gorenstein and F-rational, then R is strongly F-regular.*

Proof. By Lemma 3.3, we may assume $(R, \mathfrak{m}, k)$ is local. It is enough to show that for any $c \in R$ not in any minimal prime of R, there exists $e > 0$ such that the map $E \to E \otimes_R F^e_* R$ induced by sending 1 to $F^e_* c$ is injective (see Proposition 2.2 and Corollary 2.4), where

$E = E_R(k)$ denotes the injective hull of the residue field as usual. Since R is Gorenstein, $E \cong H^d_{\mathfrak{m}}(R)$. Thus the map $E \to E \otimes_R F^e_* R$ can be identified with the map $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_* R) \xrightarrow{\cdot F^e_* c} H^d_{\mathfrak{m}}(F^e_* R)$, which is injective by the F-rationality of R. $\square$

We next give an alternative but important characterization of F-rationality (up to completion), see [Smi97] for more details. We need a definition.

Definition 4.10. Let R be a ring of prime characteristic $p > 0$ and let M be an R-module with a Frobenius action F (i.e., $F(rm) = r^p F(m)$ for all $r \in R$ and $m \in M$). An R-submodule $N \subseteq M$ is called *F-stable* if $F(N) \subseteq N$.

Proposition 4.11. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p > 0$. Then the following are equivalent:*

(1) $\widehat{R}$ is F-rational.
(2) R is Cohen-Macaulay and the only F-stable submodules of $H^d_{\mathfrak{m}}(R)$ are 0 and $H^d_{\mathfrak{m}}(R)$, i.e., $H^d_{\mathfrak{m}}(R)$ is a simple object in the category of R-modules with a Frobenius action.

Proof. Since $H^d_{\mathfrak{m}}(R)$ is Artinian, any R-submodule of $H^d_{\mathfrak{m}}(R)$ carries a canonical $\widehat{R}$-module structure, and the Frobenius structure on $H^d_{\mathfrak{m}}(R)$ is unaffected by considering it as a module over $\widehat{R}$. Thus all conditions in (2) are unaffected by replacing R by $\widehat{R}$ and so we may assume $(R, \mathfrak{m}, k)$ is complete.

Suppose (1) holds. By Proposition 4.4, we may assume $(R, \mathfrak{m}, k)$ is a complete normal local domain. Let $N \subsetneqq H^d_{\mathfrak{m}}(R)$ be a proper F-stable submodule. By Matlis duality, $H^d_{\mathfrak{m}}(R)^\vee \cong \omega_R \twoheadrightarrow N^\vee$ is a proper quotient. Since ω_R is a rank one torsion-free R-module, it follows that $N^\vee$ (and hence N) is annihilated by some $c \neq 0$ since $N^\vee \neq \omega_R$. If $N \neq 0$, then any $0 \neq \eta \in N$ satisfies $c \cdot F^e(\eta) = 0$ for all e, which contradicts that $c \cdot F^e(-)$ is injective for some e.

Suppose (2) holds. First notice that the Frobenius is injective on $H^d_{\mathfrak{m}}(R)$: otherwise the kernel is a nonzero and proper submodule (see Exercise 23) of $H^d_{\mathfrak{m}}(R)$ which contradicts (2). Now for any $c \in R$ not in any minimal prime of R, it is easy to check that

$$\{\eta \in H^d_{\mathfrak{m}}(R) \mid c \cdot F^e(\eta) = 0 \text{ for all } e \geq 0\}$$

is an F-stable submodule of $H^d_{\mathfrak{m}}(R)$. Since it is annihilated by c, it cannot be $H^d_{\mathfrak{m}}(R)$ so it must be 0 by the conditions of (2). But this is saying that for any $\eta \in H^d_{\mathfrak{m}}(R)$, there exists $e > 0$ such that $c \cdot F^e(\eta) \neq 0$. Let $N_e := \{\eta \in H^d_{\mathfrak{m}}(R) \mid c \cdot F^e(\eta) = 0\}$. Since the Frobenius is injective on $H^d_{\mathfrak{m}}(R)$, it is easy to check that $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$. Since $H^d_{\mathfrak{m}}(R)$ is Artinian and $\cap_e N_e = 0$, there exists e such that $N_e = 0$, which is precisely saying that $c \cdot F^e(-)$ is injective on $H^d_{\mathfrak{m}}(R)$. $\square$

A natural question one might ask at this point is that whether $(R, \mathfrak{m}, k)$ is F-rational implies $\widehat{R}$ is F-rational (it is easy to see that if $\widehat{R}$ is F-rational then R is F-rational). It turns out that this is not always true, but it holds if R is excellent. We will come back to this question in Chapter 6. In the proof of Proposition 4.11, we crucially used the fact that the Frobenius action is injective on $H^d_{\mathfrak{m}}(R)$. We now formally introduce F-injective singularities.

Definition 4.12. A local ring $(R, \mathfrak{m}, k)$ of prime characteristic $p > 0$ is called *F-injective* if the natural Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all i. An arbitrary ring R of prime characteristic $p > 0$ is called F-injective if $R_{\mathfrak{m}}$ is F-injective for all maximal ideals $\mathfrak{m} \subseteq R$.

It is straightforward from the definition that if R is F-rational, then R is F-injective. Since the Frobenius structure on $H^i_{\mathfrak{m}}(R)$ is the same when we consider it as a module over $\widehat{R}$, we also know that a local ring $(R, \mathfrak{m}, k)$ is F-injective if and only if $\widehat{R}$ is F-injective. We next show that F-injectivity and F-rationality are preserved under localization. For F-injectivity, the strategy is taken from [DM19], where the result is proved in its most general form.

Theorem 4.13. *Let R be a ring of prime characteristic $p > 0$. If R is F-injective then R_P is F-injective for all $P \in \operatorname{Spec}(R)$.*

Proof. We may assume $(R, \mathfrak{m}, k)$ is local with $\dim(R) = d$. First we claim that we may assume R is complete. Let $P \in \operatorname{Spec}(R)$, pick a minimal prime Q of $P\widehat{R}$, then $R_P \to \widehat{R}_Q$ is faithfully flat with $\dim(R_P) = \dim(\widehat{R}_Q)$. Thus $H^i_Q(\widehat{R}_Q) \cong H^i_P(R_P) \otimes_{R_P} \widehat{R}_Q$ for all i and it is easy to see that the isomorphism is compatible with the Frobenius actions. Hence if we can show $\widehat{R}_Q$ is F-injective, then R_P is F-injective.

Now we assume $(R, \mathfrak{m}, k)$ is complete, by Cohen's structure theorem we can write $R = S/I$ where $(S, \mathfrak{n}, k)$ is a complete regular local ring of dimension n. We can write $F_*R = \varinjlim_j R_j$ such that each R_j is module-finite over R, thus $F_*(R_P) = \varinjlim_j (R_j)_P$. We have the following (abusing notations a bit, we still use P to denote the corresponding prime ideal in S):

$$\begin{aligned}
R \text{ is } F\text{-injective} \;&\Rightarrow\; H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(F_*R) \text{ is injective for all } i \\
&\Rightarrow\; H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R_j) \text{ is injective for all } i,\ j \\
&\Rightarrow\; \operatorname{Ext}^{n-i}_S(R_j, S) \to \operatorname{Ext}^{n-i}_S(R, S) \text{ is surjective for all } i,\ j \\
&\Rightarrow\; \operatorname{Ext}^{n-i}_{S_P}((R_j)_P, S_P) \to \operatorname{Ext}^{n-i}_{S_P}(R_P, S_P) \text{ is surjective for all } i,\ j \\
&\Rightarrow\; H^{\dim(S_P)-n+i}_P(R_P) \to H^{\dim(S_P)-n+i}_P((R_j)_P) \text{ is injective for all } i,\ j \\
&\Rightarrow\; H^{\dim(S_P)-n+i}_P(R_P) \to H^{\dim(S_P)-n+i}_P(F_*(R_P)) \text{ is injective for all } i \\
&\Rightarrow\; R_P \text{ is } F\text{-injective.}
\end{aligned}$$

where the third and fifth implications are due to local duality over S and S_P respectively. $\square$

Finally, we show that F-rationality localizes.

Theorem 4.14. *Let R be a ring of prime characteristic $p > 0$. If R is F-rational then R_P is F-rational for all $P \in \operatorname{Spec}(R)$.*

Proof. We may assume $(R, \mathfrak{m}, k)$ is local with $\dim(R) = d$. By Proposition 4.4, R is a Cohen-Macaulay normal domain, and hence so is R_P. Suppose P has height h, it is then enough to show that for any $0 \neq c \in R$, there exists $e > 0$ such that $cF^e(-)$ is injective on $H^h_P(R_P)$. Suppose on the contrary, there exists $0 \neq c \in R$ such that $cF^e(-)$ is not injective for all $e > 0$. Then for all $e > 0$, we have

$$0 \neq K_e := \operatorname{Ker}(H^h_P(R_P) \xrightarrow{cF^e(-)} H^h_P(R_P)).$$

We claim that $K_{e+1} \subseteq K_e$: if $cF^{e+1}(\eta) = 0$, then $F(cF^e(\eta)) = c^p F^{e+1}(\eta) = 0$, but we know that R_P is F-injective by Theorem 4.13, thus $cF^e(\eta) = 0$. Therefore we have a descending chain of R_P-modules:

$$K_1 \supseteq \cdots \supseteq K_e \supseteq K_{e+1} \supseteq \cdots .$$

Since $H^h_P(R_P)$ is an Artinian R_P-module, this chain stabilizes and so there exists $0 \neq \eta \in \cap_e K_e$. Next we pick a system of parameters $x_1, \ldots, x_h, x_{h+1}, \ldots, x_d$ of R such that the image of $x_1, \ldots, x_h$ is a system of parameters on R_P. Note that

$$H^h_P(R_P) = \varinjlim_e \frac{R_P}{(x_1^{p^e}, \ldots, x_h^{p^e})R_P},$$

where the connection maps are multiplication by $(x_1 \cdots x_h)^{p^{e+1}-p^e}$. By replacing $x_1, \ldots, x_h$ by their powers if necessary, we may assume that $\eta \neq 0$ is the image of $\overline{y} \in R_P/(x_1, \ldots, x_h)R_P$ in $H^h_P(R_P)$. Multiplying η and y by elements in $R - P$ (which are units in R_P), we may assume that $y \in R$. We consider the following commutative diagram

$$\begin{tikzcd}
H^h_P(R_P) \arrow[r, "cF^e(-)"] & H^h_P(R_P) \\
\frac{R_P}{(x_1,\ldots,x_h)R_P} \arrow[u, hook] \arrow[r, "\overline{x} \mapsto c\overline{x}^{p^e}"] & \frac{R_P}{(x_1^{p^e},\ldots,x_h^{p^e})R_P} \arrow[u, hook]
\end{tikzcd}$$

where the vertical maps are injections since R_P is Cohen-Macaulay. Chasing the image of $\overline{y} \in R_P/(x_1, \ldots, x_h)R_P$, we find that for all $e > 0$, $c\overline{y}^{p^e} = 0$ in $R_P/(x_1^{p^e}, \ldots, x_h^{p^e})R_P$. That is, for every $e > 0$, there exists $z_e \notin P$ such that $cz_e y^{p^e} \in (x_1^{p^e}, \ldots, x_h^{p^e})$.

Let $(x_1, \ldots, x_h) = Q_1 \cap \cdots \cap Q_s$ be an irredundant primary decomposition of $(x_1, \ldots, x_h)$, with $P_i = \sqrt{Q_i}$ the corresponding associated primes. We may assume $P = P_1$. Since R is Cohen-Macaulay and $x_1, \ldots, x_h$ is a regular sequence, each P_i is a minimal prime of

$(x_1,\dots,x_h)$ and we have $\mathrm{Ass}(R/(x_1,\dots,x_h)) = \mathrm{Ass}(R/(x_1^{p^e},\cdots,x_h^{p^e}))$ for all $e>0$. Let $(x_1^{p^e},\dots,x_h^{p^e}) = Q_{1,e}\cap\cdots Q_{s,e}$ be the irredundant primary decomposition with $P_i=\sqrt{Q_{i,e}}$. We know that $Q_{i,e}$ is the contraction of $(x_1^{p^e},\dots,x_h^{p^e})R_{P_i}$ to R. Since $(x_1,\dots,x_h)^{hp^e}\subseteq (x_1^{p^e},\dots,x_h^{p^e})$, we have $Q_i^{(hp^e)}\subseteq Q_{i,e}$. Now we fix $z\in (Q_2\cap\cdots\cap Q_s)^h - P_1$, it follows that $z^{p^e}\in Q_i^{hp^e}\subseteq Q_{i,e}$ for all $i\geq 2$. Since $cz_ey^{p^e}\in (x_1^{p^e},\dots,x_h^{p^e})\subseteq Q_{1,e}$ and $z_e\notin P=P_1$, we know that $cy^{p^e}\in Q_{1,e}$. Thus we have $z\in R-P$ such that for all $e>0$,

$$cy^{p^e}z^{p^e}\in Q_{1,e}\cap Q_{2,e}\cap\cdots\cap Q_{s,e} = (x_1^{p^e},\dots,x_h^{p^e}).$$

Therefore for all $e>0$ and all $n>0$, we have

$$c(zy)^{p^e}\in (x_1^{p^e},\dots,x_h^{p^e})\subseteq (x_1^{p^e},\dots,x_h^{p^e},x_{h+1}^{np^e},\dots,x_d^{np^e}).$$

Since R is F-rational, there exists $e>0$ such that $cF^e(-)$ is injective on $H^d_\mathfrak{m}(R)$. Fix this e, we consider the following commutative diagram

$$\begin{tikzcd} H^d_\mathfrak{m}(R) \arrow[r, hook, "cF^e(-)"] & H^d_\mathfrak{m}(R) \\ \frac{R}{(x_1,\dots,x_h,x_{h+1}^n,\dots,x_d^n)} \arrow[u, hook] \arrow[r, "\overline{x}\mapsto c\overline{x}^{p^e}"] & \frac{R}{(x_1^{p^e},\dots,x_h^{p^e},x_{h+1}^{np^e},\dots,x_d^{np^e})} \arrow[u, hook] \end{tikzcd}$$

where the vertical maps are injective since R is Cohen-Macaulay. Chasing the diagram we find that the bottom map is injective. Since $\overline{zy}\in R/(x_1,\dots,x_h,x_{h+1}^n,\dots,x_d^n)$ maps to zero in $R/(x_1^{p^e},\dots,x_h^{p^e},x_{h+1}^{np^e},\dots,x_d^{np^e})$, we obtain that $zy\in(x_1,\dots,x_h,x_{h+1}^n,\dots,x_d^n)$ for all $n>0$. Thus

$$zy\in\cap_n(x_1,\dots,x_h,x_{h+1}^n,\dots,x_d^n) = (x_1,\dots,x_h),$$

which implies $y\in(x_1,\dots,x_h)R_P$. Therefore $0=\overline{y}\in R_P/(x_1,\dots,x_h)R_P$ and thus $\eta=0$, which is a contradiction. $\square$

Exercise 19. Prove that if a ring R of prime characteristic $p>0$ is F-injective, then R is reduced. (Hint: Use the fact that reduced is characterized by (R_0) and (S_1), and then use Theorem 4.13.)

Exercise 20. Let $R\to S$ be a faithfully flat extension of rings of prime characteristic $p>0$. Prove that if S is F-rational (resp., F-injective), then R is F-rational (resp., F-injective). (Hint: Use Theorem 4.14 (resp., Theorem 4.13) to reduce to the case that $\dim(R)=\dim(S)$.)

Exercise 21. Show that if R is an F-pure ring of prime characteristic $p > 0$, then R is F-injective. Conversely, show that if $(R, \mathfrak{m}, k)$ is a quasi-Gorenstein and F-injective ring of prime characteristic $p > 0$, then R is F-pure.

Exercise 22. Let R be an $\mathbb{N}$-graded ring over a field of prime characteristic $p > 0$ with homogeneous maximal ideal $\mathfrak{m}$. Show that

(1) If R is F-injective, then $[H^i_{\mathfrak{m}}(R)]_{>0} = 0$ for each i.
(2) If R is F-rational, then $[H^d_{\mathfrak{m}}(R)]_{\geq 0} = 0$.

Exercise 23. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p > 0$ and dimension d. Show that the kernel of the natural Frobenius action on $H^d_{\mathfrak{m}}(R)$ is a proper submodule of $H^d_{\mathfrak{m}}(R)$.

Exercise 24. Prove the following strengthening of Corollary 4.7: Suppose $R \to S$ is a pure map of rings of prime characteristic $p > 0$. If S is regular, then R is F-rational.

Discussion 4.15. We have seen that direct summands of F-regular rings (respectively F-pure ring) are F-regular (respectively F-pure). One can ask if a direct summand of F-rational or F-injective ring is F-rational or F-injective. This is not the case. Watanabe [Wat97] constructed an example of a direct summand of an F-rational ring that is not even F-injective. The example will be examined in Chapter 8, where we also give an example of a direct summand of an F-rational ring that is not Cohen-Macaulay.

5. The deformation problem

An interesting question in the study of singularities is how they behave under deformation. Roughly speaking, if $\operatorname{Spec}(R)$ is the total space of a fibration over a curve, then the special fiber of this fibration is a variety with coordinate ring R/xR for a nonzerodivisor x of R. The question is whether the singularity type of the total space $\operatorname{Spec}(R)$ is no worse than the singularity type as the special fiber $\operatorname{Spec}(R/xR)$.

This deformation question has been studied in details for F-singularities. The following list summarizes the best known progress.

(1) Strong F-regularity fails to deform in general [Sin99c], but it deforms for normal $\mathbb{Q}$-Gorenstein rings [AKM98].
(2) F-purity fails to deform in general [Fed83, Sin99b], but it deforms for normal $\mathbb{Q}$-Gorenstein rings [HW02, Sch09, PS20].
(3) F-rationality always deforms [HH94a].
(4) Deformation of F-injectivity remains an open problem in general. But it is known that F-injectivity deforms for Cohen-Macaulay rings [Fed83], and that F-purity always deforms to F-injectivity [HMS14].

Counterexamples to the deformation of strongly F-regular and F-pure singularities will be examined in Chapter 8, see Example 8.9. In this chapter we present the (partial) positive results on deformation of F-singularities mentioned above.

5.1. Deformation of F-rational and F-injective singularities.

We begin by proving deformation of F-injectivity in the Cohen-Macaulay case and the deformation of F-rationality.

Theorem 5.1. *Let $(R,\mathfrak{m},k)$ be a local ring of prime characteristic $p>0$ and x a nonzerodivisor on R. Then*

(1) If R/xR is Cohen-Macaulay and F-injective, then R is Cohen-Macaulay and F-injective.
(2) If R/xR is F-rational, then R is F-rational.

Proof. We first prove (1). It is clear that R is Cohen-Macaulay. It is enough to show that the natural Frobenius action on $H^d_\mathfrak{m}(R)$ is injective. The commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"'] & R \arrow[r] \arrow[d, "F^e"'] & R/xR \arrow[r] \arrow[d, "F^e"'] & 0 \\
0 \arrow[r] & R \arrow[r, "\cdot x"] & R \arrow[r] & R/xR \arrow[r] & 0
\end{tikzcd}$$

induces a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(R/xR) \arrow[r] \arrow[d, "F^e"] & H^d_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"] & H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & 0 \\
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(R/xR) \arrow[r] & H^d_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] & H^d_{\mathfrak{m}}(R) \arrow[r] & 0
\end{tikzcd}$$

If the middle map is not injective, then we pick $\eta \in \mathrm{Soc}(H^d_{\mathfrak{m}}(R)) \cap \mathrm{Ker}(x^{p^e-1}F^e)$ and it is easy to see that η comes from $H^{d-1}_{\mathfrak{m}}(R/xR)$. But this contradicts the injectivity of F^e on $H^{d-1}_{\mathfrak{m}}(R/xR)$. Thus $x^{p^e-1}F^e$ and hence F^e is injective on $H^d_{\mathfrak{m}}(R)$.

We next prove (2). Suppose we have $c \in R$ not in any minimal prime of R. It is enough to show that the F-stable submodule $\{\eta \in H^d_{\mathfrak{m}}(R) \mid c \cdot F^e(\eta) = 0 \text{ for all } e \geq 0\}$ is 0 (see the proof of Proposition 4.11, here we need to use that R is injective, which we just proved in (1)). If this submodule is nonzero, then it intersects $\mathrm{Soc}(H^d_{\mathfrak{m}}(R))$ nontrivially so we may assume there exists $0 \neq \eta \in H^d_{\mathfrak{m}}(R)$ such that $c \cdot F^e(\eta) = 0$ for all $e > 0$ and $x\eta = 0$. We can write $c = x^n c'$ where $c' \notin (x)$ and pick any $e \gg 0$ such that $p^e - 1 \geq n$. Since $c \cdot F^e(\eta) = 0$, $c' x^{p^e-1}F^e(\eta) = 0$. Since $x\eta = 0$ we know that η comes from $H^{d-1}_{\mathfrak{m}}(R/xR)$ and chasing the diagram we find that $c'F^e(\eta) = 0$ in $H^{d-1}_{\mathfrak{m}}(R/xR)$. But since R/xR is F-rational, it is a normal domain by Proposition 4.4 and hence the image of c' is nonzero in R/xR. So the F-rationality of R/xR implies that $c'F^e(-)$ is injective on $H^{d-1}_{\mathfrak{m}}(R/xR)$ for all $e \gg 0$. Thus $\eta = 0$, a contradiction. ☐

Recall that the notions of strong F-regularity and F-rationality coincide in Gorenstein rings, Proposition 4.9. Therefore we have the following result on deformation of strong F-regularity (we will generalize this result in section 5.2).

Corollary 5.2. *Let $(R, \mathfrak{m}, k)$ be an F-finite Gorenstein local ring of prime characteristic $p > 0$ and x a nonzerodivisor on R. If R/xR is strongly F-regular, then R is strongly F-regular.*

Proof. By Theorem 5.1, R is F-rational and thus strongly F-regular by Proposition 4.9. ☐

The deformation question for F-injectivity is not solved completely. To this date, the best partial result towards this question is obtained in [HMS14], where it is shown that F-purity deforms to F-injectivity (note that F-purity itself does not deform in general by Example 8.9, unless we invoke the $\mathbb{Q}$-Gorenstein hypothesis, see Theorem 5.19 or [PS20]). To prove this result, we need a result from [Ma14].

Theorem 5.3. *If $(R, \mathfrak{m}, k)$ is an F-pure local ring of prime characteristic $p > 0$, then for all i and all F-stable submodules $N \subseteq H^i_{\mathfrak{m}}(R)$, the natural Frobenius action on $H^i_{\mathfrak{m}}(R)/N$ is injective.*

Proof. We may replace R by $\widehat{R}$ to assume R is F-split (see Corollary 2.4). We then observe the following quite general claim:

Claim 5.4. *If $R \to S$ is split, η is an element of $H^i_{\mathfrak{m}}(R)$, and N is a submodule of $H^i_{\mathfrak{m}}(R)$, then $\eta \in N$ provided that the image of η in $H^i_{\mathfrak{m}S}(S)$ is contained in the S-span of the image of N in $H^i_{\mathfrak{m}S}(S)$.*

Proof. Let $\phi\colon S \to R$ be a splitting. It is easy to check that we have the following commutative diagram

$$\begin{tikzcd} S \otimes_R H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[dr, "\phi\otimes\mathrm{id}"'] & H^i_{\mathfrak{m}S}(S) \arrow[d, "H^i_{\mathfrak{m}}(\phi)"] \\ & H^i_{\mathfrak{m}}(R). \end{tikzcd}$$

Thus if the image of η is in the S-span of the image of N, say $\mathrm{Im}(1 \otimes \eta) = \sum s_i \cdot \mathrm{Im}(1 \otimes \eta_i)$ where $\eta_i \in N$. Then by the above commutative diagram, $\eta = \sum \phi(s_i)\eta_i \in N$. $\square$

We now continue the proof of the theorem. Suppose N is an F-stable submodule such that the Frobenius action on $H^i_{\mathfrak{m}}(R)/N$ is not injective, then there exists $\eta \notin N$ such that $F(\eta) \in N$. Let N_e be the R-span of $F^e(N)$. Since N is F-stable, we have a descending chain $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$. This chain stabilizes since $H^i_{\mathfrak{m}}(R)$ is Artinian. Therefore, as $F(\eta) \in N$, $F^{e+1}(\eta) \in N_e = N_{e+1}$ for $e \gg 0$. Finally we apply Claim 5.4 to the $(e+1)$-th Frobenius map $F^{e+1}\colon R \to R$ (which is split by assumption) and note that the R-span of the image of N is precisely N_{e+1}, hence we know that $\eta \in N$, a contradiction. $\square$

We now prove the aforementioned result in [HMS14], our proof proceeds very similarly as in the Cohen-Macaulay case and it differs from the original argument.

Theorem 5.5. *Let $(R, \mathfrak{m}, k)$ be a local ring and x a nonzerodivisor on R of prime characteristic $p > 0$. If R/xR is F-pure, then R is F-injective.*

Proof. The commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"'] & R \arrow[r] \arrow[d, "F^e"'] & R/xR \arrow[r] \arrow[d, "F^e"'] & 0 \\ 0 \arrow[r] & R \arrow[r, "\cdot x"] & R \arrow[r] & R/xR \arrow[r] & 0 \end{tikzcd}$$

induces a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/xR)/\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R)) \arrow[r] \arrow[d, "F^e"] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"] & H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & \cdots \\
0 \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/xR)/\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R)) \arrow[r] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] & H^i_{\mathfrak{m}}(R) \arrow[r] & \cdots
\end{tikzcd}$$

Note that $\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R))$ is an F-stable submodule of $H^{i-1}_{\mathfrak{m}}(R/xR)$. So by Theorem 5.3, F^e is injective on $H^{i-1}_{\mathfrak{m}}(R/xR)/\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R))$. Now by the same argument as in Theorem 5.1, this implies that $x^{p^e-1}F^e$ and hence F^e is injective on $H^i_{\mathfrak{m}}(R)$. □

In fact, it can be shown that $\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R)) = 0$ in the proof of Theorem 5.5. This was observed in [MQ18], and we leave it as an exercise, see Exercise 25.

Since quasi-Gorenstein F-injective rings are F-pure (see Exercise 21), we have the following result on deformation of F-purity (we will generalize this result in section 5.2).

Corollary 5.6. *Let $(R, \mathfrak{m}, k)$ be a quasi-Gorenstein F-pure local ring of prime characteristic $p > 0$ and x a nonzerodivisor on R. If R/xR is F-pure, then R is F-pure.*

Proof. By Theorem 5.5, R is F-injective and thus F-pure by and Exercise 21. □

5.2. **Deformation of strongly F-regular and F-pure singularities.** Our approach to the deformation problem of strong F-regularity and F-purity essentially follows from recent work in [PS20], and it involves the study of cyclic covers of R. To this end, we suggest that the reader who is not familiar with divisor class groups, divisorial ideals, reflexification, and the theory of (S_2)-modules over a ring which is (S_2) and (G_1), i.e., Gorenstein in codimension 1, consult Appendix A for the basic theory, notation, and language.

Before continuing forward we want to introduce the idea of the proof informally. Suppose that R is $\mathbb{Q}$-Gorenstein, $x \in R$ a nonzerodivisor such that R/xR is strongly F-regular or F-pure, and let $R \to S$ be a *cyclic cover* of R with respect to the canonical divisor. Consider the following commutative diagram:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
R/xR \arrow[r] & S/xS
\end{tikzcd}.$$

A result of Carvajal-Rojas [CR17, Theorem C], which generalizes a theorem of Watanabe [Wat91], asserts that R is strongly F-regular (resp., F-pure) if and only if a cyclic cover of R is strongly F-regular (resp., F-pure). Therefore to show R is strongly F-regular (resp., F-pure), it suffices to show that $R/xR \to S/xS$ is a cyclic cover of R/xR and that S is Gorenstein (resp., quasi-Gorenstein), since then we can invoke Corollary 5.2 (resp., Corollary 5.6) to conclude the proof.

In fact, the proof strategy in the strongly F-regular case follows exactly as outlined above, while in the F-pure case we need some modifications. We begin by presenting a self-contained and elementary proof of [CR17, Theorem C] mentioned above. We first prove a general fact on extending R-linear maps $F^e_*R \to R$ to the cyclic cover.

Proposition 5.7. *Let $(R, \mathfrak{m}, k)$ be an (S_2) and (G_1) local ring of prime characteristic $p > 0$ and D a torsion divisor of index N. Let $S = \bigoplus_{i=0}^{N-1} R(iD)t^i$ be a cyclic cover of R with respect to D and let $\pi : S \to R$ be the projection of S onto R. If $\varphi : F^e_*R \to R$ is an R-linear map then there exists an S-linear map $\psi : F^e_*S \to S$ so that the following diagram commutes:*

$$\begin{tikzcd} F^e_*S \arrow[r, dotted, "\psi"] \arrow[d, "F^e_*\pi"'] & S \arrow[d, "\pi"] \\ F^e_*R \arrow[r, "\varphi"] & R \end{tikzcd}$$

Proof. Let $e_1 : \operatorname{Hom}_R(S, R) \to R$ be the evaluation-at-1 map defined by $\psi \mapsto \psi(1)$. To find a map ψ making the above diagram commutative we utilize Proposition A.5 and instead show the existence of an S-linear map $\psi : F^e_* \operatorname{Hom}_R(S, R) \to \operatorname{Hom}_R(S, R)$ so that the following diagram commutes:

$$\begin{tikzcd} F^e_* \operatorname{Hom}_R(S, R) \arrow[r, dotted, "\psi"] \arrow[d, "F^e_*e_1"'] & \operatorname{Hom}_R(S, R) \arrow[d, "e_1"] \\ F^e_*R \arrow[r, "\varphi"] & R \end{tikzcd}$$

Given an element $\rho \in \operatorname{Hom}_R(S, R)$ and its corresponding element $F^e_*\rho \in F^e_* \operatorname{Hom}_R(S, R)$ we let $\psi(F^e_*\rho)$ be the element of $\operatorname{Hom}_R(S, R)$ which maps an element s to

$$\begin{aligned} \psi(F^e_*\rho)(s) = \varphi(F^e_*e_1(sF^e_*\rho)) &= \varphi(F^e_*e_1(F^e_*\rho(s^{p^e} \cdot -))) \\ &= \varphi(F^e_*e_1(\rho(s^{p^e} \cdot -))) \\ &= \varphi(F^e_*\rho(s^{p^e} \cdot 1)) \\ &= \varphi(F^e_*\rho(s^{p^e})). \end{aligned}$$

We leave it to the reader to verify that ψ is S-linear and makes the diagram commute. $\square$

We are now ready to prove [CR17, Theorem C].

Theorem 5.8. *Let $(R, \mathfrak{m}, k)$ be an F-finite (S_2) and (G_1) local ring of prime characteristic $p > 0$, D a torsion divisor of index N, and $S = \bigoplus_{i=0}^{N-1} R(iD)$ a cyclic cover of R with respect to D.*

(1) R is strongly F-regular if and only if S is strongly F-regular;
(2) R is F-pure if and only if S is F-pure.

Proof. The ring R is a direct summand of S. Hence if S is strongly F-regular then so is R by Theorem 3.9 and if S is F-pure then R is F-pure by Exercise 9.

Suppose that R is strongly F-regular and let c be a nonzero element of S. We aim to show the existence of an $e \in \mathbb{N}$ and an S-linear map $\psi : F^e_* S \to S$ so that $\psi(F^e_* c) = 1$.

Consider the projection $\pi : S \to R$. We claim that there exists an element $s \in S$ so that $\pi(sc) \neq 0$. Suppose that $R(ND) = R \cdot f$. Write $c = \sum_{i=0}^{N-1} c_i t^i$ with $c_i \in R(iD)$. If $c_0 \neq 0$ then $\pi(c) = c_0 \neq 0$. If $c_i \neq 0$ for some $i > 0$ then choose nonzero element $x \in R((N-i)D)$ and observe that $\pi(xt^{N-i}c) = \frac{xc_i}{f} \neq 0$. If there exists an S-linear map $\psi : F^e_* S \to S$ so that $\psi(F^e_* sc) = 1$ then the map $\varphi := \psi(F^e_* s-)$ is such that $\varphi(F^e_* c) = 1$. Therefore we can replace c by sc and assume that $c_0 := \pi(c) \neq 0$.

Since R is strongly F-regular, there exists an $e \in \mathbb{N}$ and an R-linear map $\varphi : F^e_* R \to R$ such that $\varphi(F^e_* c_0) = 1$. By Proposition 5.7 there exists an S-linear map $\psi : F^e_* S \to S$ so that the following diagram is commutative:

$$\begin{tikzcd} F^e_* S \arrow[r, "\psi"] \arrow[d, "F^e_*\pi"'] & S \arrow[d, "\pi"] \\ F^e_* R \arrow[r, "\varphi"] & R \end{tikzcd}$$

Observe that $(\pi \circ \psi)(F^e_* c) = (\varphi \circ F^e_* \pi)(F^e_* c) = \varphi(F^e_* c_0) = 1$. Moreover, by Lemma A.4, $\mathfrak{m}_S = \mathfrak{m} \oplus \bigoplus_{i=1}^{N-1} R(-iD)$ is the unique maximal ideal of S, and we have $\pi(\mathfrak{m}_S) = \mathfrak{m}$. In particular, $\psi(F^e_* c)$ must be a unit of S and thus the element $F^e_* c$ can be split out of $F^e_* S$ as desired.

