LECH'S CONJECTURE IN ALL CHARACTERISTICS

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Abstract. This article is a digestion of a proof of Lech's conjecture in all characteristics, found by OpenAI. In fact, following their methods, we prove a stronger conjecture of Iyengar, Walker, and the author on Dutta multiplicity; and as an application we prove the total rank conjecture for complexes in mixed characteristic $(0, p > 2)$.

1. Introduction

The goal is to explain the proof of the following theorem, which settles the long standing Lech's conjecture [Lec60] in commutative algebra.

Theorem 1.1 ([Ope26a])**.** *Let $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a flat local extension of Noetherian local rings. Then $e(R) \leq e(S)$.*

In our previous works [Ma17, Section 5] and [IMW22, Proposition 8.3], we have reduced Theorem 1.1 to proving the following lower bound on Dutta multiplicity (see Section 2 for a definition of χ_∞), which was conjectured in [IMW22].

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring of residue characteristic $p > 0$ and dimension d. Let $F_\bullet := 0 \longrightarrow R^{n_d} \longrightarrow \cdots \longrightarrow R^{n_1} \longrightarrow R^{n_0} \longrightarrow 0$ be a non-exact perfect complex over R supported only at $\{\mathfrak{m}\}$. Then*

$$\chi_\infty(F_\bullet) \geq \nu_R(H_0(F_\bullet)) \cdot e(R)$$

where $\nu_R(-)$ denotes the minimal number of generators over R.

The proof of Theorem 1.2 combines recent positive / mixed characteristic methods together with ideas from Boij–Söderberg theory [ES09]. Throughout the rest of this article, we fix a prime p and we will often work under the following setup (see [CLM+26, Section 4]).

Notation 1.3. Let $(R, \mathfrak{m}, k)$ be a Noetherian complete local domain of residue characteristic p such that k is algebraically closed. Choose a system of parameters $\underline{x} := x_1, x_2, \ldots, x_d$ of R where $x_1 = p$ if R is of mixed characteristic, such that the Noether–Cohen normalization $(A, \mathfrak{m}_A) \subseteq R$ (where $\underline{x}$ is a regular system of parameters of A) is generically étale.[1] Note that $A = k[[x_1, \ldots, x_d]]$ if R has characteristic p and $A = W(k)[[x_2, \ldots, x_d]]$ if R has mixed characteristic. Set

$$A_\infty := A[x_1^{1/p^\infty}, x_2^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^\wedge$$

where the completion is p-adic and let $R^{A_\infty}_{\text{perfd}} := (R \otimes_A A_\infty)_{\text{perfd}}$ where $(-)_{\text{perfd}}$ is the perfectoidization functor of Bhatt–Scholze [BS22]. We further let $0 \neq g \in A$ so that $A[1/g] \longrightarrow R[1/g]$ is finite étale and let $(g)_{\text{perfd}}$ be the kernel of the canonical map $A_\infty \longrightarrow (A_\infty/g)_{\text{perfd}}$, see [BS22, Section 7].

[1] This condition is automatic in mixed characteristic, and in characteristic p, such a choice of $\underline{x}$ exists by the Cohen–Gabber theorem [ILO14].

Remark 1.4. Note that if R has characteristic p, then $R^{A_\infty}_{\mathrm{perfd}} = R_{\mathrm{perf}}$ is independent of A_∞. But in mixed characteristic, $R^{A_\infty}_{\mathrm{perfd}}$ depends on the choice of A (more precisely, it depends on the choice of $\underline{x}$). Throughout this article, whenever we write $R^{A_\infty}_{\mathrm{perfd}}$ (or more generally, $S^{A_\infty}_{\mathrm{perfd}}$ for $S \supseteq R$ a finite domain extension), $\underline{x}$ and A_∞ will be tacitly chosen as in Notation 1.3.

We will use the normalized length function $\lambda_\infty(-)$ of Faltings (see [Fal02] or [CLM+26, Section 2.2]). We will sometimes use almost mathematics with respect to $(g)_{\mathrm{perfd}}$ as in [BS22, Section 10]. On the other hand, we point out that the essential properties we need are simply that (under Notation 1.3) there is a well-defined "length function"

$$\lambda_\infty(-): \ \mathfrak{m}_A\text{-power torsion } A_\infty\text{-modules} \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$

that is additive on short exact sequences, and that takes value zero on A_∞-modules that are g-almost zero (i.e., annihilated by $(g)_{\mathrm{perfd}}$). The following is the key technical result that underlies Theorem 1.2 and Theorem 1.1.

Theorem 1.5. *With notations as in Notation 1.3. Let*

$$F_\bullet := 0 \longrightarrow R^{n_d} \longrightarrow \cdots \longrightarrow R^{n_1} \longrightarrow R^{n_0} \longrightarrow 0$$

be a non-exact perfect complex over R supported only at $\{\mathfrak{m}\}$. Then there exists a constant C depending only on d such that if $z_1, \ldots, z_d$ is a system of parameters of R so that $\partial(F_\bullet) \subseteq (z_1, \ldots, z_d)^N F_\bullet$ for some positive integer N, then

$$\lambda_\infty\big(H_0(F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})\big) \geq e(z_1, \ldots, z_d; R) \cdot N^d \left(n_0 - \frac{C\sum_i n_i}{N}\right).$$

This article is organized as follows. In Section 2 we prove Theorem 1.2 using Theorem 1.5 together with Theorem A.1, the latter is a special case of [Ope26b, Theorem 3.1]. We show the total rank conjecture in residue characteristic $\neq 2$ in Corollary 2.11, and we also give a more direct proof of Theorem 1.1 via Theorem 1.5, following [Ope26a] (this proof avoids Theorem 1.2 and Theorem A.1). In Section 3 we prove Theorem 1.5, largely following [Ope26a, Section 2]; and in Appendix A we give a relatively self-contained proof of Theorem A.1.

Acknowledgement and AI disclosure. First of all, we make no claim to owning the ideas found in this article. Our aim is to write [Ope26a] using the language in [CLM+26], making the proofs more accessible, improving the exposition, and refining or simplifying the original arguments on some occasions. This is a preliminary version.

The author had been thinking about a perfectoid version of the Dutta multiplicity before [Ope26a, Ope26b], and some basic results along this line (Lemma 2.3 and Lemma 2.6) are known to the author. But the most crucial results, Theorem 2.7 (which relies on Theorem 1.5) and Theorem 2.9 (which relies on Theorem A.1) were discovered in [Ope26a] and (implicitly) [Ope26b] respectively. The proof of Theorem 1.5 largely follows [Ope26a, Section 2] (except that we work under the almost mathematics setting and make some simplifications, while [Ope26a] works in a slightly more general setup). The proof of Theorem A.1 is different from the argument in [Ope26b, Theorem 3.1]. The author found the current proof via additional prompts to ChatGPT Astra.

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2. Proof of Theorem 1.2

In this section we prove Theorem 1.2 (and hence Theorem 1.1) assuming Theorem 1.5. We first recall the definition of Dutta multiplicity. Let $(R, \mathfrak{m})$ be a Noetherian complete local ring of dimension d and let $F_\bullet$ be a perfect complex over R supported only at $\{\mathfrak{m}\}$. Then the *Dutta multiplicity* of $F_\bullet$ is defined as

$$\chi_\infty(F_\bullet) = \mathrm{ch}_d(F_\bullet) \cdot [R]_d$$

where $\mathrm{ch}_d(F_\bullet)$ is the d-th localized Chern character of $F_\bullet$ and $[R]_d$ is the Todd class of R in $A_d(\mathrm{Spec}(R))$. We refer the readers to [Ful98, Rob98, Kur04] for more details on localized Chern characters (we will only need minimum knowledge on this in Lemma 2.1). One can give an alternative, elementary characterization of the Dutta multiplicity, see Theorem 2.8. It is also well-known (see [Rob98, Theorem 12.7.1]) that if R has characteristic p, then

$$\chi_\infty(F_\bullet) = \lim_{e \longrightarrow \infty} \frac{\chi(\mathcal{F}_R^e(F_\bullet))}{p^{ed}}$$

where $\mathcal{F}_R^e(-)$ denotes Peskine–Szpiro's Frobenius functor. More concretely, $\mathcal{F}_R^e(F_\bullet)$ is simply the perfect complex obtained from $F_\bullet$ by replacing all the entries in its differentials by their p^e-th powers.

Lemma 2.1. *In order to prove Theorem 1.2, we may assume that R is a Noetherian complete local domain with an algebraically closed residue field.*

Proof. First of all, it is well-known that if $(R, \mathfrak{m}) \longrightarrow (R', \mathfrak{m}')$ is any flat local extension of Noetherian complete local rings with $\mathfrak{m}' = \mathfrak{m}R'$, then $\chi_\infty(F_\bullet) = \chi_\infty(F'_\bullet)$ where $F'_\bullet = F_\bullet \otimes^{\mathbb{L}}_R R'$ (for example, using [Kur93, Proposition 2.6] and [AFH94, Theorem (1.6)]). Since $\nu_R(H_0(F_\bullet)) = \nu_{R'}(H_0(F'_\bullet))$ and $e(R) = e(R')$, we may replace R by R' to assume that the residue field is algebraically closed by [Sta26, Tag 03C3].

By the top term property ([Ful98, Theorem 18.3 (5)]), we have

$$[R]_d = \sum_{\dim(R/P)=d} \ell(R_P)[R/P]_d.$$

It follows that

$$\chi_\infty(F_\bullet) = \sum_{\dim(R/P)=d} \ell(R_P)\big(\mathrm{ch}_d(F_\bullet) \cdot [R/P]_d\big) = \sum_{\dim(R/P)=d} \ell(R_P) \cdot \chi_\infty(F_\bullet \otimes^{\mathbb{L}}_R R/P).$$

Since we also have

$$\nu_R(H_0(F_\bullet)) = \nu_R(H_0(F_\bullet \otimes^{\mathbb{L}}_R R/P)) \quad \text{and} \quad e(R) = \sum_{\dim(R/P)=d} \ell(R_P)e(R/P),$$

it is enough to prove Theorem 1.2 for each R/P. Thus we arrive at the desired reduction. □

We now give a proof of Theorem 1.2 in characteristic p – this case follows almost immediately from Theorem 1.5 and results in [CLM+26, Section 3].

Proof of Theorem 1.2 in characteristic p. By Lemma 2.1, we may assume that R is a Noetherian complete local domain with algebraically closed residue field (so that we could freely use Notation 1.3). Without loss of generality, we may assume that $\partial(F_\bullet) \subseteq \mathfrak{m}F_\bullet$.

Let $z_1, \dots, z_d$ be a minimal reduction of $\mathfrak{m}$. Then there exists a constant $c > 0$ so that $\mathfrak{m}^n \subseteq (z_1, \dots, z_d)^{n-c}$ for all $n > c$. Set $F_\bullet^{(e)} := \mathcal{F}_R^e(F_\bullet)$, for $e \gg 0$ we have

$$\partial(F_\bullet^{(e)}) \subseteq \mathfrak{m}^{p^e} F_\bullet^{(e)} \subseteq (z_1, \dots, z_d)^{p^e - c} F_\bullet^{(e)}.$$

By Theorem 1.5 applied to $F_\bullet^{(e)}$ and noting that $F_\bullet^{(e)}$ has the same Betti numbers as $F_\bullet$, we obtain that

$$\begin{aligned} \lambda_\infty\big(H_0(F_\bullet^{(e)}) \otimes_R R_{\mathrm{perf}}\big) &= \lambda_\infty\big(H_0(F_\bullet^{(e)} \otimes_R^{\mathbb{L}} R_{\mathrm{perf}})\big) \\ &\geq e(z_1, \dots, z_d; R) \cdot (p^e - c)^d \left(n_0 - \frac{C \sum_i n_i}{p^e - c}\right). \end{aligned} \tag{2.1.1}$$

It remains to relate $p^{ed}\chi_\infty(F_\bullet)$ with $\lambda_\infty\big(H_0(F_\bullet^{(e)}) \otimes_R R_{\mathrm{perf}}\big)$. This is essentially well-known to experts but we give some details. The next claim is a slight generalization of [CLM+26, Theorem 3.0.7] (which is the case that M is cyclic). The idea of the proof is similar.