The proof technique above also shows that S is F-pure provided R is F-pure. One starts with a map $F_* R \to R$ sending $F_* 1 \mapsto 1$. One can then lift this map to a map of S-modules $\psi : F_* S \to S$ and then argue as above to claim that $\psi(F_* 1)$ must be a unit of S. □

Recall that there is a one-to-one correspondence between divisorial ideals of a normal domain R and isomorphism classes of finitely generated rank 1 modules satisfying Serre's condition (S_2), see Appendix A for more general situation. It is atypical for the depth of a divisorial ideal of a normal domain to exceed 2. An exception to this "rule" is that divisorial ideals corresponding to torsion divisors in a strongly F-regular ring are Cohen-Macaulay, which follows directly from Theorem 5.8. Here we record another proof of this fact[2] following [Mar20, Proposition 2.6] which we find to be more direct, elementary, and transparent, though we will not use this result explicitly in the sequel.

Lemma 5.9. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring of prime characteristic $p > 0$ and M a finitely generated torsion-free R-module. If $m \in M$ is a nonzero element then for all $e \gg 0$ there exists $\varphi \in \operatorname{Hom}_R(F^e_* M, R)$ such that $\varphi(F^e_* m) = 1$.*

Proof. By Exercise 12, it is enough to show that there exists a single natural number e and $\varphi \in \operatorname{Hom}_R(F^e_* M, R)$ so that $\varphi(F^e_* m) = 1$. Because M is torsion-free and finitely generated there exists an inclusion of M into a free module $R^{\oplus N}$. Let $m \in M$ be a non-zero element. By mapping onto an appropriate summand of $R^{\oplus N}$ we find that there exists a map $\varphi : M \to R$ so that $\varphi(m) = r \neq 0$. We are assuming R is strongly F-regular. So for all $e \gg 0$ there exists $\psi : F^e_* R \to R$ so that $\psi(F^e_* r) = 1$. In particular, $\psi \circ F^e_* \varphi \in \operatorname{Hom}_R(F^e_* M, R)$ and $\psi(F^e_* \varphi(m)) = 1$. $\square$

Proposition 5.10. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring of prime characteristic $p > 0$. If D is a torsion divisor then there exists an $e \in \mathbb{N}$ so that $R(D)$ is a direct summand of $F^e_* R$. In particular, $R(D)$ is a Cohen-Macaulay R-module.*

Proof. Up to isomorphism, the set of R-modules $\{R(iD)\}_{i \in \mathbb{Z}}$ is a finite list as D is a torsion divisor. By Lemma 5.9 there exists an $e \in \mathbb{N}$ so that $F^e_* R(iD)$ has a free R-summand for all $i \in \mathbb{Z}$. In particular, there exists an $e \in \mathbb{N}$ so that $F^e_* R(-p^e D)$ has a free summand, say

$$F^e_* R(-p^e D) \cong R \oplus M.$$

If we apply $- \otimes_R R(D)$, reflexify, and utilize part (3) of Exercise 53 we see that

$$(F^e_* R(-p^e D) \otimes_R R(D))^{**} \cong F^e_* R(-p^e D + p^e D) \cong F^e_* R \cong R(D) \oplus (M \otimes_R R(D))^{**},$$

i.e., $R(D)$ is a (finite) direct summand of $F^e_* R$ as claimed. It follows that $R(D)$ is a Cohen-Macaulay since R (and hence $F^e_* R$) is Cohen-Macaulay by Theorem 4.6. $\square$

Now we are ready to prove the deformation of strong F-regularity when the ambient ring is $\mathbb{Q}$-Gorenstein.

Theorem 5.11. *Let $(R, \mathfrak{m}, k)$ be an F-finite $\mathbb{Q}$-Gorenstein local ring of prime characteristic $p > 0$ and x a nonzerodivisor on R. If R/xR is strongly F-regular then R is strongly F-regular.*

[2] This fact was observed in several locations in the literature, for example see [PS14, Corollary 3.3] and [DS16a, Corollary 3.12], also see [Wat91, Corollary 2.9].

Proof. First of all, since R/xR is strongly F-regular, R/xR is normal by Corollary 3.8. By Lemma A.8, we can choose an effective canonical divisor K_X of $X = \operatorname{Spec}(R)$ that has no component in $V := V(x) \cong \operatorname{Spec}(R/xR)$ and K_X restricts to an (effective) canonical divisor K_V of V by Lemma A.10. Suppose K_X has index N, we set $S = \bigoplus_{i=0}^{N-1} R(iK_X)t^i$ to be the cyclic cover of R with respect to K_X.

Claim 5.12. *$R/xR \to S/xS$ is the cyclic cover of R/xR with respect to K_V.*

Proof of Claim. Fix an $1 \le i \le N$ and consider the divisor $D = iK_X$. We will show that $R(D)/xR(D)$ is an (S_2) module over R/xR, which will imply that $R(D)/xR(D) \cong (R/xR)(iK_V)$ by Lemma A.9 and thus S/xS will indeed be a cyclic cover of R/xR with respect to the canonical divisor K_V. Note that if $\dim(R) \le 2$, then $R(D)$ is Cohen-Macaulay over R and hence $R(D)/xR(D)$ is Cohen-Macaulay over R/xR. Thus we may assume that $\dim(R) \ge 3$ in what follows.

Let $D|_V$ denote the pull back of the divisor D from $X = \operatorname{Spec}(R)$ to $V = \operatorname{Spec}(R/xR)$, see Discussion A.6. By Lemma A.9, we know that $D|_V$ is torsion of index at most N. Now for every $1 \le j \le N$, tensoring the canonical map $R(D) \to (R/xR)(D|_V)$ with the composition $R \to F^e_*R \to F^e_*R(jD)$ and reflexify we obtain (see Exercise 53):

$$\begin{tikzcd}
R(D) \arrow[r] \arrow[d] & F^e_*R(jD + p^eD) \arrow[d] \\
(R/xR)(D_V) \arrow[r] & F^e_*(R/xR)(jD|_V + p^eD|_V)
\end{tikzcd}$$

Since R/xR is strongly F-regular, by Exercise 15, the bottom map above splits for $e \gg 0$ for every $1 \le j \le N$. Now we fix such an $e \gg 0$ and pick $1 \le j \le N$ such that N divides $j + p^e$. It follows that $F^e_*R(jD + p^eD) \cong F^e_*R$ and $F^e_*(R/xR)(jD|_V + p^eD|_V) \cong R/xR$, and thus we obtain a commutative diagram:

$$(\dagger)\qquad\begin{tikzcd}
R(D) \arrow[r, "\varphi"] \arrow[d] & F^e_*R \arrow[d] \\
(R/xR)(D|_V) \arrow[r, "\psi"] & F^e_*(R/xR)
\end{tikzcd}$$

where the map ψ is split.

By induction on the dimension of R, we may assume that $R(D)/xR(D)$ is (S_2) on the punctured spectrum of R/xR and hence $R(D)/xR(D) \to (R/xR)(D|_V)$ is an isomorphism on the punctured spectrum. It follows that the induced maps of local cohomology modules

$$H^i_{\mathfrak{m}}(R(D)/xR(D)) \to H^i_{\mathfrak{m}}((R/xR)(D|_V))$$

is an isomorphism for all $i \geq 2$ and thus the following composition

$$\lambda_i : H^i_{\mathfrak{m}}(R(D)/xR(D)) \to H^i_{\mathfrak{m}}((R/xR)(D|_V)) \xrightarrow{H^i_{\mathfrak{m}}(\psi)} H^i_{\mathfrak{m}}(F^e_*(R/xR))$$

is (split) injective for all $i \geq 2$, since ψ is split.

Now consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R(D) \arrow[r, "\cdot x"] & R(D) \arrow[r] \arrow[d, "\varphi"] & R(D)/xR(D) \arrow[r] \arrow[d] & 0 . \\
& & F^e_*R \arrow[r] & F^e_*(R/xR) &
\end{tikzcd}$$

There is an induced commutative diagram of local cohomology modules:

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathfrak{m}}(R(D)/xR(D)) \arrow[r] & H^2_{\mathfrak{m}}(R(D)) \arrow[r, "\cdot x"] & H^2_{\mathfrak{m}}(R(D)) \arrow[r, "\pi"] \arrow[d] & H^2_{\mathfrak{m}}(R(D)/xR(D)) . \arrow[d, "\lambda_2"] \\
& & & 0 = H^2_{\mathfrak{m}}(F^e_*R) \arrow[r] & H^2_{\mathfrak{m}}(F^e_*(R/xR))
\end{tikzcd}$$

Since R/xR is (S_2), x is a nonzerodivisor of R and $\dim(R) \geq 3$, we know that $\operatorname{depth}(R) \geq 3$ and thus $H^2_{\mathfrak{m}}(F^e_*R) = 0$. Since λ_2 is injective, chasing the diagram shows that the map π is the 0-map and so $H^2_{\mathfrak{m}}(R(D)) = xH^2_{\mathfrak{m}}(R(D))$. The module $R(D)$ is (S_2) and therefore $H^2_{\mathfrak{m}}(R(D))$ is a finitely generated R-module, see Exercise 29, by Nakayama's lemma we have $H^2_{\mathfrak{m}}(R(D)) = 0$, and therefore $H^1_{\mathfrak{m}}(R(D)/xR(D)) = 0$.

Since $R(D)/xR(D)$ is (S_2) on the punctured spectrum and that $H^1_{\mathfrak{m}}(R(D)/xR(D)) = 0$, it follows that $R(D)/xR(D)$ is an (S_2) module over R/xR as wanted. □

Finally, by Claim 5.12 and Theorem 5.8, S/xS is strongly F-regular and thus Cohen-Macaulay. It follows that S is Cohen-Macaulay. But then since S is quasi-Gorenstein by Lemma A.7, S is Gorenstein and so S is strongly F-regular by Corollary 5.2 and thus R is strongly F-regular by Theorem 5.8. □

Finally, we turn to the deformation problem of F-purity in $\mathbb{Q}$-Gorenstein rings. Indeed, Hara and Watanabe were able to notice through their efforts to compare log terminal and log canonical singularities with F-regular and F-pure singularities in [HW02] that F-purity deforms provided R is $\mathbb{Q}$-Gorenstein of index not divisible by the characteristic of R, a proof that was eventually recorded in full generality by Schwede in [Sch09]. It was not until quite recently that the deformation of $\mathbb{Q}$-Gorenstein F-pure singularities was solved, see [PS20]. We begin with the following observation of Fedder.

Lemma 5.13. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$ and $x \in R$ a nonzerodivisor. Then for each $e \in \mathbb{N}$ the R-linear maps $R \xrightarrow{\pi} R/xR$ and $F^e_*(R/xR) \xrightarrow{\cdot F^e_* x^{p^e-1}} F^e_*(R/x^{p^e}R)$ induce R-linear maps*

(1) $\Psi^e_1 : \operatorname{Hom}_R(F^e_*R, R) \xrightarrow{\pi} \operatorname{Hom}_{R/xR}(F^e_*(R/x^{p^e}R), R/xR)$

(2) $\Psi^e_2 : \operatorname{Hom}_{R/xR}(F^e_*(R/x^{p^e}R), R/xR) \xrightarrow{\cdot F^e_* x^{p^e-1}} \operatorname{Hom}_{R/xR}(F^e_*(R/xR), R/xR)$.

If R is Gorenstein, then the maps Ψ^e_1 and Ψ^e_2 are onto for every $e \in \mathbb{N}$.

Proof. The map Ψ^e_1: $\operatorname{Hom}_R(F^e_*R, R) \to \operatorname{Hom}_R(F^e_*R, R/xR)$ is obtained by applying $\operatorname{Hom}_R(F^e_*R, -)$ to the natural surjection $R \xrightarrow{\pi} R/xR$ and observing that

$$\operatorname{Hom}_R(F^e_*R, R/xR) \cong \operatorname{Hom}_{R/xR}(F^e_*R/xF^e_*R, R/xR) \cong \operatorname{Hom}_{R/xR}(F^e_*(R/x^{p^e}R), R/xR).$$

The map $\Psi^e_2 : \operatorname{Hom}_R(F^e_*R, R/xR) \to \operatorname{Hom}_{R/xR}(F^e_*(R/xR), R/xR)$ is given by applying $\operatorname{Hom}_R(-, R/xR)$ to the map $F^e_*(R/xR) \xrightarrow{\cdot F^e_* x^{p^e-1}} F^e_*(R/x^{p^e}R)$.

Suppose that R is Gorenstein. To show that Ψ^e_1 is onto consider the short exact sequence

$$0 \to R \xrightarrow{\cdot x} R \to R/xR \to 0.$$

Then $\operatorname{Ext}^1_R(F^e_*R, R) = 0$ as F^e_*R is a Cohen-Macaulay R-module and R is Gorenstein (see [BH93, Theorem 3.3.10]). Thus Ψ^e_1 is onto. Similarly, to show that Ψ^e_2 is onto consider the short exact sequence

$$0 \to F^e_*(R/xR) \xrightarrow{\cdot F^e_* x^{p^e-1}} F^e_*(R/x^{p^e}R) \to F^e_*(R/x^{p^e-1}R) \to 0.$$

Then $\operatorname{Ext}^1_{R/xR}(F^e_*(R/x^{p^e-1}R), R/xR) = 0$ as $F^e_*(R/x^{p^e-1}R)$ is a Cohen-Macaulay R/xR-module (see Exercise 28) and R/xR is Gorenstein (again, use [BH93, Theorem 3.3.10]). Thus the induced map Ψ^e_2 is onto. $\square$

If R/xR is F-pure but not strongly F-regular, then the cyclic cover $R \to S$ with respect to the canonical divisor of R will produce a quasi-Gorenstein ring, where deformation of F-purity is known to hold, see Corollary 5.6. However, it is no longer reasonable to expect a divisorial ideal associated to a torsion divisor to be of high depth and we do not expect $R/xR \to S/xS$ to remain a cyclic cover of R/xR. Our adjustment will still come in the form of expecting certain divisorial ideals to be of high depth: not all divisorial ideals associated to torsion divisors will have high depth, but those with index p to a power do.

Lemma 5.14. *Let $(R, \mathfrak{m}, k)$ be an (S_2) and (G_1) local ring which is F-finite and F-pure of prime characteristic $p > 0$. Suppose that D is a torsion divisor of index p^{e_0} for some e_0. Then $R(D)$ is a direct summand of $F^{e_0}_*R$.*

Proof. Consider a direct sum decomposition of $F^{e_0}_*R$,

$$F^{e_0}_*R \cong R \oplus M.$$

If we tensor with $R(D)$ and reflexify we find that

$$F^{e_0}_*R(p^{e_0}D) \cong F^{e_0}_*R \cong R(D) \oplus (M \otimes_R R(D))^{**}$$

where $(-)^* = \operatorname{Hom}_R(-, R)$. $\square$

Let us return to the problem of deforming F-purity in a $\mathbb{Q}$-Gorenstein ring. Let $x \in R$ be a nonzerodivisor such that R/xR is (S_2), (G_1), and F-pure. Let K_X be a choice of canonical divisor on $X = \operatorname{Spec}(R)$ that has no component in $V(x)$. Suppose that $Np^{e_0}K_X \sim 0$ and p does not divide N. Let $D = NK_X$. Observe that not only is $p^{e_0}D \sim 0$ but for any integer m we have that $p^{e_0}mD \sim 0$. In particular, if we consider the cyclic cover $S = \bigoplus_{i=0}^{N-1} R(iD)t^i$, then we expect each of the divisorial ideals $R(iD)$ to have good enough depth properties (since this will be the case if we know R is F-pure, see Lemma 5.14) so that $R/xR \to S/xS$ is the induced cyclic cover of R/xR. This will allow us to replicate the deformation of strong F-regularity proof to the deformation of F-purity problem, provided we can establish the deformation of F-purity in $\mathbb{Q}$-Gorenstein rings whose index is relatively prime p. To this end, we should try to understand the cyclic cover S associated to the divisor $D = NK_X$. We begin with a well-known lemma.

Lemma 5.15. *Let $R \to S$ be a module-finite extension of (S_2) and (G_1) rings with choice of canonical divisor K_X on $X = \operatorname{Spec}(R)$ and K_Y on $Y = \operatorname{Spec}(S)$. Then we have*

$$\operatorname{Hom}_R(S, R) \cong S(K_Y - \pi^*K_X).$$

In particular, if R is an (S_2), (G_1), and F-finite ring of prime characteristic $p > 0$, then for each $e \in \mathbb{N}$ there is an isomorphism

$$\operatorname{Hom}_R(F^e_*R, R) \cong F^e_*R((1 - p^e)K_X).$$

Proof. First note that we have $R(K_X) \cong \omega_R$, $S(K_Y) \cong \omega_S \cong \operatorname{Hom}_R(S, \omega_R)$. Now we have

$$\begin{aligned}
\operatorname{Hom}_R(S, R) &\cong \operatorname{Hom}_R(S \otimes_R \omega_R, \omega_R) \cong \operatorname{Hom}_S(S \otimes_R \omega_R, \operatorname{Hom}_R(S, \omega_R)) \\
&\cong \operatorname{Hom}_S(S \otimes_R \omega_R, \omega_S) \cong \operatorname{Hom}_S(S \otimes_R R(K_X), S(K_Y)) \\
&\cong \operatorname{Hom}_S((S \otimes_R R(K_X))^{**}, S(K_Y)) \cong \operatorname{Hom}_S(S(\pi^*K_X), S(K_Y)) \\
&\cong S(K_Y - \pi^*K_X)
\end{aligned}$$

where the first isomorphism on the third line follows from the fact that $S(K_Y)$ is reflexive. This proves the first assertion, the second assertion follows from the first by observing that the pull back of K_X under the e-th Frobenius map is $p^e K_X$. $\square$

Lemma 5.15 implies that $\operatorname{Hom}_R(F^e_* R, R)$ is a cyclic $F^e_* R$-module for infinitely many choices of e, provided R is $\mathbb{Q}$-Gorenstein of index not divisible by the characteristic of R.

Corollary 5.16. *Let $(R, \mathfrak{m}, k)$ be a normal F-finite domain of prime characteristic $p > 0$ with choice of canonical divisor K_X on $X = \operatorname{Spec}(R)$. Then the following are equivalent:*

(1) R is $\mathbb{Q}$-Gorenstein of index not divisible by p;
(2) $\operatorname{Hom}_R(F^e_* R, R) \cong F^e_* R$ for all e sufficiently divisible.

Proof. If $NK_X \sim 0$ and p does not divide N then Fermat's Little Theorem allows us to conclude that N divides $1 - p^e$ for all integers $e \in \mathbb{N}$ which are sufficiently divisible. For such an $e \in \mathbb{N}$ we use Lemma 5.15 and find that

$$\operatorname{Hom}_R(F^e_* R, R) \cong F^e_* R((1 - p^e)K_X) \cong F^e_* R.$$

Conversely, if $\operatorname{Hom}_R(F^e_* R, R) \cong F^e_* R$ is a cyclic $F^e_* R$-module then we have that $(1-p^e)K_X \sim 0$ by Lemma 5.15 again and so K_X is torsion of index not divisible by p. $\square$

Proposition 5.17. *Let $(R, \mathfrak{m}, k)$ be an F-finite $\mathbb{Q}$-Gorenstein ring of prime characteristic $p > 0$ and of $\mathbb{Q}$-Gorenstein index not divisible by p. Suppose that $x \in R$ is a nonzerodivisor such that R/xR is (S_2) and (G_1). Then the composition of the natural maps $\Psi^e_2 \circ \Psi^e_1$ described in Lemma 5.13 is onto for infinitely many integers $e \in \mathbb{N}$.*

Proof. Fix an integer $e \in \mathbb{N}$ so that the index of K_X divides $1 - p^e$ and so $\operatorname{Hom}_R(F^e_* R, R) \cong F^e_* R((1 - p^e)K_X)$ is a cyclic $F^e_* R$-module, see Corollary 5.16.

Consider the maps Ψ^e_1 and Ψ^e_2 described in Lemma 5.13 and let

$$\Psi^e = \Psi^e_2 \circ \Psi^e_1 : \operatorname{Hom}_R(F^e_* R, R) \to \operatorname{Hom}_{R/xR}(F^e_*(R/xR), R/xR)$$

be the composition of Ψ^e_1 and Ψ^e_2. Because $\operatorname{Hom}_R(F^e_* R, R) \cong F^e_* R$ we have that the image of Ψ^e in $\operatorname{Hom}_{R/xR}(F^e_*(R/xR), R/xR)$ is abstractly isomorphic to $F^e_*(R/xR)$, in particular the image of Ψ^e is an (S_2)-module over $F^e_*(R/xR)$. The module $\operatorname{Hom}_{R/xR}(F^e_*(R/xR), R/xR) \cong F^e_*(R/xR)$ is an (S_2)-module as well. By Proposition A.2 we can check that the image of Ψ^e agrees with $\operatorname{Hom}_{R/xR}(F^e_*(R/xR), R/xR)$ by checking equality when localized at a height 1 prime. This will indeed be the case since R/xR is (G_1) and the map Ψ^e is onto under the Gorenstein hypothesis by Lemma 5.13. $\square$

We are ready to prove F-purity deforms in $\mathbb{Q}$-Gorenstein rings whose $\mathbb{Q}$-Gorenstein index is not divisible by p.

Corollary 5.18. *Let $(R, \mathfrak{m}, k)$ be an F-finite $\mathbb{Q}$-Gorenstein ring of prime characteristic $p > 0$ and of $\mathbb{Q}$-Gorenstein index not divisible by p. Suppose that $x \in R$ is a nonzerodivisor such that R/xR is (S_2), (G_1), and F-pure. Then R is F-pure.*

Proof. By Proposition 5.17, we can choose e such that

$$\Psi^e : \operatorname{Hom}_R(F^e_* R, R) \to \operatorname{Hom}_{R/xR}(F^e_*(R/xR), R/xR)$$

is onto. It follows that for each R/xR-linear map $\varphi : F^e_*(R/xR) \to R/xR$ there is a commutative diagram

$$\begin{tikzcd}
F^e_*(R/xR) \arrow[r, "\varphi"] \arrow[d, "\cdot F^e_* x^{p^e-1}"] & R/xR \arrow[d, "="] \\
F^e_*(R/x^{p^e}R) \arrow[r, "\tilde{\varphi}"] & R/xR \\
F^e_* R \arrow[r, "\varphi'"] \arrow[u, "\pi"] & R. \arrow[u, "\pi"']
\end{tikzcd}$$

In particular, since R/xR is F-pure, we can choose φ to be an onto map, but then an easy diagram chasing shows that φ' is also onto and thus R is F-pure. ☐

We now have the tools necessary to prove that F-purity deforms in $\mathbb{Q}$-Gorenstein rings.

Theorem 5.19. *Let $(R, \mathfrak{m}, k)$ be a $\mathbb{Q}$-Gorenstein F-finite ring of prime characteristic $p > 0$. Suppose that $x \in R$ is a nonzerodivisor such that R/xR is (S_2), (G_1), and F-pure. Then R is F-pure.*

Proof. By Lemma A.8, we can choose a canonical divisor K_X of $X = \operatorname{Spec}(R)$ that has no component in $V := V(x) \cong \operatorname{Spec}(R/xR)$ and K_X restricts to a canonical divisor K_V of R/xR by Lemma A.10. Suppose that $p^{e_0} N K_X \sim 0$ and p does not divide N. Consider the cyclic cover $R \to S$ associated to the divisor NK_X. The ring R is a direct summand of S and thus R will be F-pure provided S is F-pure. By Lemma A.7, S is $\mathbb{Q}$-Gorenstein with index N not divisible by p and so by Corollary 5.18 it is enough to show that S/xS is F-pure.

Suppose we can show that $R/xR \to S/xS$ is a cyclic cover of R/xR with respect to NK_V, then S/xS will be F-pure by Theorem 5.8. To show that $R/xR \to S/xS$ is a cyclic cover it is enough to show that if $D = iNK_X$ for some $1 \leq i \leq p^{e_0}$ then $R(iD)/xR(iD)$ is an (S_2) R/xR-module. Now an almost identical argument as in the proof of Claim 5.12 works: to obtain the commutative diagram (†), one just need to tensor the canonical map $R(D) \to (R/xR)(D|_V)$ with the natural map $R \to F^e_* R$ for any $e \geq e_0$ and note that the divisor D, and hence $D|_V$, has torsion index divisible by p^e (see Lemma A.9), thus it follows that $R(p^e D) \cong R$ and $(R/xR)(p^e D|_V) \cong R/xR$. ☐

Exercise 25. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p > 0$ and x a nonzerodivisor on R such that R/xR is F-pure. Use Theorem 5.3 to show that the natural map $H^i_{\mathfrak{m}}(R/x^nR) \to H^i_{\mathfrak{m}}(R/xR)$ is surjective for all $n \geq 1$ and all i. Then use this to show that multiplication by x on $H^i_{\mathfrak{m}}(R)$ is surjective for all i. (Hint: Consider the long exact sequence of local cohomology induced by $0 \to R \xrightarrow{\cdot x} R \to R/xR \to 0$ and show the connection maps are injective.)

Exercise 26. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p > 0$ and x a nonzerodivisor on R such that R/xR is quasi-Gorenstein and F-pure. Use Exercise 25 to prove that R is quasi-Gorenstein and F-pure. (We caution the reader that, in general, the quasi-Gorenstein property does not deform [STT20, Theorem 4.2].)

Exercise 27. Let $(R, \mathfrak{m}, k)$ be a $\mathbb{Q}$-Gorenstein, F-finite, and F-pure local ring of prime characteristic $p > 0$. Let K_X be a choice of canonical divisor of $X = \operatorname{Spec}(R)$. Show that there exists integer $e \in \mathbb{N}$ so that $R(p^{e_0}K_X)$ is a direct summand of $F^e_*R(K_X)$ for all $e_0 \gg 0$ sufficiently divisible. Prove that for all $e_0 \gg 0$ sufficiently divisible the divisorial ideal $R(p^{e_0}K_X)$ satisfies (S_r) provided R satisfies (S_r). (Hint: Suppose that $Np^eK_X \sim 0$ and p is relatively prime to N. Show that $R(K_X)$ is a direct summand of $F^e_*R(K_X)$ and consider what happens to this direct sum decomposition when you apply $- \otimes_R (R((p^{e_0} - 1)K_X))$ and reflexify for all $e_0 \gg 0$.)

Exercise 28. Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$ and depth $g \geq 1$. Suppose that x is a nonzerodivisor of R. Show that $F^e_*(R/x^iR)$ has depth $g - 1$ as an R/xR-module for all $1 \leq i \leq p^e$.

Exercise 29. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d that admits a canonical module (i.e., R is a homomorphic image of a Gorenstein local ring). Let M be a finitely generated R-module such that $\dim(R/P) = d$ for all minimal primes P of M, and such that M satisfies Serre's condition (S_i) for some $i < d$. Show that $H^i_{\mathfrak{m}}(M)$ is a finitely generated R-module. (Hint: Mimic the proof of Lemma 4.5.)

As we already mentioned, whether F-injectivity deforms in general remains an open question. We refer the reader to [MSS17, MQ18, DSM20] for some recent progress.

Open Problem 2. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p > 0$ and x a nonzerodivisor on R. If R/xR is F-injective, then is R also F-injective?*

6. The Γ-construction and completion of F-rationality

Our goal of this chapter is to show that completion of excellent local F-rational rings are F-rational. To establish this, we need to show that in the definition of F-rationality, we actually only need to consider one (special) c. This is not difficult to prove if R is F-finite. To reduce the general case to the F-finite case, we need a powerful tool introduced by Hochster–Huneke [HH94a]: the Γ-construction.

Discussion 6.1 (Trace map). Let $(R,\mathfrak{m},k)\to(S,\mathfrak{n},\ell)$ be a module-finite extension of local rings of dimension d. Suppose ω_R is a canonical module of R (recall that this means $\omega_R^\vee\cong H^d_{\mathfrak{m}}(R)$). Then the canonical map $R\to S$ induces a trace map:

$$\omega_S\cong\operatorname{Hom}_R(S,\omega_R)\xrightarrow{\operatorname{Tr}}\omega_R.$$

The Matlis dual of this map yields

$$H^d_{\mathfrak{m}}(R)\to\operatorname{Hom}_R(\omega_S,E_R(k))\cong\operatorname{Hom}_S(\omega_S,\operatorname{Hom}_R(S,E_R(k)))\cong\operatorname{Hom}_S(\omega_S,E_S(\ell))\cong H^d_{\mathfrak{n}}(S),$$

which is precisely the natural map on top local cohomology modules induced by $R\to S$. In particular, if R is F-finite of prime characteristic $p>0$, then the natural e-th Frobenius action $H^d_{\mathfrak{m}}(R)\to F^e_*H^d_{\mathfrak{m}}(R)$ corresponds to the trace map $F^e_*\omega_R\xrightarrow{\operatorname{Tr}^e}\omega_R$, and it can be checked that $\operatorname{Tr}^{e_1+e_2}=\operatorname{Tr}^{e_1}\circ F^{e_1}_*(\operatorname{Tr}^{e_2})$. Note that here we are implicitly using that F-finite rings admit canonical modules (see Theorem 1.6). Moreover, if, in addition, $(\omega_R)_P\cong\omega_{R_P}$ (this holds for all $P\in\operatorname{Spec}(R)$ if R is equidimensional, see Remark A.1), then $(\operatorname{Tr}^e)_P$ is the corresponding trace map for R_P.

Proposition 6.2. *Let $(R,\mathfrak{m},k)$ be an F-finite Cohen-Macaulay local ring of prime characteristic $p>0$. Then R is F-rational if and only if for every $c\in R$ that is not in any minimal prime of R, there exists $e>0$ such that the composition $F^e_*\omega_R\xrightarrow{\cdot F^e_*c}F^e_*\omega_R\xrightarrow{\operatorname{Tr}^e}\omega_R$ is surjective (i.e., $\operatorname{Tr}^e: F^e_*(c\omega_R)\to\omega_R$ is surjective).*

Proof. This follows immediately from Discussion 6.1 and the definition of F-rationality. ☐

Proposition 6.2 implies that if R is F-finite and F-rational, then R_P is F-rational for all $P\in\operatorname{Spec}(R)$. Of course, we have already proved a more general Theorem 4.14 without assuming R is F-finite.

The next result is an analog of Theorem 3.11 for F-rationality. We will eventually extend this result to excellent Cohen-Macaulay local rings in Chapter 7. But at this point, we only prove it when R is F-finite.

Proposition 6.3. *Let $(R,\mathfrak{m},k)$ be an F-finite Cohen-Macaulay local ring of prime characteristic $p>0$ and of dimension d. Suppose there exists c not in any minimal prime of R*

such that R_c is F-rational (e.g., R_c is regular). Then R is F-rational if and only if there exists $e > 0$ such that the composition

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_* R) \xrightarrow{\cdot F^e_* c} H^d_{\mathfrak{m}}(F^e_* R)$$

is injective, or equivalently,

$$F^e_*\omega_R \xrightarrow{\cdot F^e_* c} F^e_*\omega_R \xrightarrow{\mathrm{Tr}^e} \omega_R$$

is surjective.

Proof. Suppose z is not in any minimal prime of R. Then z is not in any minimal prime of R_c and thus by Proposition 6.2, there exists e_0 such that $\mathrm{Tr}^{e_0}\colon F^{e_0}_*(z\omega_{R_c}) \to \omega_{R_c}$ is surjective.[3] Since R is F-finite, we know that

$$\mathrm{Hom}_{R_c}(F^{e_0}_*(z\omega_{R_c}), \omega_{R_c}) \cong \mathrm{Hom}_R(F^{e_0}_*(z\omega_R), \omega_R)_c.$$

Therefore we know that there exists $n > 0$ such that the image of $\mathrm{Tr}^{e_0}\colon F^{e_0}_*(z\omega_R) \to \omega_R$ contains $c^n\omega_R$.

Our assumption says that there exists $e > 0$ such that $c \cdot F^e$ is injective on $H^d_{\mathfrak{m}}(R)$. If we compose this map n times we get that $c^{1+p^e+\cdots+p^{ne}} \cdot F^{ne}$ is injective on $H^d_{\mathfrak{m}}(R)$, in particular, $c^n \cdot F^{ne}$ is injective on $H^d_{\mathfrak{m}}(R)$. That is, the composition

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^{ne}_* R) \xrightarrow{\cdot F^{ne}_* c^n} H^d_{\mathfrak{m}}(F^{ne}_* R)$$

is injective. But then by Discussion 6.1, we see that $\mathrm{Tr}^{ne}\colon F^{ne}_*(c^n\omega_R) \to \omega_R$ is surjective. Now the composition

$$\mathrm{Tr}^{ne+e_0} : F^{ne+e_0}_*(z\omega_R) \xrightarrow{F^{ne}_* \mathrm{Tr}^{e_0}} F^{ne}_*\omega_R \xrightarrow{\mathrm{Tr}^{ne}} \omega_R$$

is surjective and so by Proposition 6.2, R is F-rational. $\square$

As a consequence, we prove the following result on openness of F-rational locus for F-finite local rings. We will eventually extend this result to excellent local rings.