Claim 2.2. *For any finite length R-module M,*

$$\lambda_\infty(M \otimes_R R_{\mathrm{perf}}) = \lim_{e \longrightarrow \infty} \frac{\ell_R\big(\mathcal{F}_R^e(M)\big)}{p^{ed}}.$$

Proof. Since $A \longrightarrow R$ is generically étale, it is well-known that (for example see [CLM+26, (3.0.0.1)]) there exists $0 \neq g \in A$ so that for each $e \in \mathbb{N}$, there are short exact sequences:

$$0 \longrightarrow A_{\mathrm{perf}} \otimes_{A^{1/p^e}} R^{1/p^e} \longrightarrow R_{\mathrm{perf}} \longrightarrow C_e \longrightarrow 0$$

$$0 \longrightarrow R_{\mathrm{perf}} \longrightarrow A_{\mathrm{perf}} \otimes_{A^{1/p^e}} R^{1/p^e} \longrightarrow D_e \longrightarrow 0$$

such that C_e and D_e are annihilated by g^{1/p^e}. By definition of the normalized length $\lambda_\infty(-)$ (see [CLM+26, Section 2.2]), we have $\lambda_\infty(M \otimes_R R^{1/p^e} \otimes_{A^{1/p^e}} A_{\mathrm{perf}}) = \ell_R\big(\mathcal{F}_R^e(M)\big)/p^{ed}$. Thus tensoring the above short exact sequences with M and taking $\lambda_\infty(-)$, we obtain that

$$\left| \lambda_\infty(M \otimes_R R_{\mathrm{perf}}) - \frac{\ell_R\big(\mathcal{F}_R^e(M)\big)}{p^{ed}} \right| \leq \max\{\lambda_\infty(M \otimes_R C_e), \lambda_\infty(M \otimes_R D_e)\} \tag{2.2.1}$$

for each $e \in \mathbb{N}$. Fix an integer h so that $\mathfrak{m}^h M = 0$. We have

$$\lambda_\infty(M \otimes_R C_e) \leq \nu_R(M) \cdot \lambda_\infty\big(R_{\mathrm{perf}}/(\mathfrak{m}^h + g^{1/p^e})\big) = \nu_R(M) \cdot \frac{e_{\mathrm{HK}}\big((\mathfrak{m}^h)^{[p^e]} + g; R\big)}{p^{ed}}$$

where we used [CLM+26, Theorem 3.0.7] for the equality; and we have

$$\lambda_\infty(M \otimes_R D_e) \leq \nu_R(M) \cdot \lambda_\infty\big((A_{\mathrm{perf}} \otimes_{A^{1/p^e}} R^{1/p^e})/(\mathfrak{m}^h + g^{1/p^e})\big) = \nu_R(M) \cdot \frac{\ell_R(R/((\mathfrak{m}^h)^{[p^e]} + g))}{p^{ed}}.$$

It is well-known by Hilbert–Kunz theory that the right hand sides in the above two inequalities tend to 0 when $e \longrightarrow \infty$. Therefore the claim follows by taking the limit as $e \longrightarrow \infty$ in (2.2.1) and using that $\lim_{e \longrightarrow \infty} \ell_R\big(\mathcal{F}_R^e(M)\big)/p^{ed}$ exists by [Sei89, Proposition 2 (a)]. □

We now return to the proof of the theorem. We have that

$$\begin{aligned} p^{ed}\chi_\infty(F_\bullet) &= p^{ed}\cdot \lim_{e'\longrightarrow\infty} \frac{\ell_R\big(\mathcal{F}_R^{e'}(H_0(F_\bullet))\big)}{p^{e'd}} \\ &= \lim_{e'\longrightarrow\infty} \frac{\ell_R\big(\mathcal{F}_R^{e'}(H_0(F_\bullet^{(e)}))\big)}{p^{e'd}} \\ &= \lambda_\infty(H_0(F_\bullet^{(e)})\otimes_R R_{\mathrm{perf}}) \\ &\geq e(z_1,\dots,z_d;R)\cdot(p^e-c)^d\left(n_0-\frac{C\sum_i n_i}{p^e-c}\right). \end{aligned}$$

where the first equality follows from [Rob89] (see [BHM26, Theorem 5.2]); the third equality follows from Claim 2.2; and the last inequality follows from (2.1.1). Dividing by p^{ed} and letting $e\longrightarrow\infty$ we obtain that

$$\chi_\infty(F_\bullet)\geq n_0\cdot e(z_1,\dots,z_d;R)=\nu_R(H_0(F_\bullet))\cdot e(R)$$

where the equality follows as $z_1,\dots,z_d$ is a reduction of $\mathfrak{m}$. □

In the rest of this section, we shall prove the general case of Theorem 1.2. We divide the proof into two steps. In the first step, we temporarily introduce a perfectoid version of the Dutta multiplicity which we denote by $\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet)$, and we show that under the hypothesis of Theorem 1.2, we have $\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet)\geq\nu_R(H_0(F_\bullet))\cdot e(R)$. In the second step, we will show that $\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet)=\chi_\infty(F_\bullet)$ and thus complete the proof of Theorem 1.2.

2.1. **Perfectoid Dutta multiplicity.** Throughout this section, we will follow Notation 1.3. Most of the preliminary results in this section are essentially consequences of some results and methods in [CLM+26]. We start with the following lemma.

Lemma 2.3. *With notations as in Notation 1.3. Suppose*

$$G_\bullet := 0\longrightarrow G_n\longrightarrow\cdots\longrightarrow G_1\longrightarrow G_0\longrightarrow 0$$

is a perfect complex over $R^{A_\infty}_{\mathrm{perfd}}$ so that for all i, $H_i(G_\bullet)$ is annihilated by a power of $\mathfrak{m}$. Then

$$\begin{cases}\lambda_\infty\big(H_i(G_\bullet)\big)=0 & \text{for } i>n-d;\text{ and}\\ \lambda_\infty\big(H_i(G_\bullet)\big)<\infty & \text{for all } i.\end{cases}$$

In fact, for all $i>n-d$, $H_i(G_\bullet)$ is g-almost zero.