Proposition 6.4. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$. Then the F-rational locus of $\mathrm{Spec}(R)$ is open.*

Proof. Suppose R_P is F-rational, it is enough to show that there exists $f \notin P$ such that R_f is F-rational. Since R_P is a domain by Proposition 4.4, there is a unique minimal prime P' of R that is contained in P. Suppose we can find $f \notin P$ such that $(R/P')_f$ is F-rational, then

[3]Here we are using Proposition 6.2 for an F-finite but not necessarily local ring R_c, we leave it to the reader to check that the proposition is still valid in our context: the point is that the trace map is Tr^e is globally defined and it localizes to the corresponding trace map for R_P for all $P \in \mathrm{Spec}(R_c)$.

after replacing f by a multiple not in P so that $P'R_f = 0$, we have that $R_f \cong (R/P')_f$ is F-rational. Thus we may replace R by R/P' to assume that $(R, \mathfrak{m}, k)$ is an F-finite domain.

Since R is excellent by Theorem 1.7, there exists $c \neq 0$ such that R_c is regular. Since R_P is F-finite and F-rational, by Proposition 6.2 we know that there exists $e > 0$ such that Tr^e: $F^e_*(c\omega_{R_P}) \to \omega_{R_P}$ is surjective, where Tr^e can be viewed as the trace map of ω_R localized at P, see Discussion 6.1 (here R is a local domain and hence equidimensional). It follows that there exists $f \notin P$ such that Tr^e: $F^e_*(c\omega_{R_f}) \to \omega_{R_f}$ is surjective. Since R_P is Cohen-Macaulay, we can replace f by a multiple to assume that R_f is Cohen-Macaulay.[4] Now by Proposition 6.3 (applied to each R_Q such that $Q \in D(f)$), we see that R_f is F-rational. $\square$

Remark 6.5. In the proof of Proposition 6.3, we are implicitly using R is local since we need a global trace map Tr: $F^e_*\omega_R \to \omega_R$. It is well-known that this holds as long as R is F-finite and "sufficiently affine" (see [BB11], we will not make this precise here).[5] Now for any F-finite ring R, we can find a finite cover of $\mathrm{Spec}(R)$ by sufficiently affine open subsets $\cup D(f_i)$, then a small modification of the proof of Proposition 6.3 works for each R_{f_i}. But a subset of $\mathrm{Spec}(R)$ is open if and only if its intersection with each $D(f_i)$ is open. Therefore for any F-finite (not necessarily local) ring R, the F-rational locus of $\mathrm{Spec}(R)$ is open.

We next introduce the Γ-construction of Hochster–Huneke [HH94a] – a very useful technique to reduce questions from complete local rings to the case of F-finite local rings. The results presented here: Lemma 6.9 – Lemma 6.14, originate from [HH94a] and [EH08].

Let k be a field of positive characteristic $p > 0$ with a p-base Λ. Let Γ be a fixed cofinite subset of Λ. For $e \in \mathbb{N}$ we denote by $k^{\Gamma,e}$ the purely inseparable field extension of k that is the result of adjoining p^e-th roots of all elements in Γ to k.

Discussion 6.6 (The Γ-construction). Let $(R, \mathfrak{m}, k)$ be a complete local ring of prime characteristic $p > 0$. Abusing notations a bit, we also fix $k \subseteq R$ to be a coefficient field of R. Let $x_1, \ldots, x_d$ be a system of parameters for R. By Cohen's structure theorem we know that R is module-finite over $A = k[[x_1, \ldots, x_d]] \subseteq R$. We define

$$A^\Gamma := \bigcup_{e \in \mathbb{N}} k^{\Gamma,e}[[x_1, \ldots, x_d]],$$

[4] It is well-known that the Cohen-Macaulay locus is open for excellent rings. In our context, we can argue as follows: Since $(R, \mathfrak{m}, k)$ is F-finite, we know that $(R, \mathfrak{m}, k)$ is a homomorphic image of a regular local ring $(S, \mathfrak{n}, k)$ by Theorem 1.6, it is easy to check that R_P is Cohen-Macaulay if and only if $\mathrm{Ext}^j_S(R, S)_P = 0$ for all $j \neq n - d$ where $n = \dim(S)$ and $d = \dim(R)$, but if these Ext groups vanish when localized at P, then they vanish when inverting f for some $f \notin P$.

[5] In fact, Karl Schwede communicated to us that there exists a global trace map Tr: $F^e_*\omega_R \to \omega_R$ for all F-finite rings, but to the best of our knowledge, this result has not been written down.

which is a regular local ring faithfully flat and purely inseparable over A (note that A^Γ is Noetherian, we leave this as Exercise 31). The maximal ideal of A expands to that of A^Γ. Set $R^\Gamma := A^\Gamma \otimes_A R$. Then R^Γ is module-finite over the regular local ring A^Γ, and R^Γ is faithfully flat and purely inseparable over R. The maximal ideal of R expands to the maximal ideal of R^Γ and the residue field of R^Γ is $k^\Gamma := \bigcup_{e\in\mathbb{N}} k^{\Gamma,e}$. Note that, since $R \to R^\Gamma$ is purely inseparable, $\operatorname{Spec}(R^\Gamma)$ can be identified with $\operatorname{Spec}(R)$. For every $Q \in \operatorname{Spec}(R)$, we use Q^Γ to denote the unique prime ideal in R^Γ corresponds to Q, i.e., $Q^\Gamma = \sqrt{QR^\Gamma}$.

Remark 6.7. With notation as in Discussion 6.6, it is easy to see that $R^\Gamma = \bigcup_{e\in\mathbb{N}} R\widehat{\otimes}_k k^{\Gamma,e}$. In particular, the definition of R^Γ depends only on the choice of the coefficient field k (and the choice of p-base of k), but not on the choice of $x_1, \ldots, x_d$.

Remark 6.8. With notation as in Discussion 6.6, we have $\operatorname{depth} R_Q = \operatorname{depth} R^\Gamma_{Q^\Gamma}$ since $R_Q \to R^\Gamma_{Q^\Gamma}$ is purely inseparable. In particular, R_Q is Cohen-Macaulay if and only if $R^\Gamma_{Q^\Gamma}$ is Cohen-Macaulay.

Lemma 6.9. *With notation as in Discussion 6.6, R^Γ is F-finite.*

Proof. It is enough to show that A^Γ is F-finite, that is, F_*A^Γ is finitely generated as an A^Γ-module, or equivalently, $(A^\Gamma)^{1/p}$ is finitely generated over A^Γ – it will be convenient to use this latter notation in the proof.

Let $\theta_1, \ldots, \theta_n$ be the finitely many elements in $\Lambda - \Gamma$. Then the following finite set

$$\Theta := \{\theta_1^{i_1/p} \cdots \theta_n^{i_n/p} \cdot x_1^{j_1/p} \cdots x_d^{j_d/p} | 0 \le i_t, j_t \le p-1\}$$

is a generating set of $(A^\Gamma)^{1/p}$ over A^Γ. To see this, note that

$$(A^\Gamma)^{1/p} = \bigcup_{e\in\mathbb{N}} (k^{\Gamma,e}[[x_1, \ldots, x_d]])^{1/p},$$

and it is easy to check that $(k^{\Gamma,e}[[x_1, \ldots, x_d]])^{1/p}$ is generated over $k^{\Gamma,e+1}[[x_1, \ldots, x_d]]$ by Θ. Thus after passing to the union, we see that Θ is a generating set of $(A^\Gamma)^{1/p}$ over A^Γ. □

Lemma 6.10. *With notation as in Discussion 6.6, if Q is a prime ideal of R, then for all sufficiently small choices of Γ, we have $Q^\Gamma = QR^\Gamma$.*

Proof. Replacing R by R/Q, it is enough to show that if R is a complete local domain, then R^Γ is a domain for all sufficiently small choices of Γ (see Remark 6.7).

We let L, L^Γ, L_R denote the fraction field of A, A^Γ, R respectively. Since A^Γ is purely inseparable over A, we know that $L^\Gamma = L \otimes_A A^\Gamma$. Also note that L_R is a finite extension of L. We first observe that it suffices to show $L_R \otimes_L L^\Gamma$ is a field for sufficiently small choices of Γ: for if this is true, then we have

$$R^\Gamma = R \otimes_A A^\Gamma \hookrightarrow L_R \otimes_A A^\Gamma = L_R \otimes_L L \otimes_A A^\Gamma = L_R \otimes_L L^\Gamma$$

and hence R^Γ is a domain as desired (the injection above follows because A^Γ is flat over A). We next note that, since $A^\Gamma \hookrightarrow k^\Gamma[[x_1, \dots, x_d]]$, we have $L^\Gamma \subseteq \mathrm{Frac}(k^\Gamma[[x_1, \dots, x_d]])$ and thus

$$\bigcap_{\Gamma\subseteq\Lambda \text{ cofinite}} L^\Gamma \subseteq \bigcap_{\Gamma\subseteq\Lambda \text{ cofinite}} \mathrm{Frac}(k^\Gamma[[x_1, \dots, x_d]]) = \mathrm{Frac}(k[[x_1, \dots, x_d]]) = L$$

where the middle equality follows from [Mat70, 30.E] since $\bigcap k^\Gamma = k$. Thus we have $\bigcap L^\Gamma = L$. Let $\{\lambda_1, \dots, \lambda_n\}$ be a basis of L_R over L. To show $L_R \otimes_L L^\Gamma$ is a field, it is enough to show $\{\lambda_1, \dots, \lambda_n\}$ are linearly independent over L^Γ (view all fields in a fixed ambient $\overline{L}$). We pick Γ such that the number of linearly independent vectors of $\{\lambda_1, \dots, \lambda_n\}$ over L^Γ is maximum among all the L^Γ. If this number is $h < n$, then without loss of generality we can assume $\{\lambda_1, \dots, \lambda_h\}$ are linearly independent over L^Γ but $\lambda_{h+1} = \ell_1\lambda_1 + \cdots \ell_h\lambda_h$ where $\ell_i \in L^\Gamma$ and at least one of the ℓ_i, say ℓ_1, is not in L. Since $\bigcap L^\Gamma = L$, we can pick $\Gamma' \subseteq \Gamma$ such that $\ell_1 \notin L^{\Gamma'}$. But then λ_{h+1} cannot be written as a linear combination of $\lambda_1, \dots, \lambda_h$ over $L^{\Gamma'}$ (if so then we have two expressions of λ_{h+1} as linear combinations of $\lambda_1, \dots, \lambda_h$ over L^Γ which contradict the linear independency of $\{\lambda_1, \dots, \lambda_h\}$ over L^Γ), it follows that $\{\lambda_1, \dots, \lambda_{h+1}\}$ are linearly independent over $L^{\Gamma'}$ contradicting our choice of Γ. Therefore, for all sufficiently small choices of Γ, $L_R \otimes_L L^\Gamma$ is a field. □

Remark 6.11. With notation as in Discussion 6.6, if R is a domain, then we have $\mathrm{Frac}(R) = \bigcap \mathrm{Frac}(R^\Gamma)$ where the intersection is taken over all sufficiently small Γ such that R^Γ is a domain. In fact, following the notation as in the proof of Lemma 6.10, we have

$$\bigcap \mathrm{Frac}(R^\Gamma) = \bigcap (L_R \otimes_L L^\Gamma) = L_R \otimes_L \bigcap L^\Gamma = L_R \otimes_L L = L_R = \mathrm{Frac}(R)$$

where the second equality is because L_R is a finite field extension of L and the third equality uses $\bigcap L^\Gamma = L$ as in the proof of Lemma 6.10.

Lemma 6.12. *With notation as in Discussion 6.6, if R_Q is regular then $R^\Gamma_{Q^\Gamma}$ is regular for all sufficiently small choices of Γ. In fact, the regular locus of* $\mathrm{Spec}(R)$ *can be identified with the regular locus of* $\mathrm{Spec}(R^\Gamma)$ *for all sufficiently small choices of* Γ.

Proof. By Lemma 6.10, for sufficiently small Γ, $QR^\Gamma = Q^\Gamma$ is a prime ideal. Thus $R_Q \to R^\Gamma_{Q^\Gamma}$ is a faithfully flat extension whose closed fiber is a field, so it follows that $R^\Gamma_{Q^\Gamma}$ is regular.

We use $\mathrm{Reg}(R)$ to denote the regular locus of $\mathrm{Spec}(R)$. For any $\Gamma' \subseteq \Gamma$ two cofinite subsets of Λ, we have a faithfully flat purely inseparable extension $R^{\Gamma'} \to R^\Gamma$ which induces a faithfully flat extension $R^{\Gamma'}_{P^{\Gamma'}} \to R^\Gamma_{P^\Gamma}$. Thus if $P^\Gamma \in \mathrm{Reg}(R^\Gamma)$, then $P^{\Gamma'} \in \mathrm{Reg}(R^{\Gamma'})$. Thus after we identify $\operatorname{Spec}(R^\Gamma)$ with $\operatorname{Spec}(R)$, we have $\operatorname{Reg}(R^\Gamma) \subseteq \operatorname{Reg}(R^{\Gamma'})$ (note that these are open subsets of $\operatorname{Spec}(R)$ since all R^Γ are F-finite by Lemma 6.9 and hence excellent). Since open subsets of $\operatorname{Spec}(R)$ satisfy ascending chain condition, we know that for all sufficiently small choices of Γ, $\operatorname{Reg}(R^\Gamma) = \operatorname{Reg}(R^{\Gamma'})$ for all $\Gamma' \subseteq \Gamma$. Fix such a Γ, we will show that $\operatorname{Reg}(R) = \operatorname{Reg}(R^\Gamma)$. Clearly $\operatorname{Reg}(R^\Gamma) \subseteq \operatorname{Reg}(R)$. Suppose there exists $Q \in \operatorname{Reg}(R)$ but $Q^\Gamma \notin \operatorname{Reg}(R^\Gamma)$. Then by the first part of the lemma we can pick a sufficiently small $\Gamma' \subseteq \Gamma$ such that $Q^{\Gamma'} \in \operatorname{Reg}(R^{\Gamma'})$, but then $\operatorname{Reg}(R^{\Gamma'}) \neq \operatorname{Reg}(R^\Gamma)$ which is a contradiction. $\square$

Lemma 6.13. *With notation as in Discussion 6.6, if $Q \in \operatorname{Spec}(R)$ and W is an Artinian R_Q-module with an injective Frobenius action, then for all sufficiently small choices of Γ the induced Frobenius action is injective on $W^\Gamma := W \otimes_{R_Q} R^\Gamma_{Q^\Gamma}$.*

Proof. By Lemma 6.10, we may assume Γ is small enough such that $Q^\Gamma = QR^\Gamma$. Then we have $\kappa(Q^\Gamma) = \operatorname{Frac}(R^\Gamma/QR^\Gamma)$ and $\bigcap \kappa(Q^\Gamma) = \kappa(Q)$ (see Remark 6.11).

Let V be the socle of W. Since W is Artinian, V is a finite dimensional vector space over $\kappa(Q)$ and $V^\Gamma := V \otimes_{R_Q} R^\Gamma_{Q^\Gamma} = V \otimes_{\kappa(Q)} \kappa(Q^\Gamma)$ is the socle of W^Γ (as a module over $R^\Gamma_{Q^\Gamma}$). Let F be the given Frobenius action on W and let F^Γ be the induced Frobenius action on W^Γ. Set $U^\Gamma := V^\Gamma \cap \operatorname{Ker}(F^\Gamma)$ which is a $\kappa(Q^\Gamma)$-subspace of V^Γ.

Note that $U^{\Gamma'} \subseteq U^\Gamma$ whenever $\Gamma' \subseteq \Gamma$ and F^Γ is injective on W^Γ if and only if $U^\Gamma = 0$. We pick Γ sufficiently small such that $\dim(U^\Gamma)$ is the smallest. We next fix a basis $v_1, \ldots, v_n$ of V over $\kappa(Q)$. If $\dim(U^\Gamma) > 0$, then we choose a basis of U^Γ over $\kappa(Q^\Gamma)$ and write each basis vector as $\sum a_{ij} v_j$ where $a_{ij} \in \kappa(Q^\Gamma)$. Now the reduced row echelon form of (a_{ij}) is *uniquely determined* by U^Γ, and in this reduced row echelon form, each row must contain an entry not in $\kappa(Q)$ since $U^\Gamma \cap V = 0$ (as F is injective on W). But since $\bigcap \kappa(Q^\Gamma) = \kappa(Q)$, there exists $\Gamma' \subseteq \Gamma$ such that at least one of these entries is not in $\kappa(Q^{\Gamma'})$, it follows that $U^{\Gamma'}$ must have dimension strictly smaller than $\dim(U^\Gamma)$ (choose a basis of $U^{\Gamma'}$ and look at the reduced row echelon form with respect to $v_1, \ldots, v_n$ again, it must have fewer rows). This contradicts our choice of Γ. Thus for all sufficiently small Γ, $U^\Gamma = 0$ and so F^Γ is injective as desired. $\square$

Lemma 6.14. *With notation as in Discussion 6.6, if R_Q is F-rational, then $R^\Gamma_{Q^\Gamma}$ is F-rational for all sufficiently small choices of Γ. In fact, the F-rational locus of $\operatorname{Spec}(R)$ can be identified with the F-rational locus of $\operatorname{Spec}(R^\Gamma)$ for all sufficiently small choices of Γ.*

Proof. Since R_Q is an excellent local domain (by Proposition 4.4), there exists $c \in R$ whose image in R_Q is nonzero such that $(R_Q)_c$ is regular. Since $\operatorname{Reg}(R) = \operatorname{Reg}(R^\Gamma)$ for sufficiently small choices of Γ by Lemma 6.12, $\operatorname{Reg}(R_c) = \operatorname{Reg}(R^\Gamma_c)$ and thus $(R^\Gamma_{Q^\Gamma})_c$ is regular. Since $R^\Gamma_{Q^\Gamma}$ is F-finite and Cohen-Macaulay, by Proposition 6.3 it is enough to show there exists

$e > 0$ such that for all sufficiently small choices of Γ,

$$H^h_{Q^\Gamma}(R^\Gamma_{Q^\Gamma}) \to F^e_* H^h_{Q^\Gamma}(R^\Gamma_{Q^\Gamma}) \xrightarrow{\cdot F^e_* c} F^e_* H^h_{Q^\Gamma}(R^\Gamma_{Q^\Gamma})$$

is injective, where $h = \operatorname{ht}(Q)$. This follows from Lemma 6.13 since R_Q is F-rational and $H^h_{Q^\Gamma}(R^\Gamma_{Q^\Gamma}) \cong H^h_Q(R_Q) \otimes_{R_Q} R^\Gamma_{Q^\Gamma}$.

The rest of the proof is very similar to Lemma 6.12. We use $\operatorname{Frat}(R)$ to denote the F-rational locus of $\operatorname{Spec}(R)$. For any $\Gamma' \subseteq \Gamma$ two cofinite subsets of Λ, we have a faithfully flat extension $R^{\Gamma'} \to R^\Gamma$ which induces a faithfully flat extension $R^{\Gamma'}_{P^{\Gamma'}} \to R^\Gamma_{P^\Gamma}$. Thus if $P^\Gamma \in \operatorname{Frat}(R^\Gamma)$, then $P^{\Gamma'} \in \operatorname{Frat}(R^{\Gamma'})$ by Exercise 20. Thus after we identify $\operatorname{Spec}(R^\Gamma)$ with $\operatorname{Spec}(R)$, we have $\operatorname{Frat}(R^\Gamma) \subseteq \operatorname{Frat}(R^{\Gamma'})$ (note that these are open subsets of $\operatorname{Spec}(R)$ by Proposition 6.4). Since open subsets of $\operatorname{Spec}(R)$ satisfy ascending chain condition, we know that for all sufficiently small choices of Γ, $\operatorname{Frat}(R^\Gamma) = \operatorname{Frat}(R^{\Gamma'})$ for all $\Gamma' \subseteq \Gamma$. Fix such a Γ, we will show that $\operatorname{Frat}(R) = \operatorname{Frat}(R^\Gamma)$. Clearly $\operatorname{Frat}(R^\Gamma) \subseteq \operatorname{Frat}(R)$. Suppose there exists $Q \in \operatorname{Frat}(R)$ but $Q^\Gamma \notin \operatorname{Frat}(R^\Gamma)$. Then by the first part of the lemma we can pick a sufficiently small $\Gamma' \subseteq \Gamma$ such that $Q^{\Gamma'} \in \operatorname{Frat}(R^{\Gamma'})$, but then $\operatorname{Frat}(R^{\Gamma'}) \neq \operatorname{Frat}(R^\Gamma)$ which is a contradiction. □

Corollary 6.15. *Let $(R, \mathfrak{m}, k)$ be a complete local ring of prime characteristic $p > 0$. Then the F-rational locus of $\operatorname{Spec}(R)$ is open.*

Proof. By Lemma 6.9, for all sufficiently small choices of Γ, R^Γ is F-finite. Thus by Proposition 6.4, the F-rational locus of $\operatorname{Spec}(R^\Gamma)$ is open. Hence so is the F-rational locus of $\operatorname{Spec}(R)$ by Lemma 6.14. □

We can now prove the following.

Theorem 6.16. *Let $(R, \mathfrak{m}, k)$ be an excellent Cohen-Macaulay local ring of prime characteristic $p > 0$. Suppose there exists c not in any minimal prime of R such that R_c is regular. Then $\widehat{R}$ is F-rational (and hence R is F-rational) if there exists $e > 0$ such that the composition*

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_* R) \xrightarrow{\cdot F^e_* c} H^d_{\mathfrak{m}}(F^e_* R)$$

is injective. In particular, if R is excellent, then R is F-rational if and only if $\widehat{R}$ is F-rational.

Proof. We first note that $H^d_{\mathfrak{m}}(R) = H^d_{\mathfrak{m}}(\widehat{R})$ and if $c \in R$ is not in any minimal prime of R, then c is not in any minimal prime of $\widehat{R}$. Thus it is clear that $\widehat{R}$ is F-rational implies R is F-rational (this is also a special case of Exercise 20, and we do not need to assume R is excellent).

Since R is excellent, $R \to \widehat{R}$ has geometrically regular fibers and hence we know that $\widehat{R}_c$ is also regular. By Lemma 6.12, $\widehat{R}^\Gamma_c$ is regular for sufficiently small choices of Γ. Moreover, since

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_*R) \xrightarrow{\cdot F^e_* c} H^d_{\mathfrak{m}}(F^e_*R)$$

is injective, By Lemma 6.13, it follows that for sufficiently small choices of Γ,

$$H^d_{\mathfrak{m}}(\widehat{R}^\Gamma) \to F^e_* H^d_{\mathfrak{m}}(\widehat{R}^\Gamma) \xrightarrow{\cdot F^e_* c} F^e_* H^d_{\mathfrak{m}}(\widehat{R}^\Gamma)$$

is injective.

Since $\widehat{R}^\Gamma$ is F-finite and Cohen-Macaulay, and $\widehat{R}^\Gamma_c$ is regular (note that c is not in any minimal prime of $\widehat{R}^\Gamma$ since $R \to \widehat{R}^\Gamma$ is flat), Proposition 6.3 shows that $\widehat{R}^\Gamma$ is F-rational. But then since $\widehat{R} \to \widehat{R}^\Gamma$ is faithfully flat, $\widehat{R}$ is F-rational by Exercise 20. The last conclusion follows since the assumptions are clearly satisfied if R is F-rational. $\square$

Remark 6.17. There are examples of non-excellent F-rational local rings $(R, \mathfrak{m}, k)$ of prime characteristic $p > 0$ such that $\widehat{R}$ is not F-rational, see [LR01].

Exercise 30. Let R be an F-finite ring of prime characteristic $p > 0$. Prove that the F-injective, F-pure and strongly F-regular locus of $\operatorname{Spec}(R)$ are open.

Exercise 31. With notation as in Discussion 6.6, prove that $A^\Gamma \to k^\Gamma[[x_1, \dots, x_d]]$ is faithfully flat, and use this to show that A^Γ is Noetherian. (Hint: Prove the more general fact that if $A \to B$ is a faithfully flat extension of rings such that B is Noetherian, then A is Noetherian.)

Exercise 32. With notation as in Discussion 6.6, use Lemma 6.10 to prove that if J is a radical ideal of R, then for all sufficiently small choices of Γ, we have JR^Γ is radical (in particular if R is reduced then R^Γ is reduced for all sufficiently small Γ).

In Proposition 6.3, Remark 6.5, and Exercise 30, we have seen that for F-finite rings, the loci of $\operatorname{Spec}(R)$ such that R is F-rational (resp., F-injective, F-pure) is open. In Chapter 7, we will show that the same holds for excellent *local* rings, and with some further work this can be shown to hold for all rings essentially of finite type over excellent local rings – this is basically because the theory of Γ-construction can extended to this set up (see [HH94a] or [Mur21]). It is thus natural to ask the following question.

Open Problem 3. *Let R be an excellent ring of prime characteristic $p > 0$. Is the F-rational (resp., F-injective, F-pure) locus of* $\operatorname{Spec}(R)$ *open?*[6]

[6]We caution the reader that, one cannot expect the openness of loci for these F-singularities without the excellent assumption, for example see [DM19, Theorem 5.10] (which is based on [Hoc73]).

7. F-singularities under faithfully flat base change

The goal of this chapter is to study F-singularities under faithfully flat base change. The general question we are interested is the following : Suppose $(R,\mathfrak{m},k)\to(S,\mathfrak{n},\ell)$ is a flat local extension such that the base ring R and the closed fiber $S/\mathfrak{m}S$ have certain type of F-singularities, then whether S has the same type of F-singularities? For example, if R is a DVR with uniformizer t, then $S/\mathfrak{m}S\cong S/tS$ where t is a nonzerodivisor of S, and this is precisely the deformation question we studied in Chapter 5.

Since even the deformation question has a negative answer in general (e.g., for F-pure and strongly F-regular singularities, see Chapter 8, Example 8.9), one cannot expect the general question hold without additional assumptions. We will present what is known and point out some open questions in this area. We first recall a well-known lemma.

Lemma 7.1 ([Mat70, Section 21]). *Let $(R,\mathfrak{m},k)\to(S,\mathfrak{n},\ell)$ be a flat local extension such that $S/\mathfrak{m}S$ is Cohen-Macaulay. Let $\underline{x}:=x_1,\dots,x_d$ be a system of parameters of $S/\mathfrak{m}S$. Then $x_1,\dots,x_d$ is a regular sequence on S and $S/(\underline{x})S$ is faithfully flat over R. In particular, $H^d_{(\underline{x})}(S)$ is faithfully flat over R.*

We also recall the following result on the behavior of injective hull under faithfully flat extension with Gorenstein closed fiber, which is due to Hochster–Huneke [HH94a, Lemma 7.10] in the generality we need.

Lemma 7.2. *Let $(R,\mathfrak{m},k)\to(S,\mathfrak{n},\ell)$ be a flat local extension such that $S/\mathfrak{m}S$ is Gorenstein. Let $\underline{x}:=x_1,\dots,x_d$ be a system of parameters of $S/\mathfrak{m}S$. Then $E_S(\ell)\cong E_R(k)\otimes_R H^d_{(\underline{x})}(S)$. Moreover, if u is a socle representative of $E_R(k)$ and the image of $\frac{v}{x_1\cdots x_d}\in H^d_{(\underline{x})}(S)$ in $H^d_{(\underline{x})}(S/\mathfrak{m}S)$ is a socle representative of $H^d_{(\underline{x})}(S/\mathfrak{m}S)$, then $u\otimes\frac{v}{x_1\cdots x_d}$ is a socle representative of $E_S(\ell)\cong E_R(k)\otimes_R H^d_{(\underline{x})}(S)$.*

Proof. We have $E_R(k)=\cup_h \operatorname{Ann}_{E_R(k)}\mathfrak{m}^h\cong\cup_h E_{R/\mathfrak{m}^h}(k)$ and similarly $E_S(\ell)\cong\cup_h E_{S/\mathfrak{m}^hS}(\ell)$. Thus we can replace $R\to S$ by $R/\mathfrak{m}^h\to S/\mathfrak{m}^hS$ to assume that $(R,\mathfrak{m},k)$ is Artinian (note that the socle representative doesn't change when we do this replacement).

By Lemma 7.1, we know that $S_t:=S/(x_1^t,\dots,x_d^t)S$ is faithfully flat over R with $S_t/\mathfrak{m}S_t$ Gorenstein. If we can show that $E_R(k)\otimes_R S_t\cong E_{S_t}(\ell)$, then we would have

$$E_R(k)\otimes_R H^d_{(\underline{x})}(S)\cong E_R(k)\otimes_R\varinjlim_t S_t\cong\varinjlim E_{S_t}(\ell)\cong E_S(\ell).$$

Note that $\frac{v}{x_1\cdots x_d}\in H^d_{(\underline{x})}(S)$ is the image of $v(x_1\cdots x_d)^{t-1}\in S_t$ whose image in $S_t/\mathfrak{m}S_t$ is a socle representative of $S_t/\mathfrak{m}S_t$. Therefore, replacing S by S_t and $\frac{v}{x_1\cdots x_d}$ by $v(x_1,\dots,x_d)^{t-1}$

and noting that for Artinian local rings, the injective hull of the residue field coincides with the canonical module, it is enough to establish the following claim:

Claim 7.3. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a flat local extension of Artinian local rings such that $S/\mathfrak{m}S$ is Gorenstein. Then we have $\omega_R \otimes_R S \cong \omega_S$. Moreover, if u is a socle representative of ω_R and $v \in S$ whose image in $S/\mathfrak{m}S$ is a socle representative of $S/\mathfrak{m}S$, then $u \otimes v$ is a socle representative of $\omega_R \otimes_R S \cong \omega_S$.*

Proof of Claim. Since $R \to S$ is flat local, we have $\ell_S(\omega_R \otimes_R S) = \ell_S(R \otimes_R S) = \ell_S(S)$. Thus to show $\omega_R \otimes_R S \cong \omega_S$, it is enough to show that $\omega_R \otimes_R S$ has a one-dimensional socle. But note that

$$\begin{aligned} \operatorname{Hom}_S(\ell, \omega_R \otimes_R S) &\cong \operatorname{Hom}_S(\ell, \operatorname{Hom}_S(S/\mathfrak{m}S, \omega_R \otimes_R S)) \\ &\cong \operatorname{Hom}_S(\ell, \operatorname{Hom}_R(k, \omega_R) \otimes_R S) \\ &\cong \operatorname{Hom}_S(\ell, k \otimes_R S) \cong \operatorname{Hom}_S(\ell, S/\mathfrak{m}S) \cong \ell, \end{aligned}$$

which is exactly what we want to show. We leave it to the reader to check through the above isomorphisms that the socle elements are matched as in the claim. □

□

7.1. **The case of strongly F-regular and F-pure singularities.** We first prove the base change results on F-pure and strongly F-regular singularities. These results, in the generality we presented, are originally due to Aberbach [Abe01] using methods from tight closure theory. Our arguments are more streamlined and do not depend on the knowledge of tight closure. In what follows, we will use E_R and E_S to denote the injective hull of the residue field of R and S respectively. We begin with the F-pure case.

Theorem 7.4. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a flat local extension of rings of prime characteristic $p > 0$ such that R is F-pure and $S/\mathfrak{m}S$ is Gorenstein and F-pure.*[7] *Then S is F-pure.*

Proof. By Lemma 7.2 and Proposition 2.2, it is enough to show that

$$E_R \otimes_R H^d_{(\underline{x})}(S) \to E_R \otimes_R H^d_{(\underline{x})}(S) \otimes_S F^e_* S \cong E_R \otimes_R F^e_* R \otimes_{F^e_* R} F^e_* H^d_{(\underline{x})}(S)$$

is injective for all $e > 0$. Now the image of the socle representative $u \otimes \frac{v}{x_1 \cdots x_d}$ under the map is $u \otimes F^e_* 1 \otimes F^e_*(\frac{v^{p^e}}{x_1^{p^e} \cdots x_d^{p^e}})$. Thus it is enough to show this element is nonzero in $E_R \otimes_R$

[7]Note that if R is a DVR (or more generally, a regular local ring), then we only need to assume $S/\mathfrak{m}S$ is quasi-Gorenstein and F-pure, see Exercise 26. The authors do not know whether one can relax the Gorenstein hypothesis to quasi-Gorenstein in general.

$F^e_*R \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S)$. Since R is F-pure, $u \otimes F^e_*1 \neq 0$ in $E_R \otimes_R F^e_*R$. Thus there exists a nonzero (F^e_*R)-linear map $F^e_*R \to E_R \otimes_R F^e_*R$ sending F^e_*1 to $u \otimes_R F^e_*1$, say with kernel F^e_*J. Since $F^e_*H^d_{(\underline{x})}(S)$ is faithfully flat over F^e_*R by Lemma 7.1, we have an injection:

$$(F^e_*R/F^e_*J) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \hookrightarrow E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S).$$

The image of $F^e_*1 \otimes F^e_*(\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}})$ under this map is precisely $u \otimes F^e_*1 \otimes F^e_*(\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}})$. Thus to show the latter one is nonzero, it is enough to show $F^e_*1 \otimes F^e_*(\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}}) \neq 0$. But

$$(F^e_*R/F^e_*J) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \twoheadrightarrow (F^e_*R/F^e_*\mathfrak{m}) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \cong F^e_*(H^d_{(\underline{x})}(S/\mathfrak{m}S)),$$

thus it is enough to show that $F^e_*(\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}}) \neq 0$ in $F^e_*(H^d_{(\underline{x})}(S/\mathfrak{m}S))$, that is, $\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}} \neq 0$ in $H^d_{(\underline{x})}(S/\mathfrak{m}S)$. But $S/\mathfrak{m}S$ is F-pure, in particular F-injective by Exercise 21, hence the Frobenius action on $H^d_{(\underline{x})}(S/\mathfrak{m}S)$ is injective. Since $\frac{v}{x_1\cdots x_d} \neq 0$, $\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}} = F^e(\frac{v}{x_1\cdots x_d}) \neq 0$ in $H^d_{(\underline{x})}(S/\mathfrak{m}S)$. ☐

We next prove the general base change result for strong F-regularity. One difficulty in establishing this compared with the F-pure case is that we need to choose c carefully to detect the strong F-regularity of the target ring.

Theorem 7.5. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a flat local extension of F-finite rings of prime characteristic $p > 0$ such that R is strongly F-regular and $S/\mathfrak{m}S$ is Gorenstein and strongly F-regular. Then S is strongly F-regular.*

Proof. Since $S/\mathfrak{m}S$ is strongly F-regular, it is a normal domain by Proposition 3.8. Thus $\mathfrak{m}S$ is a prime ideal in S. We first show that $S' = S_{\mathfrak{m}S}$ is strongly F-regular. We know that $R \to S'$ is a flat local extension such that $S'/\mathfrak{m}S'$ is a field. Moreover, by Proposition 3.12, we may replace R and S' by their completion to assume R and S' are both complete.

Suppose there exists $c \in S'$ not in any minimal prime of S' such that for all $e > 0$, the map $S' \to F^e_*S'$ sending 1 to F^e_*c is not split, then by Corollary 2.4 and Proposition 2.2, the map $E_{S'} \to E_{S'} \otimes_{S'} F^e_*S'$ induced by sending 1 to F^e_*c is not injective for all $e > 0$, thus the socle of $E_{S'}$ maps to zero under this map. By Lemma 7.2, $E_{S'} \cong E_R \otimes_R S'$ and a socle representative is $u \otimes 1$ where u is a socle representative of E_R. It follows that

$$E_{S'} \otimes_{S'} F^e_*S' \cong E_R \otimes_R S' \otimes_{S'} F^e_*S' \cong E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*S'$$

and that $u \otimes F^e_*1 \otimes F^e_*c = 0$ in $E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*S'$ for all $e > 0$. Thus

$$F^e_*c \in \operatorname{Ann}_{E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*S'}(u \otimes F^e_*1 \otimes F^e_*1) \cong (\operatorname{Ann}_{E_R \otimes_R F^e_*R}(u \otimes F^e_*1)) \otimes_{F^e_*R} F^e_*S'$$

for all $e > 0$ where the isomorphism follows from that F^e_*S' is flat over F^e_*R. However, since R is strongly F-regular, we know that for all $0 \neq z \in R$, there exists $e > 0$ such that the map $R \to F^e_*R$ sending 1 to F^e_*z is split, thus $F^e_*z \notin \mathrm{Ann}_{E_R \otimes_R F^e_*R}(u \otimes F^e_*1)$ (again by Corollary 2.4 and Proposition 2.2). Therefore, if we define $F^e_*I_e := \mathrm{Ann}_{E_R \otimes_R F^e_*R}(u \otimes F^e_*1)$, then $\cap_e I_e = 0$ and $0 \neq c \in \cap_e(I_e \otimes_R S')$. But by Chevalley's lemma, for all $n > 0$, there exists $e(n)$ such that $I_{e(n)} \subseteq \mathfrak{m}^n$, thus $\cap_e(I_e \otimes_R S') \subseteq \cap_n \mathfrak{m}^n S' = 0$ which is a contradiction.

So far we have proved that $S_{\mathfrak{m}S}$ is strongly F-regular. By Exercise 30, there exists $c \notin \mathfrak{m}S$ such that S_c is strongly F-regular. Note that c is a nonzerodivisor on $S/\mathfrak{m}S$ and thus it is a nonzerodivisor on S by Lemma 7.1, in particular, c is not in any minimal prime of S. By Theorem 3.11, it is enough to show that there exists $e > 0$ such that the map $S \to F^e_*S$ sending 1 to F^e_*c is split. The rest of the proof is very similar to the proof of Theorem 7.4. By Corollary 2.4 and Proposition 2.2, it is enough to show that the map $E_S \to E_S \otimes_S F^e_*S$ induced by sending 1 to F^e_*c is injective for some $e > 0$. By Lemma 7.2, this is the same as the map

$$E_R \otimes_R H^d_{(\underline{x})}(S) \to E_R \otimes_R H^d_{(\underline{x})}(S) \otimes_S F^e_*S \cong E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S).$$

Now the image of the socle representative $u \otimes \frac{v}{x_1 \cdots x_d}$ under the map is $u \otimes F^e_*1 \otimes F^e_*(\frac{cv^{p^e}}{x_1^{p^e} \cdots x_d^{p^e}})$. Thus it is enough to show this element is nonzero in $E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S)$. Since R is strongly F-regular (in particular F-pure), $u \otimes F^e_*1 \neq 0$ in $E_R \otimes_R F^e_*R$. Thus there exists a nonzero (F^e_*R)-linear map $F^e_*R \to E_R \otimes_R F^e_*R$ sending F^e_*1 to $u \otimes_R F^e_*1$, say with kernel F^e_*J. Since $F^e_*H^d_{(\underline{x})}(S)$ is faithfully flat over F^e_*R by Lemma 7.1, we have an injection:

$$(F^e_*R/F^e_*J) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \hookrightarrow E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S).$$

The image of $F^e_*1 \otimes F^e_*(\frac{cv^{p^e}}{x_1^{p^e} \cdots x_d^{p^e}})$ under this map is precisely $u \otimes F^e_*1 \otimes F^e_*(\frac{cv^{p^e}}{x_1^{p^e} \cdots x_d^{p^e}})$. Thus to show the latter one is nonzero for some $e > 0$, it is enough to show $F^e_*1 \otimes F^e_*(\frac{cv^{p^e}}{x_1^{p^e} \cdots x_d^{p^e}}) \neq 0$ for some $e > 0$. But we have

$$(F^e_*R/F^e_*J) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \twoheadrightarrow (F^e_*R/F^e_*\mathfrak{m}) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \cong F^e_*(H^d_{(\underline{x})}(S/\mathfrak{m}S)).$$

Thus it is enough to show that $F^e_*(\frac{cv^{p^e}}{x_1^{p^e} \cdots x_d^{p^e}}) \neq 0$ in $F^e_*(H^d_{(\underline{x})}(S/\mathfrak{m}S))$ for some $e > 0$, that is, $\frac{cv^{p^e}}{x_1^{p^e} \cdots x_d^{p^e}} \neq 0$ in $H^d_{(\underline{x})}(S/\mathfrak{m}S)$ for some $e > 0$. But $S/\mathfrak{m}S$ is strongly F-regular, and hence F-rational by Theorem 4.6. Therefore since $\frac{v}{x_1 \cdots x_d} \neq 0$, $\frac{cv^{p^e}}{x_1^{p^e} \cdots x_d^{p^e}} = cF^e(\frac{v}{x_1 \cdots x_d}) \neq 0$ in $H^d_{(\underline{x})}(S/\mathfrak{m}S)$ for some $e > 0$ as desired. □

7.2. **The case of F-rational and F-injective singularities.** We next prove the general base change result on F-injective and F-rational singularities. We slightly deviate from the historical discoveries of these results: we first prove the F-injective case, which is due to Datta–Murayama quite recently [DM19], and then we will make use of the F-injective case along with other techniques to establish the result on base change of F-rationality.

Theorem 7.6. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a flat local extension of rings of prime characteristic $p > 0$ such that R is F-injective and $S/\mathfrak{m}S$ is Cohen-Macaulay and geometrically F-injective over k. Then S is F-injective.*

Proof. Let $\underline{x} := x_1, \ldots, x_d$ be a system of parameters of $S/\mathfrak{m}S$. We first claim the following:

Claim 7.7. *For any Artinian R-module M, the map $F^e_* M \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(M \otimes_R H^d_{(\underline{x})}(S))$ sending $F^e_* m \otimes \eta \to F^e_*(m \otimes F^e(\eta))$ is injective for all $e > 0$, where $F^e(-)$ is the natural Frobenius action on $H^d_{(\underline{x})}(S)$.*

Proof. By taking a direct limit, it suffices to prove the claim for all R-modules of finite length. Moreover, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence, then since $F^e_*(-)$ and $\otimes_R H^d_{(\underline{x})}(S)$ are both exact (by Lemma 7.1), we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^e_* M_1 \otimes_R H^d_{(\underline{x})}(S) \arrow[r] \arrow[d] & F^e_* M_2 \otimes_R H^d_{(\underline{x})}(S) \arrow[r] \arrow[d] & F^e_* M_3 \otimes_R H^d_{(\underline{x})}(S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^e_*(M_1 \otimes_R H^d_{(\underline{x})}(S)) \arrow[r] & F^e_*(M_2 \otimes_R H^d_{(\underline{x})}(S)) \arrow[r] & F^e_*(M_3 \otimes_R H^d_{(\underline{x})}(S)) \arrow[r] & 0.
\end{tikzcd}$$

Thus to prove the claim for M_2, it is enough to prove it for M_1 and M_3. So by induction on the length of M, it is enough to prove the claim for $M = k$. But we have the following commutative diagram

$$\begin{tikzcd}
F^e_* k \otimes_R H^d_{(\underline{x})}(S) \arrow[r] \arrow[d, "\cong"] & F^e_*(k \otimes_R H^d_{(\underline{x})}(S)) \arrow[d, "\cong"] & \\
H^d_{(\underline{x})}(F^e_* k \otimes_k S/\mathfrak{m}S) \arrow[r] & F^e_*(H^d_{(\underline{x})}(S/\mathfrak{m}S)) \arrow[r] & F^e_*(H^d_{(\underline{x})}(F^e_* k \otimes_k S/\mathfrak{m}S))
\end{tikzcd}.$$

The composition map in the second row is injective, because it is a direct limit of the natural Frobenius map $H^d_{(\underline{x})}(k' \otimes_k S/\mathfrak{m}S) \to F^e_*(H^d_{(\underline{x})}(k' \otimes_k S/\mathfrak{m}S))$ (where k' is a finite extension of k in $F^e_* k$), which is injective since $S/\mathfrak{m}S$ is geometrically F-injective over k. Thus the map in the first row is injective as desired. $\square$

Now Claim 7.7 implies that the natural map

$$F^e_* H^i_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(H^i_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S)) \cong F^e_* H^{i+d}_{\mathfrak{n}}(S)$$

is injective (the last isomorphism follows from Lemma 7.1 and a simple computation using the spectral sequence $H^i_{\mathfrak{m}}H^j_{(\underline{x})}(S) \Rightarrow H^{i+j}_{\mathfrak{n}}(S)$). But $H^i_{\mathfrak{m}}(R) \to F^e_* H^i_{\mathfrak{m}}(R)$ is injective since R is injective, thus as $H^d_{(\underline{x})}(S)$ is faithfully flat over R by Lemma 7.1, we know that

$$H^{i+d}_{\mathfrak{n}}(S) \cong H^i_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_* H^i_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S)$$

is injective. Composing the two maps we find that $H^{i+d}_{\mathfrak{n}}(S) \to F^e_* H^{i+d}_{\mathfrak{n}}(S)$ is injective for all i (we leave it to the reader to check that this map is precisely the natural Frobenius action on $H^{i+d}_{\mathfrak{n}}(S)$). Thus S is F-injective. $\square$

It will take us considerable effort to prove the corresponding base change result for F-rationality. We first prove a special case, that is, when $S/\mathfrak{m}S$ is geometrically regular. This result was originally obtained by Vélez [Vél95] (which extended some results in [HH94a]).

Theorem 7.8. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a flat local extension of excellent rings of prime characteristic $p > 0$ such that R is F-rational and $S/\mathfrak{m}S$ is geometrically regular over k. Then S is F-rational.*

Proof. Since $S/\mathfrak{m}S$ is geometrically regular over k (so clearly Cohen-Macaulay and geometrically F-injective over k), by Claim 7.7 we know that

$$F^e_* H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S))$$

is injective for all $e > 0$, where $n = \dim(R)$ and $d = \dim(S/\mathfrak{m}S)$.

Furthermore, since R is excellent and $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ is flat local with $S/\mathfrak{m}S$ geometrically regular over k, $\kappa(P) \otimes_R S$ is geometrically regular over $\kappa(P)$ for all $P \in \operatorname{Spec}(R)$ by [And74, Thm on page 297]. In particular, there exists $0 \neq c \in R$ such that R_c and S_c are both regular (note that R is a domain by Proposition 4.4 and thus c is not in any minimal prime of S since $R \to S$ is flat). Now since R is F-rational, there exists $e > 0$ such that

$$H^n_{\mathfrak{m}}(R) \to F^e_* H^n_{\mathfrak{m}}(R) \xrightarrow{\cdot F^e_* c} F^e_* H^n_{\mathfrak{m}}(R)$$

is injective. This injection is preserved after tensoring with $H^d_{(\underline{x})}(S)$ since the latter is flat over R by Lemma 7.1, and thus the composition

$$H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_* H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \xrightarrow{\cdot F^e_* c} F^e_* H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S))$$

is injective. After identifying $H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S)$ with $H^{n+d}_{\mathfrak{n}}(S)$ (again, this follows from Lemma 7.1 and the spectral sequence $H^i_{\mathfrak{m}}H^j_{(\underline{x})}(S) \Rightarrow H^{i+j}_{\mathfrak{n}}(S)$), the above injection is precisely

$$H^{n+d}_{\mathfrak{n}}(S) \to F^e_* H^{n+d}_{\mathfrak{n}}(S) \xrightarrow{\cdot F^e_* c} H^{n+d}_{\mathfrak{n}}(S).$$

Since S is excellent Cohen-Macaulay and S_c is regular (and c is not in any minimal prime of S), S is F-rational by Theorem 6.16. $\square$

The above theorem allows us to prove the following criterion for F-rationality. This is a full generalization of Proposition 6.3 and Theorem 6.16.

Theorem 7.9. *Let $(R, \mathfrak{m}, k)$ be an excellent Cohen-Macaulay local ring of prime characteristic $p > 0$. Suppose there exists c not in any minimal prime of R such that R_c is F-rational. Then R is F-rational if and only if there exists $e > 0$ such that the composition*

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_*R) \xrightarrow{\cdot F^e_*c} H^d_{\mathfrak{m}}(F^e_*R)$$

is injective.

Proof. Since $R_c \to \widehat{R}_c$ has geometrically regular fibers (as R is excellent), we know that $\widehat{R}_c$ is F-rational by Theorem 7.8. It follows that for sufficiently small choices of Γ, $\widehat{R}^\Gamma_c$ is F-rational by Lemma 6.14. Moreover, since

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_*R) \xrightarrow{\cdot F^e_*c} H^d_{\mathfrak{m}}(F^e_*R)$$

is injective, it follows that for sufficiently small choices of Γ,

$$H^d_{\mathfrak{m}}(\widehat{R}^\Gamma) \to F^e_*H^d_{\mathfrak{m}}(\widehat{R}^\Gamma) \xrightarrow{\cdot F^e_*c} F^e_*H^d_{\mathfrak{m}}(\widehat{R}^\Gamma)$$

is injective by Lemma 6.13. Since $\widehat{R}^\Gamma$ is F-finite and $\widehat{R}^\Gamma_c$ is F-rational (and c is not in any minimal prime of $\widehat{R}^\Gamma$ since $R \to \widehat{R}^\Gamma$ is flat), Proposition 6.3 shows that $\widehat{R}^\Gamma$ is F-rational. But then since $R \to \widehat{R}^\Gamma$ is faithfully flat, R is F-rational by Exercise 20. $\square$

We can also extend Proposition 6.4 to the case of excellent local rings, this was also originally proved by Vélez [Vél95].

Theorem 7.10. *Let $(R, \mathfrak{m}, k)$ be an excellent local ring of prime characteristic $p > 0$. Then the F-rational locus of* $\operatorname{Spec}(R)$ *is open.*

Proof. By Corollary 6.15, we know that the F-rational locus of $\operatorname{Spec}(\widehat{R})$ is open. Let $V(I) \subseteq \operatorname{Spec}(\widehat{R})$ be the non-F-rational locus where $I \subseteq \widehat{R}$ is a radical ideal. We claim that the non-F-rational locus of $\operatorname{Spec}(R)$ is precisely $V(I \cap R)$.

To see this, first note that if $P \in \operatorname{Spec}(R)$ such that P does not contain $I \cap R$, then any prime $Q \in \operatorname{Spec}(\widehat{R})$ lying over P does not contain I and thus $\widehat{R}_Q$ is F-rational, which implies R_P is F-rational by Exercise 20 since $R_P \to \widehat{R}_Q$ is faithfully flat.

Now suppose $P \in \operatorname{Spec}(R)$ contains $I \cap R$, we want to show R_P is not F-rational. Write $I = Q_1 \cap \cdots \cap Q_n$ where $Q_1, \ldots, Q_n$ are minimal primes of I. Then $I \cap R = P_1 \cap \cdots \cap P_n$

where $P_i = Q_i \cap R$. Since $I \cap R \subseteq P$, we know $P_i \subseteq P$ for some i. If R_P is F-rational, then R_{P_i} is F-rational by Theorem 4.14. But then as Q_i contracts to P_i and R is excellent, $R_{P_i} \to \widehat{R}_{Q_i}$ is a faithfully flat extension of excellent local rings with geometrically regular fibers. Thus Theorem 7.8 implies that $\widehat{R}_{Q_i}$ is F-rational, which is a contradiction to $I \subseteq Q_i$ (recall that $V(I)$ is the non-F-rational locus of $\operatorname{Spec}(\widehat{R})$). □

Remark 7.11. In fact, the proof of Theorem 7.10 follows from a more general result: if $R \to S$ is a faithfully flat extension and $U \subseteq \operatorname{Spec}(R)$, then U is open if and only if the pre-image of U in $\operatorname{Spec}(S)$ is open, see [sta16, Lemma 29.25.12].

The behavior of F-rational singularities under flat local extension was studied extensively by Enescu [Ene00] and Aberbach–Enescu [AE03] (which extends some results in [HH94a, HH94c, Vél95]). The theorem we present here seems to be most general version, and was originally proved in [AE03] using sophisticated arguments involving tight closure. Our treatment, based on similar ideas, is more streamlined.

Theorem 7.12. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a flat local extension of excellent rings of prime characteristic $p > 0$ such that R is F-rational and $S/\mathfrak{m}S$ is geometrically F-rational over k. Then S is F-rational.*

Proof. Since $S/\mathfrak{m}S$ is F-rational, it is a normal domain by Proposition 4.4. Thus $\mathfrak{m}S$ is a prime ideal in S. We first show that $S' := S_{\mathfrak{m}S}$ is F-rational: since $S/\mathfrak{m}S$ is geometrically F-rational over k, we know that $R \to S'$ is a flat local extension such that $S'/\mathfrak{m}S'$ is geometrically F-rational and thus geometrically regular over k (since $\dim(S'/\mathfrak{m}S')$=0) and so by Theorem 7.8, S' is F-rational.

Since S is an excellent local domain and $S_{\mathfrak{m}S}$ is F-rational, by Theorem 7.10 we know that there exists $c \notin \mathfrak{m}S$ such that S_c is F-rational. Note that c is a nonzerodivisor on $S/\mathfrak{m}S$ and thus it is a nonzerodivisor on S by Lemma 7.1, in particular, c is not in any minimal prime of S. Let $\underline{x} := x_1, \ldots, x_d$ be a system of parameters of $S/\mathfrak{m}S$. In analogy with Claim 7.7, we have the following.

Claim 7.13. *For any Artinian R-module M, the map $F^e_* M \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(M \otimes_R H^d_{(\underline{x})}(S))$ sending $F^e_* m \otimes \eta \to F^e_*(m \otimes cF^e(\eta))$ is injective for some $e > 0$, where $F^e(-)$ is the natural Frobenius action on $H^d_{(\underline{x})}(S)$.*

Proof. This follows from the same argument as in Claim 7.7, using $S/\mathfrak{m}S$ is geometrically F-rational instead of geometrically F-injective. □

As a consequence, we see that there exists $e > 0$ such that the map $F^e_* H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S))$ sending $F^e_*\eta' \otimes \eta \to F^e_*(\eta' \otimes cF^e(\eta))$ is injective. But since R is F-injective and $H^d_{(\underline{x})}(S)$ is faithfully flat over R (see Lemma 7.1), composing this injection with the injection

$$H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_* H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S)$$

and using that $H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \cong H^{n+d}_{\mathfrak{n}}(S)$, we find that $H^{n+d}_{\mathfrak{n}}(S) \to F^e_* H^{n+d}_{\mathfrak{n}}(S)$ sending η to $F^e_*(cF^e(\eta))$ is injective. Since S is excellent Cohen-Macaulay and S_c is F-rational (and c is not in any minimal prime of S), by Theorem 7.9, we see that S is F-rational. $\square$

Exercise 33. With notation as in Discussion 6.6, prove that if R_Q is F-injective (resp., F-pure), then $R^\Gamma_{Q^\Gamma}$ is F-injective (resp., F-pure) for all sufficiently small choices of Γ. Furthermore, prove that the F-injective (resp., F-pure) locus of $\operatorname{Spec}(R)$ can be identified with the F-injective (resp., F-pure) locus of $\operatorname{Spec}(R^\Gamma)$ for all sufficiently small choices of Γ. (Hint: Mimic the proof of Lemma 6.14: in the F-injective case use Lemma 6.13, while in the F-pure case, Theorem 7.4 could be helpful.)

Exercise 34. Let $(R, \mathfrak{m}, k)$ be an excellent local ring of prime characteristic $p > 0$. Prove that the F-pure and F-injective locus of $\operatorname{Spec}(R)$ are open. (Hint: First mimic the proof of Theorem 7.10, replacing the use of Theorem 7.8 by using Theorem 7.4 and Theorem 7.6, to reduce to the case that R is complete. Then mimic the proof of Corollary 6.15 by using Exercise 30 and Exercise 33.)

The ideal I_e that shows up in the proof of Theorem 7.5 plays an important role in the study of F-singularities (e.g., see Chapter 9).

Exercise 35. Let $(R, \mathfrak{m}, k)$ be an F-finite ring of prime characteristic $p > 0$. Let E_R be the injective hull of the residue field and let u be a socle representative. Recall that $F^e_* I_e := \operatorname{Ann}_{E_R \otimes_R F^e_* R}(u \otimes F^e_* 1)$. Prove that

$$I_e = \{r \in R \mid \text{for all } \phi \in \operatorname{Hom}_R(F^e_* R, R), \phi(F^e_* r) \in \mathfrak{m}\}.$$

It is natural to ask whether we can drop "geometrically" in Theorem 7.8 or Theorem 7.12. This is unfortunately not known, in fact, the following question is open.

Open Problem 4. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a flat local extension of excellent rings of prime characteristic $p > 0$ such that R is F-rational and $S/\mathfrak{m}S$ is regular. Then is S also F-rational?*

On the other hand, it is known that we cannot drop "geometrically" in Theorem 7.6 (though there are no known examples with R normal), this is due to Enescu [Ene09, Proposition 4.2], which was based on [EH08, Example 2.16]. We leave the details in the next couple of exercises.

Exercise 36. Let K be an F-finite field of prime characteristic $p > 0$ and let $K \to L$ be a finite field extension that is not separable such that $L^p \cap K = K^p$. Let x be an indeterminate and let $R = K + xL[[x]] \subseteq L[[x]]$. Prove the following:

(1) R is a (Noetherian) complete local domain with $\dim(R) = 1$.
(2) R is F-injective.
(3) $K^{1/p} \otimes_K R$ is not reduced, and hence not F-injective.

In particular, $R \to S := K^{1/p} \otimes_K R$ is a flat local extension such that R is F-injective and the closed fiber is a field, but S is not F-injective.

Exercise 37. Let k be a perfect field of prime characteristic $p > 0$. Set $K = k(u, v)$ and $L = K[y]/(y^{2p} + uy^p + v)$. Prove that K, L satisfy the assumptions of Exercise 36.

Remark 7.14. Though the results in this chapter are stated for local rings, one can immediately deduce the corresponding global results (as these F-singularities are local properties). Namely, if $R \to S$ is a faithfully flat extension such that R is F-pure (resp., F-injective, excellent and F-rational, F-finite and strongly F-regular) and all fibers of $R \to S$ are Gorenstein and F-pure (resp., Cohen-Macaulay and geometrically F-injective, geometrically F-rational, Gorenstein F-finite and strongly F-regular), then S is F-pure (resp., F-injective, F-rational if S is excellent, strongly F-regular if S is F-finite).

8. Examples

We start this chapter with a quick summary of the relations between the F-singularities we have introduced so far (all the arrows that go to strongly F-regular also require F-finite assumption as usual):

$$\begin{tikzcd}
\text{regular} \arrow[r, Rightarrow] & \text{strongly } F\text{-regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F\text{-rational} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[l, Rightarrow, bend right, "+\text{Gorenstein}"'] & \text{normal} \\
& F\text{-pure} \arrow[r, Rightarrow] & F\text{-injective} \arrow[r, Rightarrow] \arrow[l, Rightarrow, bend right, "+\text{quasi-Gorenstein}"'] & \text{reduced}
\end{tikzcd}$$

A natural question one might ask is that whether there are other implications between these F-singularities: for example, whether there are relations between F-rational and F-pure singularities. However, Watanabe [Wat91] constructed examples of F-rational rings that are not F-pure, and examples of F-rational and F-pure rings that are not strongly F-regular. To study these examples, we first prove a very useful criterion of F-rationality for graded rings [FW89] (the analogous criterion for rational singularities in characteristic 0 was proved by Watanabe [Wat83]).

Theorem 8.1 (Fedder–Watanabe's criterion)**.** *Let R be an $\mathbb{N}$-graded ring over a field k of prime characteristic $p > 0$ with homogeneous maximal ideal $\mathfrak{m}$. Then R is F-rational if and only if*

(1) R is Cohen-Macaulay.

(2) R_P is F-rational for all homogeneous prime $P \neq \mathfrak{m}$.

(3) $a(R) := \max\{n | H^d_{\mathfrak{m}}(R)_n \neq 0\} < 0$.

(4) R is F-injective.

Proof. If R is F-rational, then (1) and (4) clearly hold, (2) holds since F-rationality localizes by Theorem 4.14, and (3) holds by Exercise 22.

Now we suppose R satisfies (1) – (4) and we want to prove R is F-rational. We first assume R is F-finite, that is, k is an F-finite field. Note that R has a canonical module ω_R such that $(\omega_R)_P \cong \omega_{R_P}$ for all $P \in \operatorname{Spec}(R)$. Moreover we can choose ω_R such that it is graded (see [BS98, Chapter 14]). Similar to Discussion 6.1, we have a graded trace map $F^e_*\omega_R \xrightarrow{\mathrm{Tr}^e} \omega_R$, and it is easy to verify that the graded analogs of Proposition 6.2 and Proposition 6.3 (the statments involving ω_R) hold in this set up.

Condition (2) implies R_P is a field for all minimal primes of R, so there exists a homogeneous $c \in R$ not in any minimal prime of R such that R_c is regular. By condition (2) again, for each homogenous prime $P \neq \mathfrak{m}$, there exists $e > 0$ such that $F^e_*(c\omega_R)_P \xrightarrow{\mathrm{Tr}^e} (\omega_R)_P$

is surjective. Thus there exists a homogenous $f_P \notin P$ such that $F^e_*(c\omega_R)_{f_P} \xrightarrow{\mathrm{Tr}^e} (\omega_R)_{f_P}$ is surjective. At this point, we note that $\cup D(f_P) = \mathrm{Spec}(R) - \{\mathfrak{m}\}$ where the union is taken over all homogenous primes $P \neq \mathfrak{m}$. Since $\mathrm{Spec}(R) - \{\mathfrak{m}\}$ is quasi-compact, there exists a finite collection $\{f_1, \ldots, f_n\}$ that generates $\mathfrak{m}$ up to radical such that for each f_i there is an associated e_i such that $F^{e_i}_*(c\omega_R)_{f_i} \xrightarrow{\mathrm{Tr}^{e_i}} (\omega_R)_{f_i}$ is surjective. Pick $e \gg e_i$ for all i, it follows that $F^e_*(c\omega_R)_{f_i} \xrightarrow{\mathrm{Tr}^e} (\omega_R)_{f_i}$ is surjective for all f_i.[8] But then we know that

$$\mathrm{Coker}(F^e_*(c\omega_R) \xrightarrow{\mathrm{Tr}^e} \omega_R)$$

is a graded finite length module supported only at $\mathfrak{m}$. It is enough to show that this cokernel is 0, since then $F^e_*(c\omega_R) \xrightarrow{\mathrm{Tr}^e} \omega_R$ is surjective, and by the graded analog of Proposition 6.3 we will be done.

But the graded Matlis dual of this cokernel is $N_e = \{\eta \in H^d_{\mathfrak{m}}(R) \mid cF^e(\eta) = 0\}$. For $e \gg 0$, we know that N_e is a graded F-stable submodule of $H^d_{\mathfrak{m}}(R)$, see the proof of Proposition 4.11, where we need to use the Artinianness of $H^d_{\mathfrak{m}}(R)$ and that R is F-injective by (4). But then by (4) again, any graded F-stable submodule of finite length must concentrate in degree 0, but then it vanishes by (3). We have completed the proof when k is an F-finite field.

Finally, if k is not F-finite, we can replace k by k^Γ (and R by $R^\Gamma := R \otimes_k k^\Gamma$) for Γ sufficiently small and run the above argument for R^Γ (it can be shown, in analogy with the local case, that (1) – (4) are preserved[9]). The outcome is that R^Γ is F-rational and hence R is F-rational by Exercise 20. $\square$

The rest of this chapter requires some knowledge of algebraic geometry, see [Har77, Chapter II and III]. We will present Watanabe's examples and we give more details than [Wat91]. We first collect some basic facts about section rings of divisors with rational coefficients. Let X be a normal projective variety over an algebraically closed field $k = \overline{k}$ and let D be an effective $\mathbb{Q}$-divisor such that mD is an ample Cartier divisor on X. Then

$$R = R(X, D) := \oplus_{n \geq 0} H^0(X, O_X(\lfloor nD \rfloor)) \cdot t^n$$

is a normal $\mathbb{N}$-graded ring over k. We can explicitly describe the graded canonical module of R and its symbolic powers using sheaf cohomology as follows (see [Wat91], which follows

[8]We leave it to the readers to check this carefully, the point is that $\mathrm{Tr}^e : F^e_*(\omega_R)_{f_i} \to (\omega_R)_{f_i}$ is surjective for all e since R_{f_i} is F-injective, so we can enlarge the e_i while preserving the surjectivity of $F^{e_i}_*(c\omega_R)_{f_i} \xrightarrow{\mathrm{Tr}^{e_i}} (\omega_R)_{f_i}$.

[9]Only (2) requires some work and we omit the details, as the argument is entirely similar as in the local case we carried out in Chapter 6. In fact, as we already mentioned before, the theory of Γ-construction can be extended to all rings essentially finite type over a complete local ring (e.g., a field), see [HH94a] for details. In the sequel we will apply Theorem 8.1 mainly in the case that k is perfect or algebraically closed.

from [Wat81] and [Dem88]):

$$\omega_R = \oplus_{n\in\mathbb{Z}} H^0(X, O_X(\lfloor K_X + D' + nD\rfloor))\cdot t^n, \tag{8.1}$$
$$\omega_R^{(q)} = \oplus_{n\in\mathbb{Z}} H^0(X, O_X(\lfloor q(K_X + D') + nD\rfloor))\cdot t^n,$$

where K_X is the canonical divisor of X and D' is defined as follows: if $D = \sum \frac{a_i}{b_i}E_i$ such that $(a_i, b_i) = 1$ and E_i's are prime divisors, then $D' = \sum \frac{b_i-1}{b_i}E_i$.

Example 8.2. Let $R = R(\mathbb{P}^1_k, D)$ where k is an algebraically closed field of prime characteristic $p > 0$ and let $D = \frac{1}{a}P_1 + \frac{1}{b}P_2 + \frac{1}{c}P_3$ be an effective $\mathbb{Q}$-divisor where P_1, P_2, P_3 are distinct points on $\mathbb{P}^1$. Then we have

(1) R is F-rational for all $a, b, c \geq 1$.
(2) R is not F-pure if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$.
(3) If $a = b = c = 3$, then R is F-pure if $p \equiv 1 \bmod 3$ but R is not strongly F-regular.

Proof. We first prove (1). We use Theorem 8.1. R is a two-dimensional normal $\mathbb{N}$-graded ring so it is Cohen-Macaulay and R_P is regular for all $P \neq \mathfrak{m}$. To see $a(R) < 0$, it is enough to show that $[\omega_R]_{\leq 0} = 0$, which follows from (8.1) as

$$[\omega_R]_n = H^0(\mathbb{P}^1, O(-2)\otimes O(\lfloor D' + nD\rfloor))\cdot t^n = 0$$

for $n \leq 0$. Finally we show R is F-injective. Let x be the parameter of $\mathbb{P}^1$ and let P_1, P_2, P_3 correspond to $(x-\alpha)$, $(x-\beta)$, $(x-\gamma)$. It is straightforward to check that R is generated by t, $y_1 := \frac{1}{x-\alpha}t^a$, $y_2 := \frac{1}{x-\beta}t^b$, $y_3 := \frac{1}{x-\gamma}t^c$. But then we observe that

$$R/tR \cong k[y_1, y_2, y_3]/(y_1y_2, y_1y_3, y_2y_3).$$

To see this, note that mod t, $y_1y_2 = \frac{1}{(x-\alpha)(x-\beta)}\cdot t^{a+b} = (\alpha-\beta)\cdot(\frac{1}{x-\beta}t^{a+b} - \frac{1}{x-\alpha}t^{a+b}) = 0$ and similarly $y_1y_3 = y_2y_3 = 0$. Hence R/tR is Cohen-Macaulay and F-pure and thus R is F-injective by Theorem 5.1. This completes the proof that R is F-rational.