Proof. Since each $H_i(G_\bullet)$ is annihilated by a power of $\mathfrak{m}$, there exists N such that

$$\mathfrak{m}^N\subseteq\ker\big(R^{A_\infty}_{\mathrm{perfd}}\longrightarrow\mathrm{Hom}_{D(R^{A_\infty}_{\mathrm{perfd}})}(G_\bullet,G_\bullet)\cong H^0\big(\mathbf{R}\,\mathrm{Hom}_{R^{A_\infty}_{\mathrm{perfd}}}(G_\bullet,G_\bullet)\big)\big).$$

It follows that the canonical map

$$\mathrm{Kos}(x_1^N,\dots,x_d^N;G_\bullet)\longrightarrow G_\bullet[d]$$

admits a section in $D(R^{A_\infty}_{\mathrm{perfd}})$[2] and thus $H_i(G_\bullet)$ is a direct summand of

$$H_{i+d}\big(\mathrm{Kos}(x_1^N,\dots,x_d^N;G_\bullet)\big)=H_{i+d}\big(G_\bullet\otimes^{\mathbb{L}}_{R^{A_\infty}_{\mathrm{perfd}}}\mathrm{Kos}(x_1^N,\dots,x_d^N;R^{A_\infty}_{\mathrm{perfd}})\big).$$

[2]More generally, if y annihilates $\mathrm{Hom}_D(G_\bullet,G_\bullet)$, then the exact triangle $G_\bullet\xrightarrow{\cdot y}G_\bullet\longrightarrow\mathrm{Kos}(y;G_\bullet)\longrightarrow G_\bullet[1]$ splits in D, we can then replace $G_\bullet$ by $\mathrm{Kos}(y;G_\bullet)$ and induct on the length of the sequence.

By [CLM+26, Corollary 4.0.4], we know that $\mathrm{Kos}(x_1^N,\dots,x_d^N;R^{A_\infty}_{\mathrm{perfd}})$ is g-almost isomorphic to $R^{A_\infty}_{\mathrm{perfd}}/(x_1^N,\dots,x_d^N)R^{A_\infty}_{\mathrm{perfd}}$. Thus it suffices to show that

$$\begin{cases}\lambda_\infty\big(H_{i+d}\big(G_\bullet\otimes^{\mathbb{L}}_{R^{A_\infty}_{\mathrm{perfd}}} R^{A_\infty}_{\mathrm{perfd}}/(x_1^N,\dots,x_d^N)R^{A_\infty}_{\mathrm{perfd}}\big)\big)=0 & \text{for } i>n-d;\text{ and}\\ \lambda_\infty\big(H_{i+d}\big(G_\bullet\otimes^{\mathbb{L}}_{R^{A_\infty}_{\mathrm{perfd}}} R^{A_\infty}_{\mathrm{perfd}}/(x_1^N,\dots,x_d^N)R^{A_\infty}_{\mathrm{perfd}}\big)\big)<\infty & \text{for all } i.\end{cases}$$

The first is obvious as $i+d>n$ (note that this indeed gives the stronger assertion since $H_{i+d}\big(G_\bullet\otimes^{\mathbb{L}}_{R^{A_\infty}_{\mathrm{perfd}}} R^{A_\infty}_{\mathrm{perfd}}/(x_1^N,\dots,x_d^N)R^{A_\infty}_{\mathrm{perfd}}\big)=0$ when $i>n-d$) and the second follows from the fact that

$$\lambda_\infty\big(R^{A_\infty}_{\mathrm{perfd}}/(x_1^N,\dots,x_d^N)R^{A_\infty}_{\mathrm{perfd}}\big)=e(x_1^N,\dots,x_d^N;R)<\infty$$

where the equality uses [CLM+26, Proposition 4.0.14]. □

Definition 2.4. With notations as in Notation 1.3. Let $G_\bullet$ be a perfect complex over $R^{A_\infty}_{\mathrm{perfd}}$ so that for all i, $H_i(G_\bullet)$ is annihilated by a power of $\mathfrak{m}$. By Lemma 2.3, we can define

$$\chi^{A_\infty}_\infty(G_\bullet):=\sum_i(-1)^i\lambda_\infty\big(H_i(G_\bullet)\big)<\infty.$$

Now let $F_\bullet$ be a perfect complex over R supported only at $\{\mathfrak{m}\}$. We then define the *perfectoid Dutta multiplicity* of $F_\bullet$ as

$$\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet):=\chi^{A_\infty}_\infty(F_\bullet\otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})=\sum_i(-1)^i\lambda_\infty\big(H_i(F_\bullet\otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})\big)<\infty.$$

Here the notation $\chi^{\underline{x}}_{\mathrm{perfd}}(-)$ was chosen to emphasize that the definition *a priori* depends on a choice of $\underline{x}$ in Notation 1.3 (though we will show in Section 2.2 that this definition is indeed independent of $\underline{x}$).

Corollary 2.5. *With notations as in Notation 1.3. Let*

$$F_\bullet:=0\longrightarrow R^{n_d}\longrightarrow\cdots\longrightarrow R^{n_1}\longrightarrow R^{n_0}\longrightarrow 0$$

be a non-exact perfect complex over R supported only at $\{\mathfrak{m}\}$. Then

$$\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet)=\lambda_\infty\big(H_0(F_\bullet)\otimes_R R^{A_\infty}_{\mathrm{perfd}}\big)>0.$$

In particular, if $H_0(F_\bullet)\cong R/I$ for an $\mathfrak{m}$-primary ideal $I\subseteq R$, then $\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet)=e^{\underline{x}}_{\mathrm{perfd}}(I)$ where $e^{\underline{x}}_{\mathrm{perfd}}(I)$ is the perfectoid Hilbert–Kunz multiplicity as in [CLM+26, Definition 4.0.8].

Proof. By Lemma 2.3, we know that $\lambda_\infty\big(H_i(F_\bullet\otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})\big)=0$ for all $i>0$ and thus by definition $\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet)=\lambda_\infty\big(H_0(F_\bullet)\otimes_R R^{A_\infty}_{\mathrm{perfd}}\big)$. The "In particular" part of the corollary follows from the definition of perfectoid Hilbert–Kunz multiplicity [CLM+26, Definition 4.0.8], and strict positivity follows by noting that $H_0(F_\bullet)$ always surjects onto $R/\mathfrak{m}$ and $e^{\underline{x}}_{\mathrm{perfd}}(\mathfrak{m})\geq 1$ by [CLM+26, Proposition 4.0.20]. □

We next record the following "transformation rule" for the perfectoid Dutta multiplicity under finite generically étale extensions. The essential part of the proof is similar to [CLM+26, Section 6] (see also [CFM+26]).