We next prove (2) and (3). We note that the canonical map $E_R(k) \to F^e_*R\otimes E_R(k)$ can be identified with $H^2_\mathfrak{m}(\omega_R) \to F^e_*R\otimes_R H^2_\mathfrak{m}(\omega_R) \cong H^2_\mathfrak{m}(F^e_*\omega_R^{(p^e)})$, where the isomorphism results from the fact that the natural map $F^e_*R\otimes_R\omega_R \to F^e_*\omega_R^{(p^e)}$ is an isomorphism in codimension 1 (after we localize at height one primes, ω_R is a rank one free module). We then have the degree-preserving identifications:

$$\begin{array}{ccc} E_R(k)\cong H^2_\mathfrak{m}(\omega_R) & \longrightarrow & F^e_*R\otimes E_R(k)\cong H^2_\mathfrak{m}(F^e_*\omega_R^{(p^e)}) \\ \downarrow\cong & & \downarrow\cong \\ \oplus_{n\in\mathbb{Z}}H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor K_{\mathbb{P}^1} + D' + nD\rfloor))\cdot t^n & \longrightarrow & \oplus_{n\in\mathbb{Z}}H^1(\mathbb{P}^1, F^e_*O_{\mathbb{P}^1}(\lfloor p^e(K_{\mathbb{P}^1} + D') + nD\rfloor))\cdot t^n. \end{array}$$

It is easy to check that the socle of $E_R(k)$ corresponds to $H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor K_{\mathbb{P}^1} + D'\rfloor)) \cong k$ (the point is that all the degree > 0 piece vanish by a simple computation). Thus by Proposition 2.2, R is F-pure if and only if the map

$$H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor K_{\mathbb{P}^1} + D'\rfloor)) \to H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(\lfloor p(K_{\mathbb{P}^1} + D')\rfloor))$$

is injective. But if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$, then a simple computation shows that

$$\deg(\lfloor p(K_{\mathbb{P}^1} + D')\rfloor) = -2p + \lfloor p \cdot \frac{a-1}{a}\rfloor + \lfloor p \cdot \frac{b-1}{b}\rfloor + \lfloor p \cdot \frac{c-1}{c}\rfloor \geq -1$$

and thus $H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(\lfloor p(K_{\mathbb{P}^1} + D')\rfloor)) \cong H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor p(K_{\mathbb{P}^1} + D')\rfloor)) = 0$. Hence R is not F-pure if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$, which proves (2).

Finally, the same analysis (via Proposition 2.2) shows that R is strongly F-regular if and only if for any $0 \neq f \in H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor nD\rfloor))$, there exists $e > 0$ such that the composition:

$$H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor K_{\mathbb{P}^1}+D'\rfloor)) \to H^1(\mathbb{P}^1, F^e_*O_{\mathbb{P}^1}(\lfloor p^e(K_{\mathbb{P}^1}+D')\rfloor)) \xrightarrow{\cdot F^e_* f} H^1(\mathbb{P}^1, F^e_*O_{\mathbb{P}^1}(\lfloor p^e(K_{\mathbb{P}^1}+D')+nD\rfloor))$$

is injective. Now if $a = b = c = 3$, then again a simple computation shows that for n large,

$$\deg(\lfloor p^e(K_{\mathbb{P}^1} + D') + nD\rfloor) = -2p^e + 3\lfloor \frac{2}{3}p^e + \frac{1}{3}n\rfloor \geq -1$$

for all $e > 0$ and thus $H^1(\mathbb{P}^1, F^e_*O_{\mathbb{P}^1}(\lfloor p^e(K_{\mathbb{P}^1} + D') + nD\rfloor)) = 0$. Hence R is not strongly F-regular. On the other hand, if $p \equiv 1 \bmod 3$, then one checks that

$$H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(\lfloor p(K_{\mathbb{P}^1} + D')\rfloor)) \cong H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(-2)),$$

and if we use $[z_0 : z_1]$ to denote the coordinate of $\mathbb{P}^1$, then the induced map $H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(-2)) \to H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(\lfloor p(K_{\mathbb{P}^1} + D')\rfloor))$ can be described as

$$\frac{1}{z_0z_1} \to \frac{(z_0 - \alpha z_1)^{\lfloor p\cdot\frac{a-1}{a}\rfloor}(z_0 - \beta z_1)^{\lfloor p\cdot\frac{b-1}{b}\rfloor}(z_0 - \gamma z_1)^{\lfloor p\cdot\frac{c-1}{c}\rfloor}}{z_0^p z_1^p} = \frac{u}{z_0z_1} \in H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(-2))$$

where $0 \neq u \in k$. Thus the map is injective and hence R is F-pure. $\square$

Remark 8.3. One can write some concrete examples: for instance let $P_1 = \infty$, $P_2 = 0$, $P_3 = 1$ and $a = b = c = 4$, then $R \cong k[t, xt^4, x^{-1}t^4, (x-1)^{-1}t^4]$ is F-rational but not F-pure, while if we take $a = b = c = 3$ and $p \equiv 1 \bmod 3$, then $R \cong k[t, xt^3, x^{-1}t^3, (x-1)^{-1}t^3]$ is F-rational and F-pure but not strongly F-regular. We can complete at the homogenous maximal ideal to obtain examples of complete local domains.

We next give Watanabe's example that direct summand of F-rational rings are not necessarily F-injective. Our construction slightly differs from [Wat97], and in fact, we will adapt our construction with recent work of Kovács [Kov18] (which originates from [LR97]) to obtain also an example of a direct summand of F-rational ring that is not Cohen-Macaulay. We begin with the following proposition.

Proposition 8.4. *Let R be an equidimensional $\mathbb{N}$-graded ring over a field k of prime characteristic $p > 0$ with homogeneous maximal ideal $\mathfrak{m}$ such that*

(1) R_P is regular for all homogeneous prime $P \neq \mathfrak{m}$.
(2) $a_i(R) := \max\{n | H^i_{\mathfrak{m}}(R)_n \neq 0\} < 0$ for each i.
(3) $H^0_{\mathfrak{m}}(R) = H^1_{\mathfrak{m}}(R) = 0$.

Then R is a direct summand of an F-rational ring.

Proof. Let $S_n = k[x_1, \ldots, x_n]$ be a standard graded (i.e., $\deg(x_i) = 1$) polynomial ring and let $T_n = R\#S_n$ be the Segre product, that is, $T_n = \oplus_{j \geq 0}(R_j \otimes_k (S_n)_j)$. Then R is a direct summand of T_n: we can map R to T_n by sending $r \in [R]_j$ to $r\#x_1^j$ and the map $S_n \to k[x_1]$ sending x_i to 0 for all $i \geq 2$ induces a splitting $T_n \to R\#k[x_1] \cong R$.

We claim that T_n is F-rational for all $n \gg 0$ and we use Theorem 8.1. We use the following formula to compute the local cohomology of Segre product [GW78, Theorem 4.1.5.]:

$$H^i_{\mathfrak{m}}(T_n) = H^i_{\mathfrak{m}}(R\#S_n) \cong R\#H^i_{\mathfrak{m}}(S_n) \oplus H^i_{\mathfrak{m}}(R)\#S_n \oplus \left(\oplus_{a+b=i+1} H^a_{\mathfrak{m}}(R)\#H^b_{\mathfrak{m}}(S_n)\right)$$

where we abuse notation and use $\mathfrak{m}$ to denote the corresponding homogeneous maximal ideals of R, S_n, and T_n respectively. Set $d = \dim(R)$. By assumption (2), we know that $[H^j_{\mathfrak{m}}(R)]_{\geq 0} = 0$ for all j and therefore $R\#H^i_{\mathfrak{m}}(S_n) = H^i_{\mathfrak{m}}(R)\#S_n = 0$ for all i and n. Therefore the only possible nonzero contributions for the local cohomology modules of T_n are the modules of the form $H^i_{\mathfrak{m}}(R)\#H^n_{\mathfrak{m}}(S_n)$. In particular, $H^{i+n-1}_{\mathfrak{m}}(T_n) \cong H^i_{\mathfrak{m}}(R)\#H^n_{\mathfrak{m}}(S_n)$ for all integers i as $H^j_{\mathfrak{m}}(S_n) = 0$ for all $j \neq n$. Since R is equidimensional and R_P is regular for all homogeneous prime $P \neq \mathfrak{m}$ by assumption (1), we know that $H^i_{\mathfrak{m}}(R)$ has finite length for all $i < d$ by the graded version of Lemma 4.5. Hence $H^i_{\mathfrak{m}}(R)$ only lives in finitely many (negative) degrees. Even further, the module $H^n_{\mathfrak{m}}(S_n)$ is supported in degrees no more than $-n$ and so for each $i < d$ and $n \gg 0$,

$$H^{i+n-1}_{\mathfrak{m}}(T_n) \cong H^i_{\mathfrak{m}}(R)\#H^n_{\mathfrak{m}}(S_n) = 0$$

and thus T is Cohen-Macaulay. Moreover, since $H^{d+n-1}_{\mathfrak{m}}(T) \cong H^d_{\mathfrak{m}}(R)\#H^n_{\mathfrak{m}}(S)$, we have $a(T) \leq \min\{a(R), a(S)\} < 0$.

We next show that T is F-rational for all homogeneous primes $P \neq \mathfrak{m}_T$. If we invert a homogenous element $r\#s \in \mathfrak{m}_T$, then $T_{r\#s}$ is a direct summand of $(R \otimes_k S)_{r\otimes s}$ (since T is a direct summand of $R \otimes_k S$). But $(R \otimes_k S)_{r\otimes s} \cong R[x_1, \ldots, x_n]_{rs}$ is a localization of

$R_r[x_1, \ldots, x_n]$, hence regular (because R_r is regular by assumption (1)). Therefore $T_{r\#s}$ is a direct summand of a regular ring and thus F-rational by Exercise 24.

Finally we show that T is F-injective. Our assumptions on R implies (see the proof of Theorem 8.1) that the largest proper F-stable submodule of $H^d_{\mathfrak{m}}(R)$ has finite length. In particular, there exists $m \gg 0$ such that the Frobenius action on $[H^d_{\mathfrak{m}}(R)]_{\leq m}$ is injective. Now for $n \geq m$, the Frobenius action on $H^{d+n-1}_{\mathfrak{m}}(T) \cong H^d_{\mathfrak{m}}(R)\#H^n_{\mathfrak{m}}(S) = [H^d_{\mathfrak{m}}(R)]_{\leq m}\#[H^n_{\mathfrak{m}}(S)]_{\leq m}$ is injective. □

Remark 8.5. Suppose that the field k in Proposition 8.4 is assumed to be an F-finite field. Then the hypothesis that R_P is a regular ring for all $P \neq \mathfrak{m}$ can be relaxed to the milder assumption that R_P is a strongly F-regular ring for all $P \neq \mathfrak{m}$. The proof would not need to be significantly altered and we encourage the reader to verify our claim.

Example 8.6. Perhaps the simplest example of a non-F-rational ring that is a direct summand of an F-rational ring is $R = \mathbb{F}_2[x,y,z]/(x^2+y^3+z^5)$ with $\deg(x) = 15$, $\deg(y) = 10$, and $\deg(z) = 6$. Since R is a two-dimensional normal domain with $a(R) = -1$, it satisfies all the conditions in Proposition 8.4 and thus R is a direct summand of an F-rational ring. However, it is a straightforward computation that the Čech class $[\frac{x}{yz}] \in H^2_{\mathfrak{m}}(R)$ is nonzero, but $F([\frac{x}{yz}]) = [\frac{x^2}{y^2z^2}] = 0$ since $x^2 \in (y^2, z^2)R$. Hence R is not F-injective (thus not F-rational). Again, we can complete at $\mathfrak{m}$ to obtain examples of complete local domains.

We next exhibit an example of a direct summand of F-rational ring that is not Cohen-Macaulay.

Example 8.7. In [Kov18, Theorem 1.1 and Theorem 4.7], Kovács proved that there exists a smooth projective Fano variety X over a field of characteristic 2 such that $\dim(X) = 6$, ω_X^{-1} is very ample, $H^1(X, \omega_X^{-1}) \cong H^5(X, \omega_X^2)^\vee \neq 0$ (so ω_X^{-2} violates Kodaira vanishing), and $H^i(X, O_X) = 0$ for all $1 \leq i \leq 6$.[10] Now we let $S = \oplus_{n\geq 0} H^0(X, \omega_X^{-n})$. Since ω_X^{-1} is very ample, we know that S is a standard graded normal domain of dimension 7 with homogenous maximal ideal $\mathfrak{m}$ such that

$$[H^{i+1}_{\mathfrak{m}}(S)]_n = H^i(X, \omega_X^{-n}) \text{ for all } n \in \mathbb{Z} \text{ and all } 1 \leq i \leq 6.$$

Set $t = \max\{n | [H^{i+1}_{\mathfrak{m}}(S)]_n \neq 0 \text{ for some } 1 \leq i \leq 6\}$. Then as $H^1(X, \omega_X^{-1}) \neq 0$, we know that $t \geq 1$. Let $R = S^{(t+1)}$ be the $(t+1)$-th Veronese subring of S.

Claim 8.8. *We have $a_i(R) < 0$ for all i and R is not Cohen-Macaulay.*

[10]The fact that $H^i(X, O_X) = 0$ for all $1 \leq i \leq 6$ is mentioned on [Kov18, top of page 2], and can be easily verified since the X constructed in [Kov18, Theorem 1.1] is certain $\mathbb{P}^n$-bundle over a $\mathbb{P}^n$-bundle over a projective space, so $H^{>0}(X, O_X) = 0$ follows as the same is true for projective space.

Proof. Since R is the $(t+1)$-th Veronese subring of S, it is easy to check that

$$[H^i_{\mathfrak{m}}(R)]_n = \begin{cases} 0 & \text{if } (t+1) \nmid n \\ [H^i_{\mathfrak{m}}(S)]_n & \text{if } (t+1) | n. \end{cases}$$

It follows that $a_i(R) < 0$ for all i by our choice of t (note that $H^0_{\mathfrak{m}}(R) = H^1_{\mathfrak{m}}(R) = 0$ and $[H^i_{\mathfrak{m}}(R)]_0 = [H^i_{\mathfrak{m}}(S)]_0 = H^{i-1}(X, O_X) = 0$ for $i \geq 2$). To see that R is not Cohen-Macaulay, note that by our choice of t, $H^i(X, \omega_X^{-t}) \neq 0$ for some $1 \leq i \leq 5$ ($H^6(X, \omega_X^{-t}) = H^0(X, \omega_X^{t+1})^\vee = 0$ since ω_X is anti-ample and $t \geq 1$), which implies that $H^{6-i}(X, \omega_X^{t+1}) \neq 0$. Thus

$$[H^j_{\mathfrak{m}}(R)]_{-(t+1)} = [H^j_{\mathfrak{m}}(S)]_{-(t+1)} = H^{j-1}(X, \omega_X^{t+1}) \neq 0 \text{ for some } 2 \leq j \leq 6.$$

As $\dim(R) = \dim(S) = 7$, this shows that R is not Cohen-Macaulay. □

Finally, since $\mathrm{Proj}(R) = \mathrm{Proj}(S) = X$ is nonsingular and R can be viewed as a standard graded ring (as it is a Veronese subring of a standard graded ring), we know that R_P is regular for all $P \neq \mathfrak{m}$. This combined with Claim 8.8 shows that R satisfies the conditions of Proposition 8.4 and thus is R a direct summand of F-rational ring that is not Cohen-Macaulay.

We next give Singh's example [Sin99c] showing that if we drop the $\mathbb{Q}$-Gorenstein assumption on R, then R/xR is strongly F-regular does not even imply R is F-pure.

Example 8.9. Let m and n be positive integers satisfying $m - m/n > 2$. Consider the ring $R = k[a, b, c, d, t]/I$ where k is an F-finite field of characteristic $p > 2$ and I is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} a^2 + t^m & b & d \\ c & a^2 & b^n - d \end{pmatrix}.$$

Then t is a nonzerodivisor on R and the ring R/tR is strongly F-regular. But if p and m are relatively prime, then R is not F-pure.

Proof. Let $S = k[a, b, c, d, b', c', d', t]/J$ where J is the generic 2×3 matrix

$$\begin{pmatrix} c' & b & d \\ c & b' & d' \end{pmatrix}.$$

Obviously we have $R = S/(c' - a^2 - t^m, b' - a^2, d' - b^n + d)$ and $R/(t, c, d) \cong k[a, b]/(a^4, b^{n+1})$ is Artinian. Since S is Cohen-Macaulay of dimension 6, it follows that R is Cohen-Macaulay and t, c, d is a system of parameters of R. In particular, t is a nonzerodivisor on R. We note that

$$R/tR \cong k[a,b,c,d]/(a^4-bc, a^2(b^n-d)-cd, b(b^n-d)-a^2d).$$

Note that we can assign weights to the variables to make R/tR $\mathbb{N}$-graded. We first claim that R/tR is normal, and hence a domain because it is $\mathbb{N}$-graded. We know that R/tR is Cohen-Macaulay of dimension 2 and c, d is a homogeneous system of parameters, thus it is enough to show that $(R/tR)_c$ and $(R/tR)_d$ are regular. These are straightforward to check: $(R/tR)_d \cong k[a,b,d][\frac{1}{d}]/(\frac{b^{n+1}}{d}-b-a^2)$ and $(R/tR)_c \cong k[a,c,d][\frac{1}{c}]/(\frac{a^{4n+2}}{c^{n+1}}-\frac{a^2d}{c}-d)$ are both regular.

We next claim that R/tR is isomorphic to the $(2n+1)$-th Veronese subring of $k[a,x,y]/(a^2-xy(x^n-y))$ where the variables a, x, y have weights $2n+1, 2, 2n$ respectively. To see this, we define a map

$$R/tR \cong \frac{k[a,b,c,d]}{(a^4-bc, a^2(b^n-d)-cd, b(b^n-d)-a^2d)} \to (\frac{k[a,x,y]}{(a^2-xy(x^n-y))})^{(2n+1)}$$

by sending b, c, d to $xy^2, x(x^n-y)^2, y^{2n+1}$ respectively. One easily checks that the map is well-defined and is surjective: the Veronese subring is generated over k by $a, y^2x, x^{n+1}y, x^{2n+1}, y^{2n+1}$ and it is straightforward to check that all these generators are in the image (modulo the equation $a^2-xy(x^n-y)$). Now both rings have dimension 2 and we know that R/tR is a domain, it follows that the map is injective and hence an isomorphism.

To prove R/tR is strongly F-regular, it is enough to show that $k[a,x,y]/(a^2-xy(x^n-y))$ is strongly F-regular by Theorem 3.9. We now apply Exercise 16 with $c = x$ (since x is part of a system of parameters and after inverting x the ring becomes regular), it is enough to show that there exists $e > 0$ such that $x(a^2-xy(x^n-y))^{p^e-1} \notin (a^{p^e}, x^{p^e}, y^{p^e})$. Since $p > 2$, for $e = 1$, the term $a^{p-1}x^{\frac{p+1}{2}}y^{p-1}$ appears in $x(a^2-xy(x^n-y))^{p-1}$ with nonzero coefficient, this term is not in (a^p, x^p, y^p).

It remains to prove that R is not F-pure if p and m are relatively prime. The key is the following elementary but tricky computation.

Claim 8.10 ([Sin99c, Lemma 4.2]). *If s is a positive integer such that $s(m-m/n-2) \geq 1$, then*

$$(b^n t^{m-1})^{2ms+1} \in (a^{2ms+1}, d^{2ms+1}).$$

Proof. Let $\tau = a^2 + t^m$ and $\alpha = a^2$. It suffices to work in the polynomial ring $k[\tau, \alpha, b, c, d]$ and establish that

$$b^{n(2ms+1)}(\tau-\alpha)^{2s(m-1)} \in (\alpha^{ms+1}, d^{2ms+1}) + I'$$

where I' is the ideal generated by 2×2 minors of the matrix

$$\begin{pmatrix} \tau & b & d \\ c & \alpha & b^n - d \end{pmatrix}.$$

Taking the binomial expansion of $(\tau - \alpha)^{2s(m-1)}$, it is enough to show that for all $1 \leq i \leq ms + 1$, we have

$$b^{n(2ms+1)}\alpha^{ms+1-i}\tau^{ms-2s+i-1} \in (\alpha^{ms+1}, d^{2ms+1}) + I'.$$

Thus it is enough to show that

$$b^{n(2ms+1)}\tau^{ms-2s+i-1} \in (\alpha^i, d^{2ms+1}) + I'.$$

Since $\alpha d - b(b^n - d)$ and $b^n\tau - d(c+\tau)$ belongs to I', it suffices to establish that

$$b^{n(2ms+1)}\tau^{ms-2s+i-1} \in (b^i(b^n - d)^i, d^{2ms+1}, b^n\tau - d(c+\tau)).$$

Now we work modulo the element $b^i(b^n - d)^i$, we may reduce $b^{n(2ms+1)}$ to a polynomial in b and d such that the highest power of b that occurs is less than $i(n+1)$. Thus it suffices to show that

$$b^{n(2ms+1-j)}\tau^{ms-2s+i-1}d^j \in (d^{2ms+1}, b^n\tau - d(c+\tau))$$

where $n(2ms+1-j) < i(n+1)$, i.e., $j \geq 2ms + (1-i)(1+1/n)$. So it is enough to check

$$b^{n(2ms+1-j)}\tau^{ms-2s+i-1} \in (d^{2ms+1-j}, b^n\tau - d(c+\tau)).$$

At this point, it only needs to check that $ms - 2s + i - 1 \geq 2ms + 1 - j$, since modulo $b^n\tau - d(c+\tau)$, we can then express $b^{n(2ms+1-j)}\tau^{ms-2s+i-1}$ as a multiple of $d^{2ms+1-j}$. But

$$\begin{aligned} ms - 2s + i - 1 - (2ms + 1 - j) &= j - ms - 2s + i - 2 \\ &\geq ms + (1-i)(1+1/n) - 2s + i - 2 \\ &= ms - 2s + (1-i)/n - 1 \\ &\geq ms - 2s - (ms)/n - 1 \\ &= s(m - m/n - 2) - 1 \geq 0 \end{aligned}$$

where the second $\geq$ is because $i \leq ms + 1$ and the last $\geq$ follows from our assumption that $s(m - m/n - 2) \geq 1$. ◻

Finally, since p and m are relatively prime, $p > 2$, and $m - m/n > 2$ by our assumptions, there exists $e \gg 0$ and $s > 0$ such that $p^e = 2ms + 1$ and $s(m - m/n - 2) \geq 1$. Claim 8.10 then shows that $(b^n t^{m-1})^{p^e} \in (a^{p^e}, d^{p^e})$. If R is F-pure, then $R \to F^e_* R$ is pure and hence the induced Frobenius map $R/(a,d) \to F^e_*(R/(a^{p^e}, d^{p^e}))$ is injective. Thus $(b^n t^{m-1})^{p^e} \in (a^{p^e}, d^{p^e})$

implies $b^n t^{m-1} \in (a,d)$. But $R/(a,d) \cong k[b,c,t]/(b^{n+1}, b^n t^m, bc)$ and it is clear that $b^n t^{m-1} \neq 0$ in this ring, which is a contradiction. $\square$

Remark 8.11. Take $m = 5$, $n = 2$, $k = \mathbb{F}_3$ in Example 8.9, we have

$$R = \frac{\mathbb{F}_3[a,b,c,d,t]}{((a^2+t^5)a^2 - bc, (a^2+t^5)(b^2-d) - cd, b(b^2-d) - a^2 d)}$$

is not F-pure, but R/tR is strongly F-regular. We leave the interested and diligent reader to check these using Theorem 2.5 and Exercise 16.

Finally, we explain that generic determinantal rings over a field k are F-rational [HH94c], in fact strongly F-regular if k is F-finite.

Example 8.12. Let $S = k[x_{ij} | 1 \leq i \leq m, 1 \leq j \leq n]$ be a polynomial ring in $m \times n$ variables with $m \leq n$. Let I_t be the ideal of S generated by $t \times t$ minors of the matrix $[x_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$. Then $R = S/I_t$ is F-rational. Moreover, if k is F-finite then R is strongly F-regular.

Proof. We will use Theorem 8.1 to show R is F-rational. First of all, property (1) and (3) are well-known: for example see [HE71] or [BV88].

We now prove (2). For any homogeneous prime $P \neq \mathfrak{m}$, there exists $x_{ij} \notin P$. Without loss of generality we may assume $x_{11} \notin P$. After inverting the element x_{11}, we may perform row and column operations to transform our matrix:

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11} & 0 & \dots & 0 \\ 0 & x'_{22} & \dots & x'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x'_{m2} & \dots & x'_{mn} \end{bmatrix}$$

where $x'_{ij} = x_{ij} - \frac{x_{i1}x_{1j}}{x_{11}}$. The ideal $I_t S_{x_{11}}$ is generated by $(t-1) \times (t-1)$ minors of the second displayed matrix. Therefore,

$$R_{x_{11}} = S_{x_{11}}/I_t S_{x_{11}} \cong (S'/I'_{t-1})[x_{11}, \frac{1}{x_{11}}, x_{12}, \dots, x_{1n}, x_{21}, \dots, x_{m1}]$$

where $S' = k[x'_{ij} | 2 \leq i \leq m, 2 \leq j \leq n]$ and I'_{t-1} denotes the ideal generated by the $(t-1) \times (t-1)$ minors of the matrix $[x'_{ij}]$. By induction, we know that S'/I'_{t-1} is F-rational, thus so is $R_{x_{11}}$ by Theorem 7.8 and Theorem 4.14. Since R_P can be viewed as a localization of $R_{x_{11}}$, R_P is F-rational by Theorem 4.14 again.

It remains to prove (4). In fact the method below will also reprove (1) along the way we prove (4). We need the following result from combinatorial commutative algebra:

Theorem 8.13 ([Stu90])**.** *The $t \times t$ minors of $[x_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$ form a Gröbner basis of I_t with respect to the term order $x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots\cdots > x_{mn}$.*

At this point we follow the standard construction as in [Eis95, 15.16 and 15.17]. We choose an appropriate weight function λ such that $\mathrm{in}_\lambda(I_t) = \mathrm{in}_>(I_t)$. Let $\widetilde{I}$ be the λ-homogenization of I_t in $S[z]$. We have

$$(S[z]/\widetilde{I}) \otimes_{k[z]} k(z) \cong R \otimes_k k(z) \text{ and } (S[z]/\widetilde{I})/z \cong S/\mathrm{in}_>(I_t).$$

Therefore if we can show that $S/\mathrm{in}_>(I_t)$ is Cohen-Macaulay and F-injective, then so is $S[z]/\widetilde{I}$ by the graded version of Theorem 5.1. But then $R \otimes_k k(z)$ is Cohen-Macaulay and F-injective by Theorem 4.13 and so R is Cohen-Macaulay and F-injective by Exercise 20. But since the $t \times t$ minors form a Gröbner basis by Theorem 8.13,

$$\mathrm{in}_>(I_t) = (x_{i_1j_1}x_{i_2j_2}\cdots x_{i_tj_t} | 1 \le i_1 < i_2 < \cdots < i_t \le m, 1 \le j_1 < j_2 < \cdots < j_t \le n)$$

is a square-free monomial ideal. Thus $S/\mathrm{in}_>(I_t)$ is F-pure and hence F-injective, and one can check that $S/\mathrm{in}_>(I_t)$ is Cohen-Macaulay using Hochster's criterion [Hoc72] that a Stanley-Reisner ring is Cohen-Macaulay if the corresponding simplicial complex is shellable (note that even without knowing $S/\mathrm{in}_>(I_t)$ is Cohen-Macaulay, we can show that $S[z]/\widetilde{I}$ is F-injective because we showed $S/\mathrm{in}_>(I_t)$ is F-pure and so we can invoke Theorem 5.5 instead of Theorem 5.1). This completes the proof that R is F-rational.

Finally we prove that S/I_t is strongly F-regular when k is F-finite. Note that we can enlarge the $m \times n$ generic matrix to an $n \times n$ generic matrix and consider the corresponding quotients S'/I_t of $t \times t$ minors in the $n \times n$ matrix. Then $S/I_t \to S'/I_t$ splits (we can map the new variables to zero to obtain a splitting), thus S/I_t is strongly F-regular provided S'/I_t is strongly F-regular by Theorem 3.9. But S'/I_t is F-finite and Gorenstein (see [BV88]) and thus F-rationality of S'/I_t implies the strong F-regularity of S'/I_t by Proposition 4.9. □

Exercise 38. With notation as in Example 8.2, prove that R is not F-pure if $a = b = c = 3$ and $p \equiv 2 \bmod 3$.

Exercise 39. With notation as in Example 8.9, prove that R is not strongly F-regular without assuming p and m are relatively prime.

Exercise 40 ([Sin99b, Example 3.2]). Let k be a field of prime characteristic $p > 0$ and $R = k[[x, y, z, w]]/(xy, xz, y(z - w^2))$. Prove the following:

(1) R is Cohen-Macaulay and w is a nonzerodivisor on R.
(2) R/wR is F-pure but R is not F-pure.

9. F-signature: measuring Frobenius splittings

Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$. Kunz's theorem, Theorem 1.1, tells us that R is regular if and only if $F^e_* R$ is a finite free R-module for some (or equivalently, all) $e \in \mathbb{N}$. It is thus natural to consider the number of free summands the R-modules $F^e_* R$ admit as e ranges through the natural numbers. In doing so, we develop the theory of F-signature to numerically measure the severity of a strongly F-regular singularity.

Suppose M is a finitely generated R-module. We let $\operatorname{frk}_R(M)$ denote the largest number of free summands appearing in all various direct sum decompositions of M into irreducible R-modules. Equivalently, $\operatorname{frk}_R(M)$ is the largest rank of a free module F so that there exists a surjective R-linear map $M \to F$. The free ranks of $F^e_* R$ as e varies through the natural numbers are called the *Frobenius splitting numbers of* R, denoted by $a_e(R) := \operatorname{frk}_R(F^e_* R)$. Observe that if R is a domain then $a_e(R) \leq \operatorname{rank}_R(F^e_* R)$. The F-signature of R, $s(R)$, is defined to be

$$s(R) := \lim_{e \to \infty} \frac{a_e(R)}{\operatorname{rank}_R(F^e_* R)}.$$

We will discuss more precise information of $\operatorname{rank}_R(F^e_* R)$ below. We point out that, since $0 \leq \frac{a_e(R)}{\operatorname{rank}_R(F^e_* R)} \leq 1$ for all $e \in \mathbb{N}$, we have $0 \leq s(R) \leq 1$ provided $s(R)$ exists as a limit.

The purpose of this chapter is to cover three fundamental theorems on F-signature:

(1) [Tuc12, Main Result]: F-signature exists, i.e., the sequence of numbers $\left\{ \frac{a_e(R)}{\operatorname{rank}(F^e_* R)} \right\}_{e \in \mathbb{N}}$ is a Cauchy sequence and $s(R)$ is well-defined.
(2) [HL02, Corollary 16]: F-signature detects regularity, i.e., $s(R) = 1$ if and only if R is a regular local ring.
(3) [AL03, Main Result] F-signature detects strong F-regularity, i.e., $s(R) > 0$ if and only if R is strongly F-regular.

The origins of F-signature theory can be found in [SVdB97] and was formally developed by Huneke and Leuschke in [HL02]. Researchers understood that F-signature served as a numerical measurement of singularities long before it was shown to exist in full generality. Under the assumption of existence, it was first shown in the early 2000's that $s(R) = 1$ if and only if R is regular by Huneke and Leuschke, and that $s(R) > 0$ if and only if R is strongly F-regular by Aberbach and Leuschke. Tucker's proof of the existence of F-signature came nearly 10 years later.

Our presentation of F-signature theory will significantly deviate from the historical development of the theory. We will not present the fundamental theorems of F-signature in the order they were discovered nor we will follow the original techniques. We will utilize modern techniques developed in [PT18, PS19, Pol20] to present streamlined and elementary proofs of (1), (2), and (3) respectively.

Before continuing with the theory of F-signature the reader should first observe that computing the Frobenius splitting numbers of R does not require looking at all possible choices of direct sum decompositions of F^e_*R into irreducibles and then counting free summands. More specifically, we have the following lemma.

Lemma 9.1. *Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that M is a finitely generated R-module and $M \cong R^{\oplus t_1} \oplus N_1 \cong R^{\oplus t_2} \oplus N_2$ are choices of direct sum decompositions of M so that N_1, N_2 do not admit a free summand. Then $t_1 = t_2$.*

In particular, if $(R, \mathfrak{m}, k)$ is an F-finite local ring of prime characteristic $p > 0$ and $F^e_*R \cong R^{\oplus t} \oplus M$ is any choice of direct sum decomposition of F^e_*R so that M does not admit a free summand, then $t = a_e(R)$.

Proof. There exists onto map $\varphi : R^{\oplus t_1} \oplus N_1 \to R^{\oplus t_2}$. Because we are assuming that N_1 does not admit a free summand we must have that $\varphi(0 \oplus N_1) \subseteq \mathfrak{m}R^{\oplus t_2}$. In particular, if we base change to the residue field k we find that there is an onto map $k^{\oplus t_1} \twoheadrightarrow k^{\oplus t_2}$. Therefore $t_1 \geq t_2$. By symmetry we conclude that $t_1 = t_2$. The second assertion follows by applying the first assertion to $M = F^e_*R$. $\square$

To establish the theory of F-signature, we first need to investigate the rank of F^e_*R. Suppose that K is an F-finite field. Consider the Frobenius map $F : K \to F_*K$; an element $F_*r \in F_*K$ satisfies the monic polynomial equation $x^p - r = 0$. Therefore the degree of the minimal polynomial of every element of F_*K divides p. It follows that $[F_*K : K] = p^\gamma$ for some $\gamma \in \mathbb{N}$ and $[F^e_*K : K] = p^{e\gamma}$ for every $e \in \mathbb{N}$. If R is an F-finite domain with fraction field K then we define $\gamma(R)$ to be the unique integer such that $[F^e_*K : K] = p^{e\gamma(R)}$ for all $e \in \mathbb{N}$, i.e., $\gamma(R)$ is unique integer such that $\operatorname{rank}_R(F^e_*R) = p^{e\gamma(R)}$ for all $e \in \mathbb{N}$.

Lemma 9.2. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$. Then $F^e_*\widehat{R} \cong (F^e_*R) \otimes_R \widehat{R}$ for all $e > 0$. As a consequence, $\widehat{R}$ is reduced if R is reduced.*

Proof. Since F^e_*R is a finitely generated R-module, we have $(F^e_*R) \otimes_R \widehat{R} \cong \widehat{(F^e_*R)}$. But $\widehat{(F^e_*R)} \cong F^e_*\widehat{R}$: if we identify F^e_*R with R, then $\widehat{(F^e_*R)}$ is the completion of R with respect to the ideal $\mathfrak{m}^{[p^e]}$ while $F^e_*\widehat{R}$ is the completion of R with respect to $\mathfrak{m}$, so they are the same since $\sqrt{\mathfrak{m}^{[p^e]}} = \mathfrak{m}$. If R is reduced, then $R \hookrightarrow F^e_*R$ and thus $\widehat{R} \hookrightarrow F^e_*R \otimes \widehat{R} \cong F^e_*\widehat{R}$, which implies $\widehat{R}$ is reduced by Exercise 2. $\square$

Lemma 9.3. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of prime characteristic $p > 0$ and let K be the fraction field of R. Let P be a minimal prime of $\widehat{R}$ and let $L = \widehat{R}_P$. Then L is a field and $F^e_* L \cong F^e_* K \otimes_K L$. In particular, $[F^e_* L : L] = [F^e_* K : K]$.*

Proof. By Lemma 9.2, $\widehat{R}$ is reduced so L is a field. Now we have

$$F^e_* L \cong (F^e_* \widehat{R})_P \cong F^e_* \widehat{R} \otimes_{\widehat{R}} \widehat{R}_P \cong F^e_* R \otimes_R \widehat{R} \otimes_{\widehat{R}} \widehat{R}_P \cong F^e_* R \otimes_R \widehat{R}_P \cong F^e_* K \otimes_K L$$

where the third isomorphism follows from Lemma 9.2. ☐

Theorem 9.4. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of prime characteristic $p > 0$ and of dimension d. Then for each $e \in \mathbb{N}$ we have that $\operatorname{rank}_R(F^e_* R) = [F^e_* k : k]p^{ed}$.*

Proof. We first suppose that R is complete. By Cohen's structure theorem, R module-finite over $A = k[[x_1, x_2, \ldots, x_d]]$. Consider the following commutative diagram of local domains:

$$\begin{array}{ccc} A & \longrightarrow & R \\ \downarrow & & \downarrow \\ F^e_* A & \longrightarrow & F^e_* R \end{array} \quad .$$

Since rank is multiplicative across compositions, we have

$$\operatorname{rank}_A(F^e_* R) = \operatorname{rank}_R(F^e_* R) \operatorname{rank}_A(R) = \operatorname{rank}_{F^e_* A}(F^e_* R) \operatorname{rank}_A(F^e_* A).$$

The extension of local domains $A \to R$ is isomorphic to $F^e_* A \to F^e_* R$. Therefore $\operatorname{rank}_A(R) = \operatorname{rank}_{F^e_* A}(F^e_* R)$ and hence $\operatorname{rank}_R(F^e_* R) = \operatorname{rank}_A(F^e_* A)$. As mentioned in the proof of Theorem 1.1 it is straightforward to check that $F^e_* A$ is a free A-module with basis

$$\{F^e_*(\lambda x_1^{i_1} \cdots x_d^{i_d}) \mid 0 \le i_j < p^e, \text{ where } \{F^e_* \lambda\} \text{ is a free basis of } F^e_* k \text{ over } k\}.$$

Therefore $\operatorname{rank}_A(F^e_* A) = [F^e_* k : k]p^{ed}$ as wanted.

Now we suppose that R is not necessarily complete. Let P be a minimal prime of $\widehat{R}$ such that $d = \dim(R) = \dim(\widehat{R}/P)$. Let K be the fraction field of R and L the fraction field of $\widehat{R}/P$. By Lemma 9.3 we have that $[F^e_* K : K] = [F^e_* L : L]$, i.e., $\operatorname{rank}_R(F^e_* R) = \operatorname{rank}_{\widehat{R}/P}(F^e_*(\widehat{R}/P))$. This completes the proof as we already showed that for the complete local domain $\widehat{R}/P$ that $\operatorname{rank}_{\widehat{R}/P}(F^e_* \widehat{R}/P) = [F^e_* k : k]p^{ed}$. ☐

Remark 9.5. The proof of Theorem 9.4 shows something more. It shows that if $(R, \mathfrak{m}, k)$ is an F-finite local domain of dimension d with fraction field K then $\widehat{R}$ is (reduced and) equidimensional. That is, for each minimal prime $Q \in \operatorname{Spec}(\widehat{R})$ we have that $\dim(\widehat{R}/Q) = d$. Indeed, if Q is a minimal prime of $\widehat{R}$ and L_Q is the fraction field of $\widehat{R}/Q$ then $[F^e_* K :$

$K] = [F_*^e L_Q : L_Q]$ by Lemma 9.3. But by Theorem 9.4, $[F_*^e K : K] = [F_*^e k : k]p^{ed}$ and $[F_*^e L_Q : L_Q] = [F_*^e k : k]p^{e\dim(\widehat{R}/Q)}$. Therefore $d = \dim(\widehat{R}/Q)$.

This observation that the completion of an F-finite local domain is reduced and equidimensional is not surprising. Indeed, by Theorem 1.7 every F-finite ring is excellent (we will prove this in Chapter 10), and the completion of any excellent local domain is known to be reduced and equidimensional.

Corollary 9.6. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$ and $P \subsetneq Q$ be prime ideals. Then $\gamma(R/Q) < \gamma(R/P)$.*

Proof. Since $\dim(R/Q) < \dim(R/P)$, $\gamma(R/Q) < \gamma(R/P)$ by Theorem 9.4. □

9.1. **F-signature exists.** Let R be an F-finite ring of prime characteristic $p > 0$, not necessarily a domain. We set $\gamma(R) = \max\{\gamma(R/P) \mid P \in \operatorname{Spec}(R)\}$. Corollary 9.6 implies that $\gamma(R) = \max\{\gamma(R/P) \mid P \in \operatorname{Min}(R)\}$. If R is not necessarily a domain, so that the notion of generic rank is not necessarily well-defined, then in the spirit of Theorem 9.4 we set $\operatorname{rank}_R(F_*^e R) = p^{e\gamma(R)}$. Equivalently, we set $\operatorname{rank}_R(F_*^e R)$ to be the maximal generic rank of $F_*^e(R/P)$ over R/P as P varies through the (minimal) prime ideals of R.

Lemma 9.7. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$ and M a finitely generated R-module. There exists a constant $C \in \mathbb{R}$ so that for all $e \in \mathbb{N}$,*

$$\mu_R(F_*^e M) \le C \operatorname{rank}_R(F_*^e R).$$

Proof. Begin by considering a prime filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ so that $M_i/M_{i-1} \cong R/P_i$ for some prime $P_i \in \operatorname{Spec}(R)$. Counting minimal generators is subadditive on short exact sequences, see Exercise 41, therefore $\mu_R(F_*^e R) \le \sum_{i=1}^t \mu_R(F_*^e(R/P_i))$. Thus we may assume $M = R$ is an F-finite local domain.

We induct on $\gamma(R)$, the unique integer so that $\operatorname{rank}_R(F_*^e R) = p^{e\gamma(R)}$ for all $e \in \mathbb{N}$. If $\gamma(R) = 0$ is minimal then R is a perfect field by Theorem 9.4 and there is nothing to show. Suppose that $\gamma(R) > 0$. Because we may assume that R is a domain we have that F_*R is generically free of rank $p^{\gamma(R)}$ and hence there exists a short exact sequence

$$0 \to R^{\oplus p^{\gamma(R)}} \to F_*R \to T \to 0$$

where T is a finitely generated torsion R-module. In particular, T is a module over $R/(c)$ for some $c \neq 0$. Since $\gamma(R/(c)) < \gamma(R)$ by Corollary 9.6, we may assume by induction that there exists a constant C so that $\mu(F_*^e T) \le Cp^{e(\gamma(R)-1)}$ for all $e \in \mathbb{N}$.

Applying $F_*^{e-1}(-)$ to the above short exact sequence we find new short exact sequences

$$0 \to F_*^{e-1} R^{\oplus p^{\gamma(R)}} \to F_*^e R \to F_*^{e-1} T \to 0.$$

Counting minimal number of generators is sub-additive on short exact sequences hence

$$\begin{aligned}\mu(F^e_*R) &\leq \mu(F^{e-1}_*R^{\oplus p^{\gamma(R)}}) + \mu(F^{e-1}_*T)\\ &= p^{\gamma(R)}\mu(F^{e-1}_*R) + \mu(F^{e-1}_*T)\\ &\leq p^{\gamma(R)}\mu(F^{e-1}_*R) + Cp^{e(\gamma(R)-1)}.\end{aligned}$$

Dividing by $\operatorname{rank}_R(F^e_*R) = p^{e\gamma(R)}$ we find that

$$\frac{\mu(F^e_*R)}{\operatorname{rank}_R(F^e_*R)} \leq \frac{\mu(F^{e-1}_*R)}{\operatorname{rank}_R(F^{e-1}_*R)} + \frac{C}{p^e}. \tag{9.1}$$

Similarly, there is an inequality

$$\frac{\mu(F^{e-1}_*R)}{\operatorname{rank}_R(F^{e-1}_*R)} \leq \frac{\mu(F^{e-2}_*R)}{\operatorname{rank}_R(F^{e-2}_*R)} + \frac{C}{p^{e-1}}. \tag{9.2}$$

Applying the inequality of (9.2) to (9.1) we find that

$$\frac{\mu(F^e_*R)}{\operatorname{rank}_R(F^e_*R)} \leq \frac{\mu(F^{e-2}_*R)}{\operatorname{rank}_R(F^{e-2}_*R)} + \frac{C}{p^{e-1}} + \frac{C}{p^e}.$$

Inductively, we derive the inequality

$$\frac{\mu(F^e_*R)}{\operatorname{rank}_R(F^e_*R)} \leq 1 + \frac{C}{p} + \cdots + \frac{C}{p^{e-1}} + \frac{C}{p^e} \leq C\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{e-1}} + \frac{1}{p^e}\right) \leq \frac{C}{1-\frac{1}{p}} \leq 2C.$$

Therefore $\mu(F^e_*R) \leq 2C\operatorname{rank}_R(F^e_*R)$ for all $e \in \mathbb{N}$. ☐

Corollary 9.8. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$ and let T be a finitely generated R-module not supported at any minimal prime of R. Then there exists a constant C so that*

$$\mu_R(F^e_*T) \leq Cp^{e(\gamma(R)-1)}.$$

Proof. Let $I = \operatorname{Ann}_R(T)$. Clearly we have $\mu_R(F^e_*T) = \mu_{R/I}(F^e_*T)$. By Lemma 9.7 there exists an constant C so that $\mu_{R/I}(F^e_*T) \leq Cp^{e\gamma(R/I)}$. But $\gamma(R/I) < \gamma(R)$ by Corollary 9.6. ☐

Lemma 9.9. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$. If R is not strongly F-regular then $s(R) = 0$.*

Proof. Let $F^e_*R \cong R^{\oplus a_e(R)} \oplus M_e$ be a choice of direct sum decomposition of F^e_*R so that M_e does not have a free summand, see Lemma 9.1. Set $N_e = \mathfrak{m}^{\oplus a_e(R)} \oplus M_e$. In particular, $N_e \subseteq F^e_*R$ is an R-submodule, $F^e_*R/N_e \cong k^{\oplus a_e(R)}$, and $a_e(R) = \ell_R(F^e_*R/N_e)$.

We are assuming R is not strongly F-regular. So there exists an element $c \in R$ not in any minimal prime of R such that $R \xrightarrow{\cdot F^e_*c} F^e_*R$ does not split for all $e \in \mathbb{N}$. Observe then that

$\mathfrak{m}F^e_*R + \operatorname{span}_{F^e_*R}\{F^e_*c\} \subseteq N_e$ for all $e \in \mathbb{N}$. Therefore we can estimate

$$\begin{aligned} a_e(R) = \ell(F^e_*R/N_e) \leq & \ell_R(F^e_*R/(\mathfrak{m}F^e_*R + \operatorname{span}_{F^e_*R}\{F^e_*c\})) \\ = & \ell_R(F^e_*(R/cR) \otimes_R R/\mathfrak{m}) = \mu_R(F^e_*(R/cR)). \end{aligned}$$

By Corollary 9.8 there is a constant C such that

$$\mu(F^e_*(R/cR)) \leq Cp^{e(\gamma(R)-1)}.$$

Dividing by $p^{e\gamma(R)}$ and taking a limit as $e \to \infty$ shows that

$$0 \leq s(R) = \lim_{e\to\infty} \frac{a_e(R)}{p^{e\gamma(R)}} \leq \lim_{e\to\infty} \frac{C}{p^e} = 0. \qquad \square$$

The following is the key lemma in establishing the existence of F-signature.

Lemma 9.10. *Let $(R, \mathfrak{m}, k)$ be a local ring and let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of finitely generated R-modules. Then

$$\operatorname{frk}_R(M_2) \leq \operatorname{frk}_R(M_1) + \mu_R(M_3).$$

Proof. Begin by choosing direct sum decompositions $M_1 \cong R^{\oplus \operatorname{frk}_R(M_1)} \oplus \overline{M_1}$ and $M_2 \cong R^{\oplus \operatorname{frk}_R(M_2)} \oplus \overline{M_2}$ where $\overline{M_1}$ and $\overline{M_2}$ are R-modules without a free summand. Because $\overline{M_1}$ is a module without a free summand we have that $0 \oplus \overline{M_1} \subseteq \mathfrak{m}^{\oplus \operatorname{frk}_R(M_2)} \oplus \overline{M_2}$. In particular, there is an induced map

$$\frac{M_1}{0 \oplus \overline{M_1}} \to \frac{M_2}{\mathfrak{m}^{\oplus \operatorname{frk}_R(M_2)} \oplus \overline{M_2}}.$$

Equivalently, there is a right exact sequence

$$R^{\oplus \operatorname{frk}_R(M_1)} \to k^{\oplus \operatorname{frk}_R(M_2)} \to M_3' \to 0$$

and the cokernel M_3' is a homomorphic image of M_3. Counting minimal generators is subadditive on right exact sequences and therefore

$$\operatorname{frk}_R(M_2) \leq \operatorname{frk}_R(M_1) + \mu_R(M_3') \leq \operatorname{frk}_R(M_1) + \mu_R(M_3). \qquad \square$$

Now we can prove the first main result of this chapter.

Theorem 9.11. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$. Then the F-signature of R exists, i.e., the sequence of numbers $\left\{\frac{a_e(R)}{p^{e\gamma(R)}}\right\}_{e\in\mathbb{N}}$ defines a Cauchy sequence.*

Proof. By Lemma 9.9 we are reduced to the scenario that R is strongly F-regular. By Lemma 3.2 we know that R is a domain. Let

$$s_+(R) = \limsup_{e\to\infty} \frac{a_e(R)}{p^{e\gamma(R)}}, \text{ and } s_-(R) = \liminf_{e\to\infty} \frac{a_e(R)}{p^{e\gamma(R)}}.$$

We aim to show $s_+(R) \leq s_-(R)$.

Since $\operatorname{rank}_R(F_*R) = p^{\gamma(R)}$, we have a short exact sequence

$$0 \to F_*R \to R^{\oplus p^{\gamma(R)}} \to T \to 0$$

where T is a finitely generated torsion R-module. Applyig $F^e_*(-)$ gives us a short exact sequence

$$0 \to F^{e+1}_*R \to F^e_*R^{\oplus p^{\gamma(R)}} \to F^e_*T \to 0.$$

By Lemma 9.10 we have that for each $e \in \mathbb{N}$ the inequality

$$\operatorname{frk}_R(F^e_*R^{\oplus p^{\gamma(R)}}) \leq \operatorname{frk}_R(F^{e+1}_*R) + \mu_R(F^e_*T),$$

that is,

$$p^{\gamma(R)}a_e(R) \leq a_{e+1}(R) + \mu_R(F^e_*T).$$

By Corollary 9.8 there exists a constant C so that $\mu_R(F^e_*T) \leq Cp^{e(\gamma(R)-1)}$. Dividing by $p^{(e+1)\gamma(R)}$ yields that

$$\frac{a_e(R)}{p^{e\gamma(R)}} \leq \frac{a_{e+1}(R)}{p^{(e+1)\gamma(R)}} + \frac{C}{p^e}.$$

We can similarly bound the ratio $\frac{a_{e+1}(R)}{p^{(e+1)\gamma(R)}}$ from above by $\frac{a_{e+2}(R)}{p^{(e+2)\gamma(R)}} + \frac{C}{p^{e+1}}$ and therefore

$$\frac{a_e(R)}{p^{e\gamma(R)}} \leq \frac{a_{e+2}(R)}{p^{(e+2)\gamma(R)}} + \frac{C}{p^e} + \frac{C}{p^{e+1}}.$$

Inductively, we find that for all $e, e_0 \in \mathbb{N}$ that

$$\begin{aligned}\frac{a_e(R)}{p^{e\gamma(R)}} &\leq \frac{a_{e+e_0}(R)}{p^{(e+e_0)\gamma(R)}} + \frac{C}{p^e} + \frac{C}{p^{e+1}} + \cdots + \frac{C}{p^{e+e_0-1}} \\ &= \frac{a_{e+e_0}(R)}{p^{(e+e_0)}} + \frac{C}{p^e}\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{e_0-1}}\right) \leq \frac{a_{e+e_0}(R)}{p^{(e+e_0)}} + \frac{2C}{p^e}.\end{aligned}$$

Taking a limit infimum as $e_0 \to \infty$ shows that for all $e \in \mathbb{N}$ that

$$\frac{a_e(R)}{p^{e\gamma(R)}} \leq s_-(R) + \frac{2C}{p^e}.$$

Taking a limit supremum as $e \to \infty$ then shows that

$$s_+(R) \leq s_-(R),$$

i.e., the F-signature of R exists. $\square$

9.2. **F-signature and strong F-regularity.** We aim to prove that an F-finite local ring $(R, \mathfrak{m}, k)$ is strongly F-regular if and only if $s(R) > 0$. Lemma 9.9 provides one implication that $s(R) > 0$ implies that R is strongly F-regular. It remains to show that if R is strongly F-regular then $s(R) > 0$. This was first proved by Aberbach–Leuschke in [AL03]. Their approach relies on a "valuative criterion" for tight closure given by Hochster–Huneke in [HH91] (generalized by Aberbach in [Abe01]) and the Izumi–Rees theorem [Ree89] which linearly bounds any two Rees valuations centered on the maximal ideal of an analytically irreducible local domain. We shall not resort to these advanced techniques and will instead follow a novel and more elementary path laid out in [Pol20]. We begin with the module version of the celebrated Chevalley's lemma.

Lemma 9.12 ([Che43])**.** *Let $(R, \mathfrak{m}, k)$ be a complete local ring and M a finitely generated R-module. Suppose that $I \subseteq R$ is an $\mathfrak{m}$-primary ideal and $\{M_n\}_{n\in\mathbb{N}}$ is a descending sequence of submodules of M so that $\bigcap_{n\in\mathbb{N}} M_n = 0$. Then there exists an $n > 0$ such that $M_n \subseteq IM$.*

Proof. Since M/IM us Artinian, the descending chain of submodules $\{(M_n + IM)/IM\}_{n\in\mathbb{N}}$ eventually stabilizes. Thus there exists n_1 so that for all $n \geq n_1$ we have that $(M_n + IM)/IM = (M_{n_1} + IM)/IM$. Similarly, there exists $n_2 > n_1$ so that $(M_n + I^2M)/I^2M = (M_{n_2} + I^2M)/I^2M$ for all $n \geq n_2$. Inductively choose n_t so that $n_{t+1} > n_t$ and $(M_n + I^tM)/I^tM = (M_{n_t} + I^tM)/I^tM$ for all $n \geq n_t$. Replacing M_t by M_{n_t}, we may assume that the sequence of modules $\{M_n\}_{n\in\mathbb{N}}$ is such that $(M_n + I^tM)/I^tM = (M_t + I^tM)/I^tM$ for all $n \geq t$.

We claim that $M_1 \subseteq IM$. Choose an element $\eta_1 \in M_1$. Because $(M_2 + IM)/IM = (M_1 + IM)/IM$ we can choose $\eta_2 \in M_2$ so that $\eta_2 \equiv \eta_1 \mod IM$. Inductively, we choose elements $\eta_t \in M_t$ so that $\eta_{t+1} \equiv \eta_t \mod I^tM$. The sequence of elements $\{\eta_t\}$ forms a Cauchy sequence. Let $\tilde{\eta} \in M$ denote its limit (which exists since M is complete: it is a finitely generated module over a complete local ring). Because each $\eta_t \in M_t$ and $\bigcap M_t = 0$ we must have that $\tilde{\eta} = 0$. In particular, there exists a t such that $\eta_t \in IM$. Recall that $\eta_t \equiv \eta_{t-1} \mod I^{t-1}M$. Hence $\eta_t - \eta_{t-1} \in I^tM \subseteq IM$ and therefore $\eta_{t-1} \in IM$. By induction $\eta_1 \in IM$ and hence $M_1 \subseteq IM$ as claimed. □

Lemma 9.13. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$ and M a finitely generated R-module. For each $e \in \mathbb{N}$ let*

$$I_e(M) = \{\eta \in M \mid R \xrightarrow{\cdot F^e_*\eta} F^e_*M \text{ does not split}\}.$$

(1) For each $e \in \mathbb{N}$ the set $I_e(M)$ is a submodule of M containing $\mathfrak{m}^{[p^e]}M$.

(2) For each $e \in \mathbb{N}$ we have that $a_e(M) = \ell(M/I_e(M))[F^e_*k : k]$.

(3) $\{I_e(M)\}_{e\in\mathbb{N}}$ is a descending chain of submodules of M.

(4) If R is strongly F-regular and M is torsion-free then $\bigcap_{e\in\mathbb{N}} I_e(M) = 0$.

If $M = R$ then we refer to $I_e := I_e(R)$ as the eth splitting ideal *of R.*

Proof. (1) Suppose that $\eta_1, \eta_2 \in I_e(M)$ and $r \in R$ we aim to show that $r\eta_1 + \eta_2 \in I_e(M)$. Suppose by way of contradiction that there exists $\varphi \in \operatorname{Hom}_R(F^e_*M, R)$ so that $\varphi(F^e_*(r\eta_1 + \eta_2)) = \varphi(F^e_*r\eta_1) + \varphi(F^e_*\eta_2) = 1$. Because R is local we must have that either $\varphi(F^e_*r\eta_1)$ is a unit of R or $\varphi(F^e_*\eta_2)$ is a unit of R. If $\varphi(F^e_*r\eta_1)$ is a unit then $\eta_1 \notin I_e(M)$ if $\varphi(F^e_*\eta_2)$ is a unit then $\eta_2 \notin I_e(M)$.

(2) Suppose that $F^e_*M \cong R^{\oplus a_e(M)} \oplus N$ is a choice of direct sum decomposition of F^e_*M so that N does not admit a free summand. Under this choice of direct sum decomposition we have that $F^e_*I_e(M) = \mathfrak{m}^{\oplus a_e(M)} \oplus N$. Therefore

$$\ell_R(M/I_e(M)) = \ell_{F^e_*R}(F^e_*(M/I_e(M))) = \frac{\ell_R(F^e_*M/F^e_*I_e(M))}{[F^e_*k : k]} = \frac{a_e(M)}{[F^e_*k : k]}.$$

(3) We want to show $I_e(M) \supseteq I_{e+1}(M)$, this is clear if $I_e(M) = M$ and so we assume $I_e(M) \neq M$. Suppose $\eta \notin I_e(M)$ and choose splitting $\varphi : F^e_*M \to R$ so that $\varphi(F^e_*\eta) = 1$. We will show that $\eta \notin I_{e+1}(M)$. Observe that R is F-pure: consider $R \xrightarrow{\cdot\eta} M$, then ψ: $F^e_*R \xrightarrow{\cdot F^e_*\eta} F^e_*M \xrightarrow{\varphi} R$ is a splitting of $R \to F^e_*R$. Then $\psi(F_*\varphi(F^{e+1}_*\eta)) = 1$ and $\eta \notin I_{e+1}(M)$ as claimed.

(4) See Lemma 5.9. $\square$

Theorem 9.14. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring of prime characteristic $p > 0$. Then there exists an $e_0 \in \mathbb{N}$ so that if M is a finitely generated maximal Cohen-Macaulay R-module and $\eta \in M \setminus \mathfrak{m}M$ then there exists $\varphi \in \operatorname{Hom}_R(F^{e_0}_*M, R)$ so that $\varphi(F^{e_0}_*\eta) = 1$.*

Proof. First of all we observe that the finitely generated R-module $F^{e_0}_*M$ has a free R-summand if and only if $F^{e_0}_*\widehat{M}$ has a free $\widehat{R}$-summand (see Exercise 42). Therefore one can replace R by $\widehat{R}$ to assume that R is complete (note that strong F-regularity is preserved under completion by Corollary 3.12). In particular, R admits a canonical module ω_R. Given a finitely generated R-module N, we use N^* to denote the ω_R-dual $\operatorname{Hom}_R(N, \omega_R)$ for the rest of this proof.

We map a free module $R^{\oplus N}$ onto M^*, let K denote the kernel, and consider the short exact sequence

$$0 \to K \to R^{\oplus N} \to M^* \to 0.$$

The module $R^{\oplus N}$ is Cohen-Macaulay by Theorem 4.6, M^* is Cohen-Macaulay by [BH93, Theorem 3.3.10], and therefore K is seen to be Cohen-Macaulay by examining the induced long exact sequence of local cohomology modules with support in the maximal ideal $\mathfrak{m}$. If we apply $\operatorname{Hom}_R(-,\omega_R)$ to the above short exact sequence and utilize [BH93, Theorem 3.3.10] a second time we find that there is a short exact sequence of Cohen-Macaulay R-modules

$$0 \to M \to \omega_R^{\oplus N} \to K^* \to 0. \tag{9.3}$$

Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of R and let $I = (\underline{x})$. Then $\operatorname{Tor}_1^R(R/I, K^*)$ agrees with the first Koszul homology module $H_1(\underline{x}; K^*)$ and $H_1(\underline{x}; K^*) = 0$ as $\underline{x}$ is a regular sequence on K^*. Therefore if we apply $- \otimes_R R/I$ to the short exact sequence in (9.3) we produce a new short exact sequence

$$0 \to \frac{M}{IM} \to \frac{\omega_R^{\oplus N}}{I\omega_R^{\oplus N}} \to \frac{K^*}{IK^*} \to 0.$$

Consequently, if $\eta \in M \backslash IM$ then under the inclusion $M \subseteq \omega_R^{\oplus N}$ we find that $\eta \in \omega_R^{\oplus N} \backslash I\omega_R^{\oplus N}$.

For each natural number $e \in \mathbb{N}$ let

$$I_e(\omega_R) = \{m \in \omega_R \mid R \xrightarrow{\cdot F^e_* m} F^e_*\omega_R \text{ does not split}\}.$$

By Lemma 9.13, $\bigcap_{e\in\mathbb{N}} I_e(\omega_R) = 0$. By Lemma 9.12 there exists an integer e_0, depending only on ω_R and I, so that $I_{e_0}(\omega_R) \subseteq I\omega_R$. Thus if $\eta \in M \setminus IM$ then under the inclusion $M \subseteq \omega_R^{\oplus N}$ we must have that $\eta \in \omega_R^{\oplus N} \setminus I_{e_0}(\omega_R)^{\oplus N}$. In particular, there exists an R-linear map $\varphi : F^{e_0}_*\omega_R^{\oplus N} \to R$ so that $\varphi(F^{e_0}_*\eta) = 1$. Restricting the domain of φ to $F^{e_0}_* M$ then shows that $F^{e_0}_* M$ admits a free summand. $\square$

Theorem 9.15. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$. Then $s(R) > 0$ if and only if R is strongly F-regular.*

Proof. By Lemma 9.9, if $s(R) > 0$ then R is strongly F-regular. So we assume R is strongly F-regular and our goal is to show that $s(R) > 0$. Let e_0 be as in Theorem 9.14 and for each $e \in \mathbb{N}$ consider the Frobenius non-splitting ideals

$$I_e := I_e(R) = \{r \in R \mid R \xrightarrow{\cdot F^e_* r} F^e_* R \text{ does not split}\},$$

as in Lemma 9.13. We claim that $I_{e+e_0} \subseteq \mathfrak{m}^{[p^e]}$ for all $e \in \mathbb{N}$. Indeed, $r \in R \setminus \mathfrak{m}^{[p^e]}$ if and only if $F^e_* r \in F^e_* R \setminus \mathfrak{m} F^e_* R$. The modules $F^e_* R$ are maximal Cohen-Macaulay and so by Theorem 9.14 there exists $\varphi : F^{e+e_0}_* R \to R$ so that $\varphi(F^{e+e_0}_* r) = 1$, i.e., $r \in R \setminus I_{e+e_0}$. Therefore by Lemma 9.13 and Theorem 9.4,

$$\frac{a_{e+e_0}(R)}{[F^{e+e_0}_* k : k]} = \ell(R/I_{e+e_0}) \geq \ell(R/\mathfrak{m}^{[p^e]}) = \frac{\mu(F^e_* R)}{[F^e_* k : k]} \geq \frac{\operatorname{rank}_R(F^e_* R)}{[F^e_* k : k]} = p^{ed}.$$

Dividing by $p^{(e+e_0)d}$ and taking a limit as $e \to \infty$ shows that

$$s(R) \geq \frac{1}{p^{e_0 d}} > 0. \qquad \square$$

9.3. **F-signature and regularity.** Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$. Huneke–Leuschke were the first to prove in [HL02] that R is a regular local ring if and only if $s(R) = 1$. They showed that $s(R) = 1$ implies that a related numerical invariant called the Hilbert–Kunz multiplicity of R, $\mathrm{e}_{\mathrm{HK}}(R)$, must also be equal to 1. Then they appeal to a result of Watanabe–Yoshida [WY00] that analytically irreducible local rings with Hilbert–Kunz multiplicity equal to 1 must be regular. The proof of Huneke–Leuschke's theorem presented here follows the methodology of [PS19] and allows us to bypass Hilbert–Kunz theory.

Our proof that $s(R) = 1$ if and only if R is regular is a consequence of developing an equimultiplicity theory of F-signature in strongly F-regular rings. More specifically, we need to study the behavior of F-signature and Frobenius splitting numbers under localization.

Suppose that $F^e_* R \cong R^{\oplus a_e(R)} \oplus M_e$ and the module M_e does not admit a free summand. If $P \in \operatorname{Spec}(R)$ then

$$F^e_* R \otimes_R R_P \cong F^e_* R_P \cong R_P^{\oplus a_e(R)} \oplus (M_e)_P.$$

By Lemma 9.1 we find that $a_e(R_P) \geq a_e(R)$ and equality holds if and only if $(M_e)_P$ does not admit a free R_P-summand. Therefore to keep track of the differences of the Frobenius splitting numbers of R and a localization of R at a prime ideal P it is beneficial to keep track of the number of summands $F^e_* R$ isomorphic to a particular module. To this end, if M is a finitely generated R-module we let

$$a^M_e(R) = \max\{n \mid M^{\oplus n} \text{ is a direct summand of } F^e_* R\}.$$

Observe that if M does not admit a free summand and M is a direct summand of $F^e_* R$ so that M_P has at least one free R_P-summand, then $a_e(R_P) \geq a_e(R) + a^M_e(R)$.

The following lemma is an elementary observation that for a strongly F-regular local ring R, if a finitely generated R-module M is a direct summand of $F^{e_0}_* R$ for some e_0, then the numbers $a^M_e(R)$ are asymptotically comparable to the numbers $\operatorname{rank}_R(F^e_* R) = p^{e\gamma(R)}$.