Lemma 2.6. *With notations as in Notation 1.3. Suppose $R \longrightarrow S$ is a finite extension of Noetherian complete local domains that is generically étale. Then for any perfect complex $F_\bullet$ over R supported only at $\{\mathfrak{m}\}$, we have $\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet \otimes^{\mathbb{L}}_R S) = (\mathrm{rank}_R S) \cdot \chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet)$, i.e.,*

$$\chi^{A_\infty}_\infty(F_\bullet \otimes^{\mathbb{L}}_R S^{A_\infty}_{\mathrm{perfd}}) = (\mathrm{rank}_R S) \cdot \chi^{A_\infty}_\infty(F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}}).$$

Proof. First we note that both $\chi^{\underline{x}}_{\mathrm{perfd}}(- \otimes^{\mathbb{L}}_R S)$ and $\chi^{\underline{x}}_{\mathrm{perfd}}(-)$ are additive on exact triangles of perfect complexes over R supported only at $\{\mathfrak{m}\}$. Therefore, since the Grothendieck group of perfect complexes over R supported only at $\{\mathfrak{m}\}$ is generated by those of length exactly d [FH09], without loss of generality, it is enough to prove the lemma when $F_\bullet = 0 \longrightarrow R^{n_d} \longrightarrow \cdots \longrightarrow R^{n_1} \longrightarrow R^{n_0} \longrightarrow 0$. But then by Corollary 2.5, it is enough to show that for any finite length R-module M,

$$\lambda_\infty(M \otimes_R S^{A_\infty}_{\mathrm{perfd}}) = (\mathrm{rank}_R S) \cdot \lambda_\infty(M \otimes_R R^{A_\infty}_{\mathrm{perfd}}). \tag{2.6.1}$$

We first prove (2.6.1) when R has characteristic p. In this case, by Claim 2.2 we have

$$\lambda_\infty(M \otimes_R R^{A_\infty}_{\mathrm{perfd}}) = \lim_{e \longrightarrow \infty} \frac{\ell_R\big(\mathcal{F}^e_R(M)\big)}{p^{ed}} \quad \text{and} \quad \lambda_\infty(M \otimes_R S^{A_\infty}_{\mathrm{perfd}}) = \lim_{e \longrightarrow \infty} \frac{\ell_S\big(\mathcal{F}^e_S(M \otimes_R S)\big)}{p^{ed}}.$$

Thus (2.6.1) follows from [Sei89, Proposition 2 (a')].

We now assume R has mixed characteristic. Let $0 \neq g \in A$ so that $A[1/g] \longrightarrow R[1/g]$ and $R[1/g] \longrightarrow S[1/g]$ are finite étale and $R[1/g] \longrightarrow S[1/g]$ is finite free. Let $r = \mathrm{rank}_R S$ and fix an integer c so that $p^c M = 0$. By [CFM+26, Lemma 8.5 and proof of Proposition 8.6 (the diagram above (8.6.1))], there exist $f_2^\sharp, \ldots, f_d^\sharp \in \mathfrak{m}_A A_\infty$ so that $\sqrt{(pg, f_2^\sharp, \ldots, f_d^\sharp)} = \mathfrak{m}_{A_\infty}$, a sequence of elements $\epsilon_n \in (g)_{\mathrm{perfd}}$ so that $\epsilon_n = p^{1/p^n} \bmod (f_2^\sharp, \ldots, f_d^\sharp)_{\mathrm{perfd}}$, together with a commutative diagram:[3]

$$\begin{array}{ccccc}
S^{A_\infty}_{\mathrm{perfd}}/p^c & \xrightarrow{\widetilde{\phi}_{\epsilon_n}} & (R^{A_\infty}_{\mathrm{perfd}}/p^c)^r & \xrightarrow{\widetilde{\psi}_{\epsilon_n}} & S^{A_\infty}_{\mathrm{perfd}}/p^c \\
\downarrow{\scriptstyle can} & & \downarrow{\scriptstyle can} & & \downarrow{\scriptstyle can} \\
S^{A_\infty}_{\mathrm{perfd}}/(p, (f_2^\sharp, \ldots, f_d^\sharp)_{\mathrm{perfd}}) & \xrightarrow{\epsilon_n \cdot \phi_{\epsilon_n}} & (R^{A_\infty}_{\mathrm{perfd}}/(p, (f_2^\sharp, \ldots, f_d^\sharp)_{\mathrm{perfd}}))^r & \xrightarrow{\psi_{\epsilon_n}} & S^{A_\infty}_{\mathrm{perfd}}/(p, (f_2^\sharp, \ldots, f_d^\sharp)_{\mathrm{perfd}})
\end{array}$$

such that $\phi_{\epsilon_n} \circ \psi_{\epsilon_n}$ and $\psi_{\epsilon_n} \circ \phi_{\epsilon_n}$ are both multiplication by p^{1/p^n}.

We set $C_{\epsilon_n} = \mathrm{coker}(\widetilde{\psi}_{\epsilon_n})$. Since $(p, (f_2^\sharp, \ldots, f_d^\sharp)_{\mathrm{perfd}}) = (p, (f_2^\sharp)^{1/p^\infty}, \ldots, (f_d^\sharp)^{1/p^\infty})$ and C_{ϵ_n} is g-almost finitely generated, there is a finitely generated $R^{A_\infty}_{\mathrm{perfd}}/p^c$-submodule D_{ϵ_n} of C_{ϵ_n} so that $\epsilon_n C_{\epsilon_n} \subseteq D_{\epsilon_n}$. By construction we have $\epsilon_n^2 C_{\epsilon_n} \subseteq (p, (f_2^\sharp, \ldots, f_d^\sharp)_{\mathrm{perfd}}) C_{\epsilon_n}$. It follows that $\epsilon_n^2 D_{\epsilon_n} \subseteq (p, (f_2^\sharp)^{1/p^\infty}, \ldots, (f_d^\sharp)^{1/p^\infty}) C_{\epsilon_n}$ and as D_{ϵ_n} is finitely generated there exists $m \gg n$ so that $\epsilon_n^2 D_{\epsilon_n} \subseteq (p, (f_2^\sharp)^{1/p^m}, \ldots, (f_d^\sharp)^{1/p^m}) C_{\epsilon_n}$ and thus $\epsilon_n^3 \cdot C_{\epsilon_n} \subseteq (p, (f_2^\sharp)^{1/p^m}, \ldots, (f_d^\sharp)^{1/p^m}) C_{\epsilon_n}$. Thus we have

$$\epsilon_n^3 \cdot (S^{A_\infty}_{\mathrm{perfd}}/p^c) \subseteq \mathrm{im}(\widetilde{\psi}_{\epsilon_n}) + (p, (f_2^\sharp)^{1/p^m}, \ldots, (f_d^\sharp)^{1/p^m})(S^{A_\infty}_{\mathrm{perfd}}/p^c). \tag{2.6.2}$$

Increasing m if necessary, we may assume that $\epsilon_n = p^{1/p^n}$ in $A_\infty/((f_2^\sharp)^{1/p^m}, \ldots, (f_d^\sharp)^{1/p^m})$ up to units, and increasing c if necessary, we may assume that $(p^c, (f_2^\sharp)^c, \ldots, (f_d^\sharp)^c)(M \otimes_A A_\infty) = 0$.