Lemma 9.16. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring of prime characteristic $p > 0$ and let M be a finitely generated R-module. If $a^M_{e_0}(R) \geq 1$ for some $e_0 \in \mathbb{N}$ then*

$$\liminf_{e \to \infty} \frac{a^M_e(R)}{p^{e\gamma(R)}} > 0.$$

Proof. Suppose that $F^{e_0}_*R \cong M \oplus N$ and then consider a direct sum decomposition of F^e_*R as $F^e_*R \cong R^{\oplus a_e(R)} \oplus P$. Then

$$F^{e+e_0}_*R \cong F^{e_0}_*R^{\oplus a_e(R)} \oplus F^{e_0}_*P \cong (M \oplus N)^{\oplus a_e(R)} \oplus F^{e_0}_*P.$$

In particular,

$$a^M_{e+e_0}(R) \geq a_e(R).$$

Dividing by $p^{(e+e_0)\gamma(R)}$ and taking a limit infimum as $e \to \infty$ reveals that

$$\liminf_{e\to\infty} \frac{a^M_e(R)}{p^{e\gamma(R)}} \geq \frac{s(R)}{p^{e_0\gamma(R)}},$$

a quantity that is positive by Theorem 9.15. □

A consequence of Lemma 9.16 is an equimultiplicity theory of F-signature. The following corollary gives us that F-signature is unchanged under localization at a prime ideal if and only if each of the Frobenius splitting numbers too are unchanged under localization.

Corollary 9.17. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring of prime characteristic $p > 0$. Suppose that $P \in \operatorname{Spec}(R)$. Then the following are equivalent:*

(1) $a_e(R) = a_e(R_P)$ for all $e \in \mathbb{N}$;
(2) $s(R) = s(R_P)$.

Proof. If $a_e(R) = a_e(R_P)$ for all $e \in \mathbb{N}$ then $s(R) = s(R_P)$: The sequences of numbers $\{\frac{a_e(R)}{p^{e\gamma(R)}}\}$ and $\{\frac{a_e(R_P)}{p^{e\gamma(R_P)}}\}$ defining the F-signature of R and R_P respectively are identical sequences, see Exercise 43.

Suppose that $a_{e_0}(R) \neq a_{e_0}(R_P)$, or equivalently, $F^{e_0}_*R \cong R^{\oplus a_{e_0}(R)} \oplus M_{e_0}$ where M_{e_0} does not admit a free summand but $(M_{e_0})_P$ has a free R_P-summand (see Lemma 9.1). By Lemma 9.16 we have that

$$\liminf_{e\to\infty} \frac{a^{M_{e_0}}_e(R)}{p^{e\gamma(R)}} > 0.$$

For each $e \in \mathbb{N}$ consider a direct sum decomposition of the form

$$F^e_*R \cong R^{\oplus a_e(R)} \oplus M_{e_0}^{\oplus a^{M_{e_0}}_e(R)} \oplus N_e.$$

Localizing at P and counting free summands gives us

$$a_e(R_P) \geq a_e(R) + a^{M_{e_0}}_e(R).$$

Diving by $p^{e\gamma(R)} = p^{e\gamma(R_P)}$ and taking a limit infimum as $e \to \infty$ shows that

$$s(R_P) \geq s(R) + \liminf_{e\to\infty} \frac{a^{M_{e_0}}_e(R)}{p^{e\gamma(R)}} > s(R).$$ □

Now we can prove the following.

Theorem 9.18. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic $p > 0$. Then $s(R) = 1$ if and only if R is a regular local ring.*

Proof. If R is regular then $F^e_* R$ is a finite free R-module for all $e \in \mathbb{N}$ by Theorem 1.1. Hence $\frac{a_e(R)}{p^{e\gamma(R)}} = 1$ for all $e \in \mathbb{N}$ and so $s(R) = 1$.

Conversely, if $s(R) = 1$ then R is strongly F-regular by Theorem 9.15 and hence a domain by Lemma 3.2. Consider the localization of R at the prime ideal 0 and observe then that $1 = s(R) = s(R_0)$. By Corollary 9.17 we must have that $a_e(R) = a_e(R_0) = \operatorname{rank}_R(F^e_* R)$ for all $e \in \mathbb{N}$. Therefore $F^e_* R$ is a free R-module for all $e \in \mathbb{N}$ and therefore R is a regular local ring by Theorem 1.1. $\square$

We end this chapter with an application of Theorem 9.14 to the divisor class group of strongly F-regular singularities.

Proposition 9.19. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring of prime characteristic $p > 0$. Then, up to linear equivalence, there are only finitely many divisors D such that $R(p^e D)$ is maximal Cohen-Macaulay for all $e > 0$.*

Proof. Let e_0 be the constant in Theorem 9.14. Let D be a divisor such that $R(p^e D)$ is maximal Cohen-Macaulay for all $e > 0$. Then $F^{e_0}_* R(p^{e_0} D)$ admits an R-summand, that is, there exists a (split) surjection $F^{e_0}_* R(p^{e_0} D) \twoheadrightarrow R$. Tensoring with $R(-D)$ and applying $(-)^{**}$, we obtain a split surjection $F^{e_0}_* R \twoheadrightarrow R(-D)$. Thus, $R(-D)$ is a summand of $F^{e_0}_* R$. Since $F^{e_0}_* R$ can only have finitely many rank one summand up to isomorphism, we see that there are only finitely many isomorphism classes of such $R(-D)$. Hence there are only finitely many such divisors D up to linear equivalence. $\square$

Corollary 9.20. *Let $(R, \mathfrak{m}, k)$ be a two-dimensional F-finite and strongly F-regular local ring of prime characteristic $p > 0$. Then the divisor class group $\operatorname{Cl}(R)$ is finite.*

Proof. By Proposition 9.19, it is enough to observe that $R(D)$ is maximal Cohen-Macaulay for all divisors D: this is because $R(D)$ is (S_2) over a two dimensional normal domain R. $\square$

Corollary 9.21. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring of prime characteristic $p > 0$ and of dimension d. Then the torsion part of $\operatorname{Cl}(R)$ is finite.*

Proof. By Proposition 9.19, it is enough to show that $R(D)$ is maximal Cohen-Macaulay for all torsion divisors D, which follows from Proposition 5.10. $\square$

Corollary 9.20 and Corollary 9.21 are in some sense the best possible: it is not true that $\mathrm{Cl}(R)$ is finite for all strongly F-regular local rings in higher dimension. In Example 3.10, we see that $R = k[[x,y,u,v]]/(xy-uv)$ is a three-dimensional strongly F-regular local ring, and it is easy to check that $\mathrm{Cl}(R) \cong \mathbb{Z}$.

Exercise 41. Let $(R,\mathfrak{m},k)$ be a local ring and $M' \to M \to M'' \to 0$ a right exact sequence of finitely generated R-modules. Show that $\mu_R(M) \leq \mu_R(M') + \mu_R(M'')$ where $\mu_R(N)$ counts the minimal number of elements needed to generate a finitely generated R-module N.

Exercise 42. Let $(R,\mathfrak{m},k)$ be a local ring and M a finitely generated R-module. Show that $\mathrm{frk}_R(M) = \mathrm{frk}_{\widehat{R}}(\widehat{M})$.

Exercise 43. Let $(R,\mathfrak{m},k)$ be an F-finite local ring of prime characteristic $p > 0$ and $P \subseteq Q$ be prime ideals of R. Prove that $\gamma(R) \geq \gamma(R/P) = \gamma(R_Q/PR_Q)$ and that $a_e(R) \leq a_e(R_Q)$. Prove that $s(R) \leq s(R_Q)$ for all $Q \in \mathrm{Spec}(R)$.

Exercise 44. Let $(R,\mathfrak{m},k) \to (S,\mathfrak{n},\ell)$ be a flat local extension of F-finite rings of prime characteristic $p > 0$. Prove that $s(R) \geq s(S)$. (Hint: Use Exercise 43 to reduce to the case that $\dim(R) = \dim(S)$.)

In connection with Corollary 9.20 and Corollary 9.21, the following question is open.

Open Problem 5. *Let $(R,\mathfrak{m},k)$ be an F-finite and strongly F-regular local ring of prime characteristic $p > 0$. Is it true that $\mathrm{Cl}(R)$ is finitely generated?*

10. Radu–André's theorem, Kunz's theorem, and Gabber's theorem

In this chapter, we utilize modern techniques to prove some foundational results unique to prime characteristic commutative algebra. The first theorem is obtained by Radu and André [Rad92, And93] and can be viewed as a relative version of Kunz's theorem, Theorem 1.1. The second and third theorems are mentioned earlier: see Theorem 1.7 and Theorem 1.6, and they both indicate that F-finite rings have nice geometric properties and are not pathological from the view of algebraic geometry.

We begin with the Radu–André Theorem. Recall that a map $R \to S$ of (Noetherian) rings is called *regular* if it is flat and all fibers are geometrically regular, i.e., $\kappa(P) \otimes_R S$ is geometrically regular over $\kappa(P)$ for all $P \in \operatorname{Spec}(R)$.

Theorem 10.1 (Radu–André Theorem)**.** *A homomorphism $R \to S$ of (Noetherian) rings of prime characteristic $p > 0$ is regular if and only if $F^e_*R \otimes_R S \to F^e_*S$ is flat for some (equivalently, all) $e > 0$.*

The difficulty of the theorem is that it is not clear in priori that $F^e_*R \otimes_R S$ is a Noetherian ring (though it will follow from the conclusion of the theorem that $F^e_*R \otimes_R S$ is in fact Noetherian, see Exercise 46).[11] We proceed carefully. We first record some criteria for flatness, see [sta16, Tag 00MD] for more details.

Lemma 10.2 ([sta16, Lemma 10.98.11])**.** *Let $R \to S$ be a map of (Noetherian) rings. Let $I \subseteq R$ be an ideal and let M be a finitely generated S-module. Suppose for each $n \geq 1$, M/I^nM is flat over R/I^n. Then for each prime $Q \in \operatorname{Spec}(S)$ such that $I \subseteq Q$, M_Q is flat over R. In particular, if $(S, \mathfrak{n}, \ell)$ is local and $IS \subseteq \mathfrak{n}$, then M is flat over R.*

Lemma 10.3 ([sta16, Lemma 10.98.8])**.** *Let A be a ring that is not necessarily Noetherian, $I \subseteq A$ an ideal, and M an A-module. If M/IM is flat over A/I and $\operatorname{Tor}_1^A(A/I, M) = 0$, then*

(1) M/I^nM is flat over A/I^n for all $n \geq 1$.
(2) For any A-module N that is annihilated by I^m for some $m \geq 0$, $\operatorname{Tor}_1^A(N, M) = 0$.

In particular, if I is nilpotent, then M is flat over A.

The next lemma is well-known to experts, as we cannot find a good reference beyond the Noetherian set up, we deduce it from Lemma 10.3.

[11] If we know $F^e_*R \otimes_R S$ is Noetherian in priori (e.g., if R is F-finite), then at least one direction of the theorem follows quite easily from Kunz's theorem and the local criterion for flatness [sta16, Lemma 10.98.10].

Lemma 10.4 (Fiberwise criteria for flatness)**.** *Let A be a ring that is not necessarily Noetherian, and let M be an A-module. Let $t \in A$ such that t is a nonzerodivisor on both A and M. If M/tM is flat over A/tA and M_t is flat over A_t, then M is flat over A.*

Proof. By Lemma 10.3 applied to $I = (t)$, we know that $\mathrm{Tor}_1^A(N, M) = 0$ for all t^∞-torsion A-modules N (by taking a direct limit). For any t^m-torsion A-module N, we have $0 \to K \to F \to N \to 0$ where F is a free A/t^mA-module and K is t^m-torsion. Since t is a nonzerodivisor on A and M, $\mathrm{Tor}_j^A(F, M)$=0 for all $j > 0$. The long exact sequence of Tor then shows that $\mathrm{Tor}_j^A(N, M) = 0$ for all $j > 0$. By taking direct limit we know that $\mathrm{Tor}_j^A(N, M) = 0$ for all $j > 0$ and all t^∞-torsion A-modules N.

For an arbitrary A-module N, if we let $\Gamma_{(t)}N = \{n \in M \mid t^\ell n = 0 \text{ for some } \ell\}$, then we have two short exact sequences:

$$0 \to \Gamma_{(t)}N \to N \to \overline{N} \to 0, \text{ and } 0 \to \overline{N} \to \overline{N}_t \to N' \to 0$$

Now $\mathrm{Tor}_j^A(\Gamma_{(t)}N, M) = \mathrm{Tor}_j^A(N', M) = 0$ for all $j > 0$ since $\Gamma_{(t)}N, N'$ are both t^∞-torsion, and $\mathrm{Tor}_j^A(\overline{N}_t, M) \cong \mathrm{Tor}_j^{A_t}(\overline{N}_t, M_t) = 0$ for all $j > 0$ since M_t is flat over A_t. By examining the long exact sequence of Tor, it is easy to see that $\mathrm{Tor}_j^A(N, M) = 0$ for all $j > 0$. Thus M is flat over A. $\square$

Proof of Theorem 10.1. We first prove that if $F^e_*R \otimes_R S \to F^e_*S$ is flat for some $e > 0$, then $R \to S$ is regular. We observe that if $F^e_*R \otimes_R S \to F^e_*S$ is flat, then applying $F^e_*(-)$, we see that $F^{2e}_*R \otimes_{F^e_*R} F^e_*S \to F^{2e}_*S$ is flat, while applying $\otimes_{F^e_*R} F^{2e}_*R$, we see that $F^{2e}_*R \otimes_R S \to F^{2e}_*R \otimes_{F^e_*R} F^e_*S$ is flat. Thus composing these two maps we see that $F^{2e}_*R \otimes_R S \to F^{2e}_*S$ is flat. Thus iterating this process, we find that there are infinitely many $e > 0$ such that $F^e_*R \otimes_R S \to F^e_*S$ is flat.

We set $\kappa = \kappa(P)$ and aim to show $\kappa \otimes_R S$ is geometrically regular over κ. Note that for any finite and purely inseparable field extension κ' of κ, we can pick $e \gg 0$ such that $\kappa' \subseteq F^e_*\kappa$ and $F^e_*R \otimes_R S \to F^e_*S$ is flat. Base change the flat map $F^e_*R \otimes_R S \to F^e_*S$ along $F^e_*R \to F^e_*\kappa$, we know that $F^e_*\kappa \otimes_R S \to F^e_*(\kappa \otimes_R S)$ is flat. Consider the composition:

$$\kappa' \otimes_R S \to F^e_*\kappa \otimes_R S \to F^e_*(\kappa \otimes_R S) \to F^e_*(\kappa' \otimes_R S)$$

where the first and third maps are flat as they are base changed from field extensions, and the middle map is flat by previous discussion. Thus the composition is flat and so $\kappa' \otimes_R S$ is regular by Theorem 1.1. Therefore $\kappa \otimes_R S$ is geometrically regular over κ.

To show $R \to S$ is flat, we may localize a prime ideal of S and localize the contraction of that prime ideal to R. Thus we may assume that $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ is a local homomorphism. By Lemma 10.2, it is enough to show that $R/\mathfrak{m}^{[p^e]} \to S/\mathfrak{m}^{[p^e]}S$ is flat for infinitely many $e > 0$. Base change the flat map $F^e_*R \otimes_R S \to F^e_*S$ along $R \to R/\mathfrak{m}$, we see that $F^e_*(R/\mathfrak{m}^{[p^e]}) \otimes_{R/\mathfrak{m}} S/\mathfrak{m}S \to F^e_*(S/\mathfrak{m}^{[p^e]}S)$ is flat. Thus the composition:

$$F^e_*(R/\mathfrak{m}^{[p^e]}) \to F^e_*(R/\mathfrak{m}^{[p^e]}) \otimes_{R/\mathfrak{m}} S/\mathfrak{m}S \to F^e_*(S/\mathfrak{m}^{[p^e]}S)$$

is flat (the first map is flat since it is base changed over a field), and hence $R/\mathfrak{m}^{[p^e]} \to S/\mathfrak{m}^{[p^e]}S$ is flat as desired.

We now prove the other direction that if $R \to S$ is regular, then $F^e_*R \otimes_R S \to F^e_*S$ is flat for all $e > 0$. For any ideal $J \subseteq R$, consider the ideal $F^e_*J(F^e_*R \otimes_R S) \cong F^e_*J \otimes_R S \subseteq F^e_*R \otimes_R S$. Since $R \to S$ (and hence $F^e_*R \to F^e_*R \otimes_R S$) is flat, we know that

$$\mathrm{Tor}_j^{F^e_*R\otimes_R S}(F^e_*S, (F^e_*R \otimes_R S)/F^e_*J(F^e_*R \otimes_R S)) \cong \mathrm{Tor}_j^{F^e_*R}(F^e_*S, F^e_*R/F^e_*J) = 0$$

for all $j > 0$. Apply the above discussion to the nilradical J of R, since $J^n = 0$ for $n \gg 0$ as R is Noetherian, if we can show that

$$F^e_*(R/J) \otimes_{R/J} (S/JS) \cong (F^e_*R \otimes_R S)/F^e_*J(F^e_*R \otimes_R S) \to F^e_*S/F^e_*J(F^e_*S) \cong F^e_*(S/J)$$

is flat, then by Lemma 10.3 (applied to $A = F^e_*R \otimes_R S$ and $I = F^e_*J(F^e_*R \otimes_R S)$) we will get that $F^e_*R \otimes_R S \to F^e_*S$ is flat as desired. Therefore, we may replace R by R/J to assume R is reduced.

We next note that, to show $F^e_*R \otimes_R S \to F^e_*S$ is flat, it is enough to check this at each prime ideal of S. Thus we may localize S at a prime ideal and localize R at the contraction to assume $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ is a regular local homomorphism.

Now we use induction on $\dim(R)$. If $\dim(R) = 0$, then since we may assume R is local and reduced, $R = k$ is a field and our hypothesis becomes that S is geometrically regular over k. Consider the composition:

$$F^e_*k \otimes_k S \to F^e_*S \to F^{2e}_*k \otimes_{F^e_*k} F^e_*S \cong F^e_*(F^e_*k \otimes_k S).$$

This composition is flat: $F^e_*k \otimes_k S = \varinjlim_{k'} k' \otimes_k S$ where k' runs over all finite field extensions of k contained in F^e_*k, since each $k' \otimes_k S$ is regular by our assumption, $k' \otimes_k S \to F^e_*(k' \otimes_k S)$ is flat by Theorem 1.1, and a direct limit of flat maps is flat. But the second map in the composition is obviously faithfully flat as it is base changed from field extensions. Thus the first map in the composition, $F^e_*k \otimes_k S \to F^e_*S$, is flat. This proves the case $\dim(R) = 0$.

Finally, we assume $\dim(R) > 0$. We may assume $(R, \mathfrak{m}, k)$ is local and reduced. Thus there exists a nonzerodivisor $t \in \mathfrak{m}$. Since $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ is flat, $F^e_*t \otimes 1$ is a nonzerodivisor on $F^e_*R \otimes_R S$ and F^e_*t is a nonzerodivisor on F^e_*S. By Lemma 10.4, to show $(F^e_*R \otimes_R S) \to F^e_*S$ is flat, it is enough to show that

(1) $(F^e_*R \otimes_R S)/F^e_*t(F^e_*R \otimes_R S) \to F^e_*S/F^e_*t(F^e_*S)$ is flat, and

(2) $(F^e_*R \otimes_R S)[\frac{1}{F^e_*t\otimes 1}] \to (F^e_*S)[\frac{1}{F^e_*t}]$ is flat.

Now the first map is the same as $F^e_*(R/tR)\otimes_{R/tR} S/tS \to F^e_*(S/tS)$, while the second map is the same as $F^e_*(R_t)\otimes_{R_t} S_t \to F^e_*(S_t)$. Since t is a nonzerodivisor, $\dim(R/tR) < \dim(R)$ and $\dim(R_t) < \dim(R)$. Thus by induction on dimension, we know both maps are flat (note that R_t is not local, but this doesn't matter, since to show $F^e_*(R_t)\otimes_{R_t} S_t \to F^e_*(S_t)$ is flat, we can localize at primes of S_t and their contractions to R_t again). This completes the proof. ☐

Our second goal is to show the following Kunz's theorem, proved in [Kun76], that every F-finite ring is excellent and a partial converse.

Theorem 10.5. *If R is an F-finite ring of prime characteristic $p > 0$ then R is excellent. Moreover, if $(R, \mathfrak{m}, k)$ is a local ring of prime characteristic $p > 0$, then R is F-finite if and only if R is excellent and k is F-finite.*

We start with a lemma.

Lemma 10.6. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of prime characteristic $p > 0$ and let K be the fraction field of R. Then for any finite field extension L of K, $L\otimes_R \widehat{R}$ is regular.*

Proof. For all $e > 0$, we have

$$F^e_*(L\otimes_R \widehat{R}) = F^e_*L\otimes_{F^e_*R} F^e_*\widehat{R} \cong F^e_*L\otimes_{F^e_*R} F^e_*R\otimes_R \widehat{R} = F^e_*L\otimes_R \widehat{R}$$

where the isomorphism in the middle follows from Lemma 9.2. Since F^e_*L is free over L, $F^e_*L\otimes_R \widehat{R}$ is free over $L\otimes_R \widehat{R}$. Thus by Theorem 1.1, $L\otimes_R \widehat{R}$ is regular (note that here we are implicitly using that $L\otimes_R \widehat{R}$ is Noetherian: it is module-finite over $K\otimes_R \widehat{R}$, which is a localization of $\widehat{R}$). ☐

We will also need the following fact about excellent rings, see [sta16, Tag 032E] for more details.

Lemma 10.7 ([sta16, Lemma 10.160.2])**.** *Let R be an excellent reduced ring with total quotient ring K. Then the integral closure of R in any finite reduced extension L of K is module-finite over R.*

Now we are ready to prove Kunz's theorem. Recall that R is excellent if R satisfies the following:

(1) R is universally catenary.
(2) If S is an R-algebra of finite type, then the regular locus of S is open in $\mathrm{Spec}(S)$.
(3) For all $P \in \mathrm{Spec}(R)$, the map $R_P \to \widehat{R_P}$ has geometrically regular fibers.

Proof of Theorem 10.5. We first show that if R is F-finite, then R is excellent. Since any ring finite type over an F-finite ring is still F-finite (see Exercise 5), to show R is universally catenary, it is enough to show that any F-finite ring is catenary.

Now let $P \subseteq Q$ be two prime ideals in R, we want to show any saturated chain of primes between P and Q have the same length. Suppose we have two saturated chains:

$$P = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = Q, \text{ and } P = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_m = Q.$$

Applying Theorem 9.4 to $R_{P_{i+1}}/P_iR_{P_{i+1}}$, we find that

$$[F^e_*\kappa(P_i) : \kappa(P_i)] = p^e \cdot [F^e_*\kappa(P_{i+1}) : \kappa(P_{i+1})] \text{ for all } i.$$

Thus $[F^e_*\kappa(P) : \kappa(P)] = p^{en} \cdot [F^e_*\kappa(Q) : \kappa(Q)]$, but then the same argument for the other chain shows that $[F^e_*\kappa(P) : \kappa(P)] = p^{em} \cdot [F^e_*\kappa(Q) : \kappa(Q)]$. It follows that $n = m$.

We next show that for any finite type R-algebra S, the regular locus of S is an open subset of $\mathrm{Spec}(S)$. But since S is F-finite, F^e_*S is a finitely generated S-module. By Theorem 1.1, S_P is regular if and only if $(F^e_*S)_P$ is a finite free S_P-module. Since F^e_*S is finitely generated, it is easy to see that if $(F^e_*S)_P$ is finite free over S_P, then there exists $f \notin P$ such that $(F^e_*S)_f$ is finite free over S_f. Thus the regular locus of S is open in $\mathrm{Spec}(S)$ and we have completed the proof that F-finite implies excellent.

It remains to show $R_P \to \widehat{R_P}$ has geometrically regular fibers. That is, for any $Q \subseteq P$ and any finite field extension $\kappa(Q)'$ of $\kappa(Q)$, $\kappa(Q)' \otimes_{R_P} \widehat{R_P}$ is regular. This follows immediately from Lemma 10.6 applied to R_P/QR_P.

We now prove that if $(R, \mathfrak{m}, k)$ is an excellent local ring with k an F-finite field, then R is F-finite. By Exercise 4 we may assume R is reduced. Let K be the total quotient ring of R, which is a product of fields $K = K_1 \times K_2 \times \cdots \times K_s$. Since R is excellent, each $K_i \otimes_R \widehat{R}$ is regular and thus $K \otimes_R \widehat{R}$ is regular and hence reduced. But since $\widehat{R} \hookrightarrow K \otimes_R \widehat{R}$, we see that $\widehat{R}$ is reduced. By Cohen's structure theorem, $\widehat{R}$ is a homomorphic image of $k[[x_1, \ldots, x_n]]$ and so by Exercise 5, $\widehat{R}$ is F-finite since k is F-finite. We next claim the following.

Claim 10.8. *$F^e_*K \otimes_R \widehat{R}$ is finitely generated over $K \otimes_R \widehat{R}$ for all $e > 0$.*

Proof. For any $L = L_1 \times L_2 \times \cdots \times L_s$ where L_i is a finite field extension of K_i, since $R \to \widehat{R}$ has geometrically regular fibers, we know that $L \otimes_R \widehat{R}$ is regular. Thus by Theorem 1.1, $L \otimes_R \widehat{R} \to F^e_*(L \otimes_R \widehat{R})$ is faithfully flat. By considering all finite extensions L_i between K_i and $F^e_*K_i$ and taking a direct limit, we find that $F^e_*K \otimes_R \widehat{R} \to F^e_*(F^e_*K \otimes_R \widehat{R})$ is faithfully flat. But this map factors as

$$F^e_*K \otimes_R \widehat{R} \to F^e_*(K \otimes_R \widehat{R}) \to F^e_*(F^e_*K \otimes_R \widehat{R})$$

and obviously, $K \otimes_R \widehat{R} \to F^e_* K \otimes_R \widehat{R}$ is faithfully flat as K is a product of field (or one can use Theorem 1.1 since K is regular). Therefore we find that $F^e_* K \otimes_R \widehat{R} \to F^e_*(K \otimes_R \widehat{R})$ is faithfully flat, in particular it is injective. But since $\widehat{R}$ is F-finite, $K \otimes_R \widehat{R}$ is F-finite since it is a localization of $\widehat{R}$, we know that $F^e_*(K \otimes_R \widehat{R})$ is finitely generated over $K \otimes_R \widehat{R}$. Therefore $F^e_* K \otimes_R \widehat{R}$ is finitely generated over $K \otimes_R \widehat{R}$ as desired. □

Finally, since $\widehat{R}$ is faithfully flat over R, by Claim 10.8 we see that $F^e_* K$ is finitely generated over K. Now we apply Lemma 10.7, we know that the integral closure of R inside $F^e_* K$ is module-finite over R. But clearly $F^e_* R$ is contained inside this integral closure, hence $F^e_* R$ is module-finite over R, that is, R is F-finite. □

Our final goal is to explain in detail the following result of Gabber [Gab04].

Theorem 10.9. *If R is an F-finite ring of prime characteristic $p > 0$ then R is a homomorphic image of an F-finite regular ring. In particular, every F-finite ring admits a canonical module.*

Proof. Let R^p be the subring of R consisting of p-th powers of elements of R. Note that R is F-finite is equivalent to saying that R is module-finite over R^p. Let $a_1, \dots, a_s$ be generators of R as a module over R^p. Set

$$R_n := \frac{R[z_1, \dots, z_s]}{(z_1^{p^n} - a_1, \dots, z_s^{p^n} - a_s)}.$$

Consider the inverse system:

$$\cdots \twoheadrightarrow R_n \twoheadrightarrow R_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow R_0 = R$$

where each $R_n \to R_{n-1}$ is the Frobenius map on R and the identity map on $z_1, \dots, z_s$, it is easy to see that the map is surjective for all n. Set $R_\infty := \varprojlim_n R_n$ and we will show R_∞ is a (Noetherian) F-finite regular ring. By Theorem 1.1, it is enough to show:

(1) R_∞ is Noetherian
(2) R_∞ is reduced
(3) R_∞ is generated over R_∞^p freely by $\{z_{1\bullet}^{i_1} \cdots z_{s\bullet}^{i_s}\}_{0 \le i_j \le p-1}$ where $z_{j\bullet}$ denotes the constant sequence $(\cdots \to z_j \to z_j \to \cdots \to z_j) \in R_\infty$.

We first prove (3). Since R is generated by $a_1, \dots, a_s$ over R^p. By the definition of R_n, it is easy to check that R_n is generated *freely* over R_n^p by $\{z_1^{i_1} \cdots z_s^{i_s}\}_{0 \le i_j \le p-1}$ for any $n \ge 1$.[12] Thus the conclusion follows as we pass to the inverse limit.

[12]We caution the reader that one cannot invoke Theorem 1.1 to say that R_n is regular, this is because R_n is not reduced so we cannot identify $R_n^p \to R_n$ with $R_n \to F_* R_n$.

We next prove (2). To ease the presentation we will use the following notations for the rest of the argument: $\underline{i}$ denotes an s-tuple $i_1, \ldots, i_s$, $\lambda\underline{i}$ means $\lambda i_1, \ldots, \lambda i_s$, $\underline{i} \equiv \underline{j}$ means $i_k \equiv j_k$ for each k, and $\alpha \le \underline{i} \le \beta$ means $\alpha \le i_k \le \beta$ for each k. Moreover, we set $\underline{z}^{\underline{i}} := z_1^{i_1} \cdots z_s^{i_s}$ and $\underline{a}^{\underline{i}} := a_1^{i_1} \cdots a_s^{i_s}$.

Claim 10.10. $\operatorname{Ker}(R_n \to R_{n-1}) = \{x \in R_n | x^p = 0\}$.

Proof. Suppose $x = \sum_{0 \le \underline{i} < p^n} a_{\underline{i}} \underline{z}^{\underline{i}} \in R_n$ where $a_{\underline{i}} \in R$. Then $x^p = \sum_{0 \le \underline{i} < p^n} a_{\underline{i}}^p \underline{z}^{p\underline{i}}$. Write

$$\sum_{0 \le \underline{i} < p^n} a_{\underline{i}}^p \underline{z}^{p\underline{i}} = \sum_{0 \le \underline{j} < p^{n-1}} \sum_{\substack{\underline{i} \equiv \underline{j} \\ \text{mod } p^{n-1}}} a_{\underline{i}}^p \underline{z}^{p(\underline{i}-\underline{j})} \underline{z}^{p\underline{j}} = \sum_{0 \le \underline{j} < p^{n-1}} \Big(\sum_{\substack{\underline{i} \equiv \underline{j} \\ \text{mod } p^{n-1}}} a_{\underline{i}}^p \underline{a}^{\frac{1}{p^{n-1}}(\underline{i}-\underline{j})} \Big) \underline{z}^{p\underline{j}},$$

we see that

$$x^p = 0 \text{ if and only if for each } \underline{j}, \quad \sum_{\substack{\underline{i} \equiv \underline{j} \\ \text{mod } p^{n-1}}} a_{\underline{i}}^p \underline{a}^{\frac{1}{p^{n-1}}(\underline{i}-\underline{j})} = 0.$$

But this is equivalent to saying that

$$\sum_{0 \le \underline{j} < p^{n-1}} \sum_{\substack{\underline{i} \equiv \underline{j} \\ \text{mod } p^{n-1}}} a_{\underline{i}}^p \underline{a}^{\frac{1}{p^{n-1}}(\underline{i}-\underline{j})} \underline{z}^{\underline{j}} = 0 \text{ in } R_{n-1}$$

since R_{n-1} is finite free over R with basis $\{\underline{z}^{\underline{j}}\}_{0 \le \underline{j} < p^{n-1}}$. But note that in R_{n-1}, we have

$$\sum_{0 \le \underline{j} < p^{n-1}} \sum_{\substack{\underline{i} \equiv \underline{j} \\ \text{mod } p^{n-1}}} a_{\underline{i}}^p \underline{a}^{\frac{1}{p^{n-1}}(\underline{i}-\underline{j})} \underline{z}^{\underline{j}} = \sum_{0 \le \underline{i} < p^n} a_{\underline{i}}^p \underline{z}^{\underline{i}},$$

which is precisely the image of x under the map $R_n \to R_{n-1}$ (by definition of this map). Therefore $x^p = 0$ if and only if $x \in \operatorname{Ker}(R_n \to R_{n-1})$. $\square$

Claim 10.10 immediately implies that $R_\infty = \varprojlim_n R_n$ is reduced. We have completed the proof of (2).

Finally, we prove (1). This will take some work. We first let

$$K_{n+m,n} := \operatorname{Ker}(R_{n+m} \to R_n)$$

and we claim the following.