[3] In [CFM+26], the diagram was stated when $c = 1$, but the proof for general c is the same since [BS22, Theorem 10.9] holds for all powers of p.

We now have that

$$\begin{aligned}
&\lambda_\infty(M \otimes_R S^{A_\infty}_{\mathrm{perfd}}) - r \cdot \lambda_\infty(M \otimes_R R^{A_\infty}_{\mathrm{perfd}}) \\
&\leq \lambda_\infty(M \otimes_R C_{\epsilon_n}) \\
&\leq \nu_R(M) \cdot \lambda_\infty\big(S^{A_\infty}_{\mathrm{perfd}}/((p^c, (f_2^\sharp)^c, \dots, (f_d^\sharp)^c)S^{A_\infty}_{\mathrm{perfd}} + \mathrm{im}(\widetilde{\psi}_{\epsilon_n}))\big) \\
&\leq \nu_R(M) \cdot c^d p^{m(d-1)} \cdot \lambda_\infty\big(S^{A_\infty}_{\mathrm{perfd}}/((p, (f_2^\sharp)^{1/p^m}, \dots, (f_d^\sharp)^{1/p^m})S^{A_\infty}_{\mathrm{perfd}} + \mathrm{im}(\widetilde{\psi}_{\epsilon_n}))\big) \\
&\leq \nu_R(M) \cdot c^d p^{m(d-1)} \cdot \lambda_\infty\big(S^{A_\infty}_{\mathrm{perfd}}/(\epsilon_n^3, (f_2^\sharp)^{1/p^m}, \dots, (f_d^\sharp)^{1/p^m})\big) \\
&= \nu_R(M) \cdot c^d p^{m(d-1)} \cdot (\mathrm{rank}_A\, S) \cdot \lambda_\infty\big(A_\infty/(\epsilon_n^3, (f_2^\sharp)^{1/p^m}, \dots, (f_d^\sharp)^{1/p^m})\big) \\
&= \nu_R(M) \cdot c^d p^{m(d-1)} \cdot (\mathrm{rank}_A\, S) \cdot \lambda_\infty\big(A_\infty/(p^{3/p^n}, (f_2^\sharp)^{1/p^m}, \dots, (f_d^\sharp)^{1/p^m})\big) \\
&= \frac{3c^d}{p^n} \cdot \nu_R(M) \cdot (\mathrm{rank}_A\, S) \cdot \lambda_\infty\big(A_\infty/(p, f_2^\sharp, \dots, f_d^\sharp)\big)
\end{aligned}$$

where the inequality on the fifth line follows from (2.6.2), the equality on the sixth line follows from [CLM+26, Proposition 4.0.16]; the equality in the seventh line follows from our choice that $\epsilon_n = p^{1/p^n}$ up to units in $A_\infty/((f_2^\sharp)^{1/p^m}, \dots, (f_d^\sharp)^{1/p^m})$; and the equality in the last line follows from taking a filtration and noting that $p, f_2^\sharp, \dots, f_d^\sharp$ is a regular sequence on A_∞: after reducing modulo p, $(f_2^\sharp, \dots, f_d^\sharp)(A_\infty/p)$ has the same radical as $\mathfrak{m}_A(A_\infty/p)$, so the images of $f_2^\sharp, \dots, f_d^\sharp$ in A_∞/p, which we may assume are coming from $A_e := A[x_1^{1/p^e}, \dots, x_d^{1/p^e}]$ for $e \gg 0$, form a system of parameters and thus a regular sequence on A_e/p, and hence they form a regular sequence on A_∞/p. Now letting $n \longrightarrow \infty$ and noting that all the other invariants in the last line do not depend on n, we obtain that

$$\lambda_\infty(M \otimes_R S^{A_\infty}_{\mathrm{perfd}}) - r \cdot \lambda_\infty(M \otimes_R R^{A_\infty}_{\mathrm{perfd}}) \leq 0.$$

A very similar argument (examining $\mathrm{coker}(\widetilde{\phi}_\epsilon)$ instead of $\mathrm{coker}(\widetilde{\psi}_\epsilon)$) shows that

$$r \cdot \lambda_\infty(M \otimes_R R^{A_\infty}_{\mathrm{perfd}}) - \lambda_\infty(M \otimes_R S^{A_\infty}_{\mathrm{perfd}}) \leq 0.$$

This completes the proof of (2.6.1) in mixed characteristic. □

We now prove the main result of this section, which gives a lower bound on the perfectoid Dutta multiplicity $\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet)$.

Theorem 2.7. *With notations as in Notation 1.3. Let*

$$F_\bullet := 0 \longrightarrow R^{n_d} \longrightarrow \cdots \longrightarrow R^{n_1} \longrightarrow R^{n_0} \longrightarrow 0$$

be a non-exact perfect complex over R supported only at $\{\mathfrak{m}\}$. Then

$$\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet) \geq \nu_R(H_0(F_\bullet)) \cdot e(R).$$