Claim 10.11. *For all $n \ge 0$ and $m \ge 1$, $K_{n+m,n} = (K_{n+m,0})^{[p^n]}$ as ideals in R_{n+m}.*

Proof. By Claim 10.10 (and an easy induction), we have that $(K_{n+m,0})^{[p^n]} \subseteq K_{n+m,n}$. Now let $r \in K_{n+m,n}$. We write

$$r = \sum_{0 \le \underline{i} < p^{n+m}} r_{\underline{i}} \underline{z}^{\underline{i}} = \sum_{0 \le \underline{j} < p^n} \Big(\sum_{\substack{\underline{i} \equiv \underline{j} \\ \text{mod } p^n}} r_{\underline{i}} \underline{z}^{\underline{i}-\underline{j}} \Big) \underline{z}^{\underline{j}}$$

where $r_{\underline{i}} \in R$. Since $\underline{a}$ generates R over R^p, $\{\underline{a}^{\underline{k}}\}_{0\leq \underline{k}<p^n}$ generates R over R^{p^n}. Thus we can write

$$r_{\underline{i}} = \sum_{0\leq \underline{k}<p^n} b_{\underline{i},\underline{k}}^{p^n}\underline{a}^{\underline{k}} = \sum_{0\leq \underline{k}<p^n} b_{\underline{i},\underline{k}}^{p^n}\underline{z}^{p^{n+m}\underline{k}} \text{ in } R_{n+m},$$

where $b_{\underline{i},\underline{k}} \in R$. Thus we have

$$r = \sum_{0\leq \underline{j}<p^n}\Big(\sum_{\substack{\underline{i}\equiv\underline{j} \\ \mathrm{mod}\ p^n}}\sum_{0\leq \underline{k}<p^n} b_{\underline{i},\underline{k}}\underline{z}^{p^m\underline{k}}\underline{z}^{\frac{1}{p^n}(\underline{i}-\underline{j})}\Big)^{p^n}\underline{z}^{\underline{j}}.$$

In order to show $r \in K_{n+m,0}^{[p^n]}$, it is enough to show that for each $\underline{j}$,

$$\sum_{\substack{\underline{i}\equiv\underline{j} \\ \mathrm{mod}\ p^n}}\sum_{0\leq \underline{k}<p^n} b_{\underline{i},\underline{k}}\underline{z}^{p^m\underline{k}}\underline{z}^{\frac{1}{p^n}(\underline{i}-\underline{j})} \in K_{n+m,0}.$$

But its image in $R = R_0$ is (note that in R_0, $\underline{z} = \underline{a}$)

$$c_{\underline{j}} := \sum_{\substack{\underline{i}\equiv\underline{j} \\ \mathrm{mod}\ p^n}}\sum_{0\leq \underline{k}<p^n} b_{\underline{i},\underline{k}}^{p^{n+m}}\underline{a}^{p^m\underline{k}}\underline{a}^{\frac{1}{p^n}(\underline{i}-\underline{j})},$$

and our hypothesis $r \in K_{n+m,n}$ implies that

$$\sum_{0\leq \underline{j}<p^n}\Big(\sum_{\substack{\underline{i}\equiv\underline{j} \\ \mathrm{mod}\ p^n}} r_{\underline{i}}^{p^m}\underline{z}^{\underline{i}-\underline{j}}\Big)\underline{z}^{\underline{j}} = \sum_{0\leq \underline{j}<p^n}\Big(\sum_{\substack{\underline{i}\equiv\underline{j} \\ \mathrm{mod}\ p^n}} r_{\underline{i}}^{p^m}\underline{a}^{\frac{1}{p^n}(\underline{i}-\underline{j})}\Big)\underline{z}^{\underline{j}} = 0 \text{ in } R_n.$$

Since R_n is finite free over R with basis $\{\underline{z}^{\underline{j}}\}_{0\leq \underline{j}<p^n}$, this implies that for every $\underline{j}$,

$$0 = \sum_{\substack{\underline{i}\equiv\underline{j} \\ \mathrm{mod}\ p^n}} r_{\underline{i}}^{p^m}\underline{a}^{\frac{1}{p^n}(\underline{i}-\underline{j})} = \sum_{\substack{\underline{i}\equiv\underline{j} \\ \mathrm{mod}\ p^n}}\sum_{0\leq \underline{k}<p^n} b_{\underline{i},\underline{k}}^{p^{n+m}}\underline{a}^{p^m\underline{k}}\underline{a}^{\frac{1}{p^n}(\underline{i}-\underline{j})} = c_{\underline{j}},$$

which is exactly what we want. ◻

At this point, we set $J_n := \mathrm{Ker}(R_\infty \to R_n)$. Note that we have

$$J := J_0 \supseteq J_1 \supseteq \cdots \supseteq J_n \supseteq \cdots$$

We next claim the following

Claim 10.12. *For each $n \geq 0$, $J_n \subseteq \cap_{m\geq 0}(J^{[p^n]} + J_m) \subseteq \cap_{m\geq 0}(J^n + J_m)$.*

Proof. The second inclusion is trivial. We prove the first inclusion. Pick $x_\bullet \in J_n$, which can be thought of as a sequence

$$x_\bullet = \cdots \to x_{m+1} \to x_m \to \cdots \to x_n = 0 \to \cdots \to x_0 = 0.$$

In particular, $x_m \in K_{m,n} = K_{m,0}^{[p^n]}$ by Claim 10.11 and thus we can write $x_m = \sum r_{im} y_{im}^{p^n}$ where $r_{im} \in R_m$ and $y_{im} \in K_{m,0}$. Since the inverse system has surjective transition maps, r_{im}, y_{im} are images of $r_{i\bullet}, y_{i\bullet} \in R_\infty$ and $y_{i\bullet} \in J_0 = J$ by construction. Thus by looking at the m-th entry we find that $x_\bullet - \sum r_{i\bullet} y_{i\bullet}^{p^n} \in J_m$. Therefore $J_n \subseteq J^{[p^n]} + J_m$ as desired. ☐

Next we set $I_n = \cap_m (J^n + J_m)$. It is clear that $J = I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ and that $\{I_n\}_{n\geq 1}$ is a graded family of ideals in R_∞ (i.e., $I_n I_m \subseteq I_{n+m}$).

Claim 10.13. *R_∞ is complete with respect to the topology defined by $\{I_n\}_{n\geq 1}$.*

Proof. By Claim 10.12, $J_n \subseteq I_n$ for each $n \geq 1$. Consider the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & I_{n+1}/J_{n+1} \arrow[r] \arrow[d, hook] & I_n/J_n \arrow[r] \arrow[d, hook] & I_{n-1}/J_{n-1} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r, two heads] & R_{n+1} = R_\infty/J_{n+1} \arrow[r, two heads] \arrow[d] & R_n = R_\infty/J_n \arrow[r, two heads] \arrow[d] & R_{n-1} = R_\infty/J_{n-1} \arrow[r, two heads] \arrow[d] & \cdots \\
\cdots \arrow[r, two heads] & R_\infty/I_{n+1} \arrow[r, two heads] & R_\infty/I_n \arrow[r, two heads] & R_\infty/I_{n-1} \arrow[r, two heads] & \cdots .
\end{tikzcd}$$

The inverse limit of the second row is R_∞ by definition. Thus to prove the claim it is enough to show that the first row is a null system, that is, for each $n \geq 1$ there exists $k \gg 0$ such that $I_k \subseteq J_n$.

For each $y_\bullet = (\cdots \to y_{n+1} \to y_n \to \cdots \to 0) \in I_1 = J$, we have $y_{n+1} \in K_{n+1,0}$ for all $n \geq 0$. Since $K_{n+1,n} = (K_{n+1,0})^{[p^n]}$ by Claim 10.11, we can pick $k \gg 0$ (depends on n) such that $(K_{n+1,0})^k \subseteq (K_{n+1,0})^{[p^n]} = K_{n+1,n}$ (this is possible since $K_{n+1,0}$ is finitely generated, as it is an ideal in a Noetherian ring R_{n+1}). Therefore for each $x_\bullet \in I_k = \cap_{m\geq 0}(J^k + J_m) \subseteq J^k + J_{n+1}$, the $(n+1)$-th entry x_{n+1} is contained in $(K_{n+1,0})^k$ as this holds for all elements in J^k and elements in J_{n+1} have $(n+1)$-th entry 0. Thus $x_{n+1} \in K_{n+1,n}$ by our choice of k and hence $x_n = 0$, which implies $x_\bullet \in J_n$. So $I_k \subseteq J_n$ as desired. ☐

Finally, we claim the following.

Claim 10.14. *The associated graded ring $gr_{I_\bullet} R_\infty := (R_\infty/I_1) \oplus (I_1/I_2) \oplus \cdots$ is Noetherian.*

Proof. Since $R_\infty/I_1 = R_\infty/J \cong R$ is Noetherian and I_1/I_2 is finitely generated (it can be viewed as an ideal in R_∞/I_2, which is Noetherian since it is a quotient of $R_\infty/J_2 \cong R_2$). Thus to show $gr_{I_\bullet} R$ is Noetherian, it is enough to show that $I_n/I_{n+1} = (I_1/I_2)^n$, that is, $I_n \subseteq I_1^n + I_{n+1}$ for all $n \geq 1$ (the other inclusion is clear). Since $I_{n+1} \supseteq J_{n+1}$ by Claim 10.12, it is enough to show $I_n \subseteq I_1^n + I_{n+1}$ modulo J_{n+1}. But recall that $I_n = \cap_{m\geq 0}(J^n + J_m)$, thus after modulo J_{n+1}, I_n is generated by $J^n = I_1^n$. ☐

Now the conclusion of (1) that R_∞ is Noetherian follows from Claim 10.13 and Claim 10.14. For any ideal $I \subseteq R_\infty$, its image in $gr_{I_\bullet}R_\infty$ is finitely generated, say by $\overline{f}_1, \dots, \overline{f}_t$. We claim that I is generated by $f_1, \dots, f_t$: given any $x \in I$, suppose $x \in I_n - I_{n+1}$, then we can find $x_1, \dots, x_t$ such that $x' := x - (f_1x_1 + \cdots f_tx_t) \in I_{n+1} \cap I$, now pick $n' > n$ such that $x' \in I_{n'} - I_{n'+1}$, we can find $x_1', \dots, x_t'$ such that $x'' := x' - (f_1x_1' + \cdots f_tx_t') \in I_{n'+1} \cap I$, continuing this process and using R_∞ is complete with respect to $\{I_n\}_{n\geq 1}$, it is easy to check that eventually we can write $x = f_1y_1 + \cdots + f_ny_n$, so I is generated by $f_1, \dots, f_n$.

We have completed the proof that R is a homomorphic image of an F-finite regular ring, call it S. By Exercise 47, we have $\dim(R) = d < \infty$ and $\dim(S) = n < \infty$. Therefore $\operatorname{Ext}_S^{n-d}(R, S)$ is a canonical module of R. □

Remark 10.15. It is worth pointing out that not all excellent local rings admit canonical modules, for example see [Nis12, Example 6.1].

Exercise 45. Let $R \to S$ be a homomorphism of rings of prime characteristic $p > 0$ such that $F^e_*R \otimes_R S \to F^e_*S$ is pure. Prove that all fibers of $R \to S$ are F-pure.

Exercise 46. Let $R \to S$ be a regular homomorphism of (Noetherian) rings of prime characteristic $p > 0$. Prove that $F^e_*R \otimes_R S$ is a Noetherian ring. (Hint: Use Theorem 10.1 and the hint in Exercise 31.)

Exercise 47. Let R be a (not necessarily local) F-finite ring of prime characteristic $p > 0$ and let $P \subseteq Q$ be two prime ideals of R. Prove that

$$\operatorname{ht}(P) + \log_p \operatorname{rank}_{\kappa(P)}(F_*\kappa(P)) = \operatorname{ht}(Q) + \log_p \operatorname{rank}_{\kappa(Q)}(F_*\kappa(Q)).$$

Use this to show that $\dim(R) < \infty$.

It is natural to ask whether the property of being F-finite is a local property. It turns out that this is not always true! Counter-examples are constructed (quite recently) in [DI20]. On the other hand, we have the following.

Exercise 48. Let R be an excellent ring of prime characteristic $p > 0$. Prove that if R_P is F-finite for all $P \in \operatorname{Spec}(R)$, then R is F-finite. (Hint: Use Lemma 10.7.)

Appendix A. Generalized divisors and class groups

We collect the basics of generalized divisors and the divisor class group of a Noetherian ring which is not necessarily assumed to be normal. We refer to the reader to [Har94, Section 2] for details.

Let R be a (Noetherian) ring and let K denote its total ring of fractions. We assume that R satisfies Serre's condition (S_2) and is (G_1), i.e., R is Gorenstein in codimension 1. A finitely generated R-submodule I of K is a *fractional ideal*. We say that I is *non-degenerate* if $I_P = K_P$ for each minimal prime P of R. The *inverse* of a fractional ideal I is the fractional ideal $I^{-1} := \{f \in K \mid fI \subseteq R\}$. Observe that $I^{-1} = \operatorname{Hom}_R(I, R) := I^*$ and so if I is non-degenerate then so is I^{-1}. If a fractional ideal I is reflexive, i.e., $I \to I^{**}$ is an isomorphism, then I is called a *generalized divisor*. Note that since we are assuming R is (S_2) and (G_1), I is reflexive if and only if I is (S_2) as an R-module, see Exercise 52. If $I \subseteq R$ then I is called *effective*. There is a one-to-one correspondence between non-degenerate effective reflexive fractional ideals of R and codimension 1 subschemes of $\operatorname{Spec}(R)$ without embedded components.

We aim to describe the divisor class group of R. To do so, it is convenient to use additive notation. So if D_1, D_2 represents generalized divisors I_1, I_2, then we use $D_1 + D_2$ to represent the generalized divisor

$$((I_1 I_2)^{-1})^{-1} = \operatorname{Hom}_R(\operatorname{Hom}_R(I_1 I_2, R), R) = (I_1 I_2)^{**}$$

and $-D_1$ to represent I_1^{-1} (note that, with the additive notion, 0 represents R). A generalized divisor D is *almost Cartier* if its corresponding fractional ideal I is principal in codimension 1. If R is normal then every divisor is almost Cartier.

Let D be a generalized divisor correspond to a fractional ideal I. We define the divisorial ideal associated to D to be $R(D) := I^{-1}$. Note that, with this notation, D is effective if and only if $R \subseteq R(D)$. We will say $D_1 \geq D_2$ if $D_1 - D_2$ is effective. For any nonzerodivisor $f \in K$, we use $\operatorname{div}(f)$ to denote the *principal divisor* that corresponds to the fractional ideal (f), i.e., $R(\operatorname{div}(f)) = R \cdot \frac{1}{f}$. Now if D_1, D_2 are almost Cartier generalized divisors then D_1 is *linearly equivalent* to D_2, $D_1 \sim D_2$, if $D_1 - D_2$ is a principal divisor. It is easy to see that $D_1 - D_2 = \operatorname{div}(f)$ if and only if $(f) \cdot R(D_1) = R(D_2)$, in other words, $D_1 \sim D_2$ if and only if $R(D_1) \cong R(D_2)$ as R-modules.

The *divisor class group of* R, denoted by $\operatorname{Cl}(R)$, is the abelian group of almost Cartier generalized divisors modulo linear equivalence. Abusing notations a bit, a *divisor* is a choice of an almost Cartier generalized divisor that represents an element of $\operatorname{Cl}(R)$.

Suppose further that R admits a canonical module ω_R. The assumption that R is (S_2) and (G_1) insures that $\omega_R \cong R(K_X)$ for some (almost Cartier generalized) divisor K_X of $X = \operatorname{Spec}(R)$. Any such divisor is referred to as a *canonical divisor*. If K_X is a torsion element of $\operatorname{Cl}(R)$ then R is said to be $\mathbb{Q}$-*Gorenstein*. The $\mathbb{Q}$-Gorenstein index of R is the least positive integer N so that NK_X is a principal divisor. Whenever a ring is described as being $\mathbb{Q}$-Gorenstein, it is implicitly assumed that R is (S_2) and (G_1) and admits a canonical module. We say that R is quasi-Gorenstein if R is $\mathbb{Q}$-Gorenstein of $\mathbb{Q}$-Gorenstein index 1.

Remark A.1. If $(R, \mathfrak{m}, k)$ is local, then R admits a canonical module if and only if R is a homomorphic image of a Gorenstein local ring. If, in addition, R is equidimensional (which holds if R is (S_2) and is a homomorphic image of a Gorenstein local ring), then we have $(\omega_R)_P$ is a canonical module for R_P for all $P \in \operatorname{Spec}(R)$, see [Aoy83, HH94b] or [sta16, Sections 47.16–47.19] for more details.

It is important for us to understand when maps of (S_2)-modules are isomorphisms, see Exercise 53. To this end we present a proposition. For simplicity of our presentation, we make a convention that a finitely generated (S_2)-module has no associated primes that are not minimal (e.g., when R is a domain, then we are assuming (S_2)-modules are automatically torsion-free), this condition holds for all divisorial ideals $R(D)$ discussed above so there should be no ambiguity.

Proposition A.2. *Let R be an (S_2) ring and $N \to M$ a map of finitely generated (S_2) R-modules. Then the following are equivalent:*

(1) $N \to M$ is an isomorphism;
(2) $N \to M$ is an isomorphism in codimension 1, i.e., $N_P \to M_P$ is an isomorphism for each height ≤ 1 prime ideal $P \in \operatorname{Spec}(R)$.

Proof. Suppose that $N_P \to M_P$ is an isomorphism for each height ≤ 1 prime ideal $P \in \operatorname{Spec}(R)$. Let K be the kernel of $N \to M$. Then K is a submodule of N and if $K \neq 0$, then the associated primes of K has height at least two, which is not possible since K is a submodule of an (S_2) R-module (see our convention above). Thus we have a short exact sequence of the form

$$0 \to N \to M \to C \to 0$$

and the module C localizes to 0 at every height ≤ 1 prime ideal of R. Suppose by way of contradiction that $C \neq 0$ and choose $P \in \operatorname{Spec}(R)$ that is minimal as an element of the support of C. Then P has height at least 2. In particular, we can localize at P and consider the short exact sequence

$$0 \to N_P \to M_P \to C_P \to 0$$

where the modules N_P, M_P have depth at least 2 and C_P is a nonzero finite length R_P-module. This is impossible, if we consider the long exact sequence of local cohomology modules we find that there is an exact sequence

$$H^0_{PR_P}(M_P) \to H^0_{PR_P}(C_P) \to H^1_{PR_P}(N_P)$$

and hence either M_P has depth 0 or N_P has depth no more than 1. $\square$

If R is normal, then the divisor class group of R described above agrees with the standard definition. Specifically, if R is normal then a divisor can be defined to be an element of the free abelian group generated by the irreducible codimension 1 subvarieties of $X = \operatorname{Spec}(R)$ and $\operatorname{Cl}(R)$ is the group of Weil divisors modulo linear equivalence.

A.1. **Cyclic covers.** Let $(R, \mathfrak{m}, k)$ be a local (S_2) and (G_1) ring and let D be a torsion divisor of index N, i.e., ND is a principal divisor. Suppose that $R(ND) = R \cdot f$ where f is a nonzerodivisor of the total ring of fractions K of R. For every pair of natural numbers i, j we have that $R(iD)R(jD) \subseteq R((i+j)D)$ and so we can consider the following graded R-algebra

$$T := \bigoplus_{i=0}^{\infty} R(iD)t^i.$$

It is not difficult to see that T is finitely generated over R by elements of degree no more than N and $S := T/(ft^N - 1)$ decomposes as an R-module as $R \oplus R(D) \oplus \cdots \oplus R((N-1)D)$. The ring S is referred to as a *cyclic cover of* R *with respect to the divisor* D. Observe that if g is a different choice of generator of $R(ND)$ then we can form the cyclic cover $S' := T/(gt^N - 1)$. The rings S and S' need not be isomorphic.

Example A.3. Let k be a field, $R = k[[x, y, z]]/(xy + z^3)$, and consider the height 1 prime $P = (x, z)$. Consider the divisor D corresponds to the fractional ideal P^{-1} (so D is anti-effective) and observe that D is torsion of index 3 with

- $R(D) = P = (x, z)$;
- $R(2D) = P^{(2)} = (x, z^2)$;
- $R(3D) = P^{(3)} = (x)$.

As an R-module, a cyclic cover of R with respect to D decomposes as

$$R \to S = R \oplus R(D)t \oplus R(2D)t^2 = R \oplus Pt \oplus P^{(2)}t^2 = R \oplus (x, z)t \oplus (x, z^2)t^2.$$

To understand the multiplicative structure of S, with respect to the choice of generator x of $R(3D)$, let us consider the product of the elements zt and xt^2 of S as an example. Then

$$zt \cdot xt^2 = z(xt^3) = z \cdot 1 = z.$$

The following lemma shows two important pieces of information concerning cyclic covers.

Lemma A.4. *Let $(R, \mathfrak{m}, k)$ be a local (S_2) and (G_1) ring. Suppose that D is a torsion divisor of index N and $S = \bigoplus_{i=0}^{N-1} R(iD)t^i$ a cyclic cover of R by D.*

(1) *The ring S is local with unique maximal ideal $\mathfrak{m} \oplus \bigoplus_{i=0}^{N-1} R(iD)t^i$;*
(2) *If $\pi : S \to R$ is the projection of S onto the degree 0 component of S then π principally generates $\operatorname{Hom}_R(S, R)$, i.e., if $\psi \in \operatorname{Hom}_R(S, R)$ then there exists $s \in S$ so that $\psi = \pi(s-)$.*

Proof. We first check that $\mathfrak{m} \oplus \bigoplus_{i=0}^{N-1} R(iD)t^i$ is an ideal of S. Once this is established it is easy to see that $\mathfrak{m} \oplus \bigoplus_{i=0}^{N-1} R(iD)t^i$ is the unique maximal ideal of S (we leave it to the reader to check that every element not belonging to this ideal is a unit). Showing that $\mathfrak{m} \oplus \bigoplus_{i=0}^{N-1} R(iD)t^i$ is an ideal of S amounts to checking that if $1 \leq i \leq N-1$, $a \in R(iD)$, and $b \in R((N-i)D)$ then $at^i \cdot bt^{N-i}$ is an element of $\mathfrak{m}$. Suppose that f is a choice of principal generator of $R(ND)$ defining the multiplicative structure of S. Suppose by way of contradiction that $at^i \cdot bt^{N-i} = \frac{ab}{f} = u$ for some unit u of R. Then $ab = uf$ and so $\operatorname{div}(a) + \operatorname{div}(b) = \operatorname{div}(f)$. This provides us the following information:

(1) $a \in R(iD)$ and therefore $\operatorname{div}(a) \geq -iD$;
(2) $b \in R((N-i)D)$ and therefore $\operatorname{div}(b) \geq -(N-i)D$;
(3) $\operatorname{div}(f) = \operatorname{div}(ab) = \operatorname{div}(a) + \operatorname{div}(b) = -ND = -iD - (N-i)D$.

Properties $(1), (2)$, and (3) can only hold if $\operatorname{div}(a) = -iD$ and $\operatorname{div}(b) = -(N-i)D$, contradicting the initial assumption that the index of D is N.

We now aim to show that $\operatorname{Hom}_R(S, R)$ is a principal S-module. There is an isomorphism of S-modules
$$\operatorname{Hom}_R(S, R) \cong \bigoplus_{i=0}^{N-1} \operatorname{Hom}_R(R(iD)t^i, R).$$
Furthermore, $\operatorname{Hom}_R(R(iD), R) \cong R(-iD)$, i.e., if $\lambda : R(iD) \to R$ is R-linear then there exists $x \in R(-iD)$ so that $\lambda(\eta) = x\eta$ for all $\eta \in R(iD)$. To show that $\operatorname{Hom}_R(S, R)$ is principally generated as an S-module by the projection map π it is enough to show that if $\psi : S \to R$ is the composition of S projected onto $R(iD)$ followed by the multiplication map $\lambda : R(iD) \to R$ defined by $\lambda(\eta) = x\eta$ then $\psi = \pi(s-)$ for some $s \in S$. This is indeed the case, suppose that $R(ND) = R \cdot f$. Then $fx \in R((N-i)D)$ and we consider the element $s = fxt^{N-i}$ of S. Then $s \cdot \eta t^i = fx\eta t^N = x\eta$. It readily follows that $\psi = \pi(s-)$ as claimed. $\square$

Proposition A.5. *Let $(R, \mathfrak{m}, k)$ be a local (S_2) and (G_1) ring. Suppose that D is a torsion divisor of index N and $S = \bigoplus_{i=0}^{N-1} R(iD)t^i$ is a cyclic cover of R by D. Let $\pi \in \operatorname{Hom}_R(S, R)$*

be the projection of S onto R. Then $S \to \mathrm{Hom}_R(S,R)$ defined by mapping $s \mapsto \pi(s-)$ is an isomorphism. Under this isomorphism, the evaluation-at-1 map $e_1 : \mathrm{Hom}_R(S,R) \to R$ defined by $\psi \mapsto \psi(1)$ corresponds to the projection map π.

Proof. The map $S \to \mathrm{Hom}_R(S,R)$ sending $s \mapsto \pi(s-)$ is onto by Lemma A.4. We leave as an exercise to the reader to verify that $S \xrightarrow{s\mapsto\pi(s-)} \mathrm{Hom}_R(S,R)$ is injective, see Exercise 51.

Showing that π corresponds to the evaluation-at-1 map e_1 under the isomorphism

$$S \xrightarrow{s\mapsto\pi(s-)} \mathrm{Hom}_R(S,R)$$

is equivalent to observing the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "s\mapsto\pi(s-)"] \arrow[dr, "\pi"'] & \mathrm{Hom}_R(S,R) \arrow[d, "e_1"] \\ & R. \end{tikzcd}$$

□

We point out that the (S_2) and (G_1) properties are preserved when we pass from a local ring to a cyclic cover, a proof of this fact is contained in the proof of Lemma A.7 below. We caution the reader that in prime characteristic $p > 0$ it might happen that a cyclic cover of a normal domain fails to be normal (though it is always a domain in our context), see [TW92] for more detailed discussions. For this reason, it is important for us to relax ourselves to work with (S_2) and (G_1) rings.

A.2. **Pull back divisors.** Let $R \to S$ be a map of (S_2) and (G_1) rings that corresponds to a map of schemes $\pi\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$. Given a divisor D on R, we want to pull it back along π to obtain a divisor π^*D on S – this is not always possible if D is not Cartier. We thus restrict ourselves in the following two cases:

- $R \to S$ is a module-finite extension.
- $S = \overline{R} := R/xR$ where x is a nonzerodivisor of R.

Discussion A.6. Recall that a divisor D on R corresponds to a fractional ideal $R(D)$ that is principal in codimension 1. In the case $R \to S$ is a module-finite extension, we define π^*D to be the divisor on S such that $S(\pi^*D) = (R(D)S)^{**}$, where $(-)^* := \mathrm{Hom}_S(-,S)$. In the case that $S = R/xR$, we need to replace the divisor D by D' linearly equivalent to D such that D' has no component in $V(x)$ (which is always possible by Lemma A.8), and then define π^*D' to be the divisor such that $S(\pi^*D') = (R(D')S)^{**}$, in this case we will also write $\overline{D'}$ for π^*D' to indicate that $\overline{D'}$ is a divisor of $\overline{R}$. Note that in the second case, we are actually defining a map $\mathrm{Cl}(R) \to \mathrm{Cl}(S)$. We leave it to the reader in Exercise 49 to check that these are well-defined.

Lemma A.7. *Let $(R, \mathfrak{m}, k)$ be a $\mathbb{Q}$-Gorenstein local ring with choice of canonical divisor K_X on $X = \mathrm{Spec}(R)$ that has index NM, with N and M positive integers. If $D = NK_X$ and $R \to S$ the cyclic cover of R with respect to D then S is $\mathbb{Q}$-Gorenstein of index N.*

Proof. We first check that S is (S_2) and (G_1). The extension $R \to S$ is finite and S decomposes as a finite direct sum of R-modules which are (S_2), thus S is (S_2) as a ring. If $Q \in \mathrm{Spec}(S)$ is a height 1 prime then $P = R \cap Q$ is a height 1 prime of R and S_Q is a localization of $S_P := S \otimes_R R_P$. The canonical module $R(K_X)_P$ is a principal fractional ideal of R_P. Thus S_P is isomorphic to a ring of the form $R_P[Z]/(f)$ where Z is a variable. In particular, S_P and its localization S_Q are Gorenstein.

To compute the $\mathbb{Q}$-Gorenstein index of S, note that we have

$$\begin{aligned}
\omega_S \cong \mathrm{Hom}_R(S, \omega_R) &\cong \mathrm{Hom}_R(\oplus_{i=0}^{M-1} R(iD)t^i, R(K_X)) \\
&\cong \oplus_{i=0}^{M-1} R(K_X - iD)t^{-i} \\
&\cong \oplus_{i=0}^{M-1} R(K_X + (M-i)D)t^{M-i} \\
&\cong (R(K_X) \cdot S)^{**}.
\end{aligned}$$

Therefore $\pi^* K_X$ is linearly equivalent to K_Y where $\pi : Y = \mathrm{Spec}(S) \to X = \mathrm{Spec}(R)$. Thus NK_Y is linearly equivalent to $\pi^*(NK_X) = \pi^* D$ which is principal, see Exercise 50. On the other hand if $N' < N$, then we have

$$S(N'K_Y) \cong (R(N'K_X) \cdot S)^{**} \cong \oplus_{i=0}^{M-1} R(N'K_X + iNK_X)t^i.$$

It is readily checked that the right hand side is not principally generated over S: if it is, then we have $R(N'K_X + iNK_X) \cong R$ for some $0 \leq i \leq M-1$, which contradicts that the index of K_X is NM. Thus S is $\mathbb{Q}$-Gorenstein of index N. ☐

Lemma A.8. *Let R be an (S_2) and (G_1) ring and let $x \in R$ be a nonzerodivisor of R. Then for every divisor D there exists D' linearly equivalent to D such that D' has no component in $V(x)$.*

Proof. Let $P_1, \ldots, P_n$ be the associated primes of (x). Since R is (S_2), all the P_i's have height one. Set $W = R - \cup_i P_i$ and note that, as D is almost Cartier, $W^{-1}R(D) = f \cdot W^{-1}R$ for some element f in the total ring of fractions of R. Now it is easy to see that $D' := D + \mathrm{div}(f)$ does the job. ☐

Lemma A.9. *Let R be an (S_2) and (G_1) ring and let $x \in R$ be a nonzerodivisor of R such that $\overline{R} := R/xR$ is also (S_2) and (G_1). Suppose D is a torsion divisor on R with index N, such that D has no component in $V(x)$. Then $\overline{D}$ is a torsion divisor on $\overline{R}$ whose index divides N.*

Moreover, if $(R, \mathfrak{m}, k)$ is local and $R(iD)/xR(iD)$ is an (S_2) module over $\overline{R}$ for each i, then the torsion index of $\overline{D}$ equals N.

Proof. We have $R(ND) \cong R$ and thus

$$\overline{R}(N\overline{D}) = \overline{R}(\overline{ND}) = (R(ND)/xR(ND))^{**} \cong \overline{R}.$$[13]

This proves the first assertion. Now if $R(iD)/xR(iD)$ is (S_2), then $\overline{R}(i\overline{D}) = R(iD)/xR(iD)$, so if $\overline{R}(i\overline{D}) \cong \overline{R}$ for some i, then $R(iD) \cong R$ since R is local. Thus the torsion index of $\overline{D}$ equals N. □

Lemma A.10. *Let $(R, \mathfrak{m}, k)$ be an (S_2) and (G_1) local ring and let $x \in R$ be a nonzerodivisor of R such that $\overline{R} := R/xR$ is also (S_2) and (G_1). Suppose R admits a canonical module and that K_X is a choice of the canonical divisor of $X = \mathrm{Spec}(R)$ such that K_X has no component in $V(x)$. Then $\overline{K_X}$ is a canonical divisor of $\overline{R}$.*

Proof. It is enough to show that $(R(K_X)/xR(K_X))^{**} \cong \omega_{\overline{R}}$, that is, $(\omega_R/x\omega_R)^{**} \cong \omega_{\overline{R}}$. Recall that we always have $\omega_R/x\omega_R \hookrightarrow \omega_{\overline{R}}$, and as the latter module is reflexive, we have an induced map $(\omega_R/x\omega_R)^{**} \to \omega_{\overline{R}}$. Now by Proposition A.2, it is enough to observe that this map is an isomorphism in codimension 1 as $\overline{R}$ is (G_1). □

Exercise 49. With notation as in Discussion A.6, show that the definition π^*D, π^*D' induces a well-defined group homomorphism $\pi^*\colon \mathrm{Cl}(R) \to \mathrm{Cl}(S)$.

Exercise 50. Let $(R, \mathfrak{m}, k)$ be an (S_2) and (G_1) local ring. Suppose that D is a torsion divisor of index N and $S = \bigoplus_{i=0}^{N-1} R(iD)t^i$ is a cyclic cover of S with respect to D. Prove that π^*D is a principal divisor where $\pi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$.

Exercise 51. Let $(R, \mathfrak{m}, k)$ be an (S_2) and (G_1) local ring. Suppose that D is a torsion divisor of index N and $S = \bigoplus_{i=0}^{N-1} R(iD)t^i$ is a cyclic cover of S with respect to D. Let $\varphi \in \mathrm{Hom}_R(S, R)$ be as in Proposition A.4. Show that the map $S \to \mathrm{Hom}_R(S, R)$ defined by mapping $s \mapsto \varphi(s-)$ is injective.

[13]Here, the first equality actually requires one to check that the pull back of divisors yields a well-defined map $\mathrm{Cl}(R) \to \mathrm{Cl}(S)$, see Exercise 49.

Exercise 52. Let R be an (S_2) and (G_1) ring, M a finitely generated R-module. Prove the following:

(1) $M \to M^{**}$ is injective if and only if M has no associated primes that are not minimal.
(2) $M \to M^{**}$ is an isomorphism if and only if M is an (S_2)-module. (Recall our convention on (S_2)-modules in the paragraph above Proposition A.2.)

Exercise 53. Let R be an (S_2) and (G_1) ring and let D_1, D_2 be two divisors. Prove the following:

(1) $\operatorname{Hom}_R(R(D_1), R(D_2)) \cong R(D_2 - D_1)$;
(2) $(R(D_1) \otimes_R R(D_2))^{**} \cong R(D_1 + D_2)$;
(3) If R is additionally F-finite of prime characteristic $p > 0$, then

$$(F^e_* R(D_1) \otimes_R R(D_2))^{**} \cong F^e_* R(D_1 + p^e D_2).$$

(Hint: Exercise 52 and Proposition A.2 could be helpful.)

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