Proof. Without loss of generality, we may assume that $\partial(F_\bullet) \subseteq \mathfrak{m}F_\bullet$. Let $z_1, \dots, z_d$ be a minimal reduction of $\mathfrak{m}$. Fix an absolute integral closure R^+ of R and for each $q > 0$ *relatively prime* to p, let

$$S'_q := R[z_1^{1/q}, \dots, z_d^{1/q}] \subseteq R^+.$$

Note that each S'_q is a Noetherian complete local domain that is generically étale over R (this is the reason we require that q is relatively prime to p, if R has mixed characteristic, this is not necessary and the argument below works for all $q > 0$). We have

$$\mathfrak{m}S'_q \subseteq \overline{(z_1^{1/q}, \dots, z_d^{1/q})^q S'_q} \subseteq (z_1^{1/q}, \dots, z_d^{1/q})^{q-d+1} R^+ \cap S'_q$$

where the second inclusion follows from [MMRGS26, Proposition 4.11] (note that the p-adic completion there can be ignored as $(z_1, \dots, z_d)$ contains a power of p, see also [RS24, Hei97]). It follows that there exists a finite extension S_q of S'_q inside R^+ (and we may assume S_q is generically étale over S'_q in characteristic p by [Sin99, Corollary 3.4]) such that

$$\mathfrak{m}S_q \subseteq (z_1^{1/q}, \dots, z_d^{1/q})^{q-d+1} S_q.$$

Let $F_\bullet^q := F_\bullet \otimes_R^{\mathbb{L}} S_q$. Since S_q is a Noetherian complete local domain with an algebraically closed residue field, we can apply Theorem 1.5 to the perfect complex $F_\bullet^q$ over S_q (and note that $F_\bullet^q$ has the same Betti numbers as $F_\bullet$) to obtain that

$$\begin{aligned}\lambda_\infty\big(H_0(F_\bullet^q \otimes_{S_q}^{\mathbb{L}} S_{q,\mathrm{perfd}}^{A_\infty})\big) &\geq e(z_1^{1/q}, \dots, z_d^{1/q}; S_q) \cdot (q-d+1)^d \left(n_0 - \frac{C \cdot \sum n_i}{q-d+1}\right) \\ &= e(z_1, \dots, z_d; R) \cdot (\operatorname{rank}_R S_q) \cdot \frac{(q-d+1)^d}{q^d} \left(n_0 - \frac{C \cdot \sum n_i}{q-d+1}\right)\end{aligned}$$

Since S_q is generically étale over R, by Lemma 2.3 and Lemma 2.6, we have

$$\begin{aligned}\lambda_\infty\big(H_0(F_\bullet^q \otimes_{S_q}^{\mathbb{L}} S_{q,\mathrm{perfd}}^{A_\infty})\big) &= \chi_\infty^{A_\infty}\big(F_\bullet^q \otimes_{S_q}^{\mathbb{L}} S_{q,\mathrm{perfd}}^{A_\infty}\big) \\ &= \chi_\infty^{A_\infty}\big(F_\bullet \otimes_R^{\mathbb{L}} S_{q,\mathrm{perfd}}^{A_\infty}\big) \\ &= (\operatorname{rank}_R S_q) \cdot \chi_\infty^{A_\infty}\big(F_\bullet \otimes_R^{\mathbb{L}} R_{\mathrm{perfd}}^{A_\infty}\big) \\ &= (\operatorname{rank}_R S_q) \cdot \chi_{\mathrm{perfd}}^{x}(F_\bullet)\end{aligned}$$

Combining the above and letting $q \longrightarrow \infty$ we obtain that

$$\chi_{\mathrm{perfd}}^{x}(F_\bullet) \geq n_0 \cdot e(z_1, \dots, z_d; R) = \nu_R(H_0(F_\bullet)) \cdot e(R)$$

where the equality follows as $z_1, \dots, z_d$ is a reduction of $\mathfrak{m}$. □

2.2. **Comparison with Dutta multiplicity.** In this section we prove that the perfectoid Dutta multiplicity agrees with the ordinary Dutta multiplicity defined at the beginning of Section 2. We will need the following alternative characterization of the Dutta multiplicity.

Theorem 2.8 ([Kur01, Theorem 1.1])**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian complete local domain of dimension d with k algebraically closed. Let $A \longrightarrow R$ be a Noether–Cohen normalization that is generically étale. Let K and L be the fraction fields of A and R respectively. Let L' be the Galois closure of L/K and let S be the integral closure of R in L'. Then for any perfect complex $F_\bullet$ over R supported only at $\{\mathfrak{m}\}$, we have*

$$\chi_\infty(F_\bullet) = \frac{\chi(F_\bullet \otimes_R^{\mathbb{L}} S)}{\operatorname{rank}_R S}.$$

In particular, if R is normal and generically Galois over A, then $\chi_\infty(F_\bullet) = \chi(F_\bullet)$.

Theorem 2.9. *With notations as in Notation 1.3. Then for any perfect complex $F_\bullet$ over R supported only at $\{\mathfrak{m}\}$, we have*

$$\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet) = \chi_\infty(F_\bullet).$$

In particular, $\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet)$ does not depend on the choice of $\underline{x}$.

Proof. Let K and L denote the fraction fields of A and R respectively. Let L' be the Galois closure of L/K and let S be the integral closure of R in L'. By Theorem 2.8 and Lemma 2.6, we have

$$\chi_\infty(F_\bullet) = \frac{\chi(F_\bullet \otimes^{\mathbb{L}}_R S)}{\operatorname{rank}_R S} \quad \text{and} \quad \chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet) = \frac{\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet \otimes^{\mathbb{L}}_R S)}{\operatorname{rank}_R S}.$$

Thus to prove the theorem, we may replace R by S and $F_\bullet$ by $F_\bullet \otimes^{\mathbb{L}}_R S$ to assume that R is normal and L is finite Galois over K, and under these assumptions we will show that $\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet) = \chi(F_\bullet)$.

Let G be the Galois group of L/K. Since R is normal, each $\sigma \in G$ induces an A-linear automorphism $\sigma\colon R \to R$. It is straightforward to see that $\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet) = \chi^{\underline{x}}_{\mathrm{perfd}}(\sigma_* F_\bullet)$ and $\chi(F_\bullet) = \chi(\sigma_* F_\bullet)$ for each σ. Thus we have

$$|G| \cdot \chi(F_\bullet) = \sum_{\sigma \in G} \chi(\sigma_* F_\bullet) = \sum_{\sigma \in G} \chi^{\underline{x}}_{\mathrm{perfd}}(\sigma_* F_\bullet) = |G| \cdot \chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet),$$

where the critical equality in the middle follows from Theorem A.1: both $\chi(-)$ and $\chi^{\underline{x}}_{\mathrm{perfd}}(-)$ are additive on exact triangles of perfect complexes supported only at $\{\mathfrak{m}\}$ and thus induce maps $\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} \to \mathbb{R}$; since the class $\sum_{\sigma \in G}[\sigma_* F_\bullet] \in \mathrm{K}_0^{\mathfrak{m}}(R)^G_{\mathbb{Q}}$, Theorem A.1 tells us that up to a rational scalar, $\sum_{\sigma \in G}[\sigma_* F_\bullet]$ comes from $\mathrm{K}_0^{\mathfrak{m}_A}(A)$; as the latter is generated by the Koszul complex $\mathrm{Kos}(x_1, \dots, x_d; A)$, it remains to observe that

$$\chi\big(\mathrm{Kos}(x_1, \dots, x_d; R)\big) = e(x_1, \dots, x_d; R) = \chi^{\underline{x}}_{\mathrm{perfd}}\big(\mathrm{Kos}(x_1, \dots, x_d; R)\big)$$

where the second equality follows from [CLM+26, Proposition 4.0.14]. Dividing by $|G|$ then completes the proof of the theorem. □

Proof of Theorem 1.2. By Lemma 2.1, we may assume that R is a Noetherian complete local domain with algebraically closed residue field (so that we could freely use Notation 1.3). Now the result follows from Theorem 2.7 and Theorem 2.9. □

Remark 2.10. Theorem 1.2 answers [IMW22, Conjecture 8.1] when the residue characteristic of R is p. This implies the conjecture in general. For if R is a counterexample to Theorem 1.2 that contains a field of characteristic 0, then we can use [KR00, Lemma 4.4] to express $\chi_\infty(F_\bullet)$ as a rational linear combination of the usual Euler characteristics of a finite collection of perfect complexes. Furthermore, these Euler characteristics, the expression of $\chi_\infty(F_\bullet)$ as a linear combination, as well as $e(R)$, are all preserved after reduction modulo p for $p \gg 0$ (see [KR00, Section 4], [Hoc97, Appendix], and [Ma17, Section 5]). Therefore we arrive at a counterexample to Theorem 1.2 in characteristic p, a contradiction.

We end this section by exhibiting another application of the theory of (perfectoid) Dutta multiplicity developed here. This application relies crucially on Theorem 2.9 (and so requires Theorem A.1), but it does not require Theorem 2.7 or Theorem 1.5.

Corollary 2.11. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d such that the residue characteristic is not equal to 2. Let*

$$F_\bullet := 0 \longrightarrow R^{n_d} \longrightarrow \cdots \longrightarrow R^{n_1} \longrightarrow R^{n_0} \longrightarrow 0$$

be a non-exact perfect complex over R supported only at $\{\mathfrak{m}\}$. Then $\sum_i n_i \geq 2^d$.

Proof. We first assume that R has residue characteristic p. We may enlarge the residue field of R by [Sta26, Tag 03C3], replace R by its completion, and quotient by a minimal prime of dimension d to assume that $(R, \mathfrak{m})$ is a Noetherian complete local domain of dimension d with algebraically closed residue field. The rest of the proof essentially follows from the same argument as in [Wal17, Theorem 2 (2)] – simply replace the use of Frobenius by the (perfectoid) Dutta multiplicity. More specifically, we have

$$\begin{aligned}
2^d \chi_\infty(F_\bullet) &= \chi_\infty(S^2 F_\bullet) - \chi_\infty(\wedge^2 F_\bullet) \\
&= \chi^{\underline{x}}_{\mathrm{perfd}}(S^2 F_\bullet) - \chi^{\underline{x}}_{\mathrm{perfd}}(\wedge^2 F_\bullet) \\
&= \sum_i (-1)^i \lambda_\infty\big(H_i(S^2 F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})\big) - \sum_i (-1)^i \lambda_\infty\big(H_i(\wedge^2 F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})\big) \\
&\leq \sum_{i=\mathrm{even}} \lambda_\infty\big(H_i(S^2 F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})\big) + \sum_{i=\mathrm{odd}} \lambda_\infty\big(H_i(\wedge^2 F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})\big) \\
&\leq \sum_i \lambda_\infty\big(H_i(F_\bullet \otimes^{\mathbb{L}}_R F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})\big) \\
&= \sum_i \lambda_\infty\big(H_i(F_\bullet \otimes^{\mathbb{L}}_R H_0(F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}}))\big) \\
&\leq \lambda_\infty\big(H_0(F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})\big) \cdot \sum_i n_i \\
&= \chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet) \cdot \sum_i n_i,
\end{aligned}$$

where the first equality follows from [KR00, Theorem 3.1] and [Wal17, (2.1)] (here we used $p \neq 2$); the second equality follows from Theorem 2.9; the inequality on the fifth line follows as $S^2 F_\bullet$ and $\wedge^2 F_\bullet$ are direct summands of $F_\bullet \otimes^{\mathbb{L}}_R F_\bullet$ ($p \neq 2$ is used again); the equality on the sixth line follows from Lemma 2.3 (in our case it gives $F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}}$ is g-almost isomorphic to $H_0(F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})$); the inequality on the seventh line follows from a simple spectral sequence argument; and the equality on the last line follows from Corollary 2.5.

Finally, by Theorem 2.9 and Corollary 2.5, dividing the above by $\chi_\infty(F_\bullet) = \chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet) > 0$ we obtain that $\sum_i n_i \geq 2^d$ as wanted. The residue characteristic 0 case follows by reduction modulo p, as already carried out in [VW25, Theorem 5.1]. □

Remark 2.12. Corollary 2.11 is new only when R has mixed characteristic $(0, p > 2)$, and combined with [VW25], the total rank conjecture for complexes remains open only when R has mixed characteristic $(0, p = 2)$.

2.3. **Proof of Lech's conjecture.** We have proved Theorem 1.2 (and hence Theorem 1.1) modulo Theorem 1.5 and Theorem A.1. We shall prove Theorem 1.5 in Section 3, while the proof of Theorem A.1 will be given in Appendix A, which relies on nontrivial inputs from K-theory. In this section we give a direct proof of Lech's conjecture (Theorem 1.1) using

Theorem 2.7 (thus relying on Theorem 1.5) but without using Theorem 2.9 (thus avoiding K-theory). The strategy follows [Ope26a] and we make some simplifications (e.g., we treat the characteristic p and mixed characteristic cases together).

Proof of Theorem 1.1. By [Ma17, Section 5], it is enough to treat the case when the residue field of R has characteristic p. By [Ma17, Lemma 2.2], we may assume that R and S are complete, R is a domain, $\dim(R) = \dim(S) = n$, and S has algebraically closed residue field. We need the following generalization of [Ma17, Lemma 3.5].

Claim 2.13. *There exists a Cohen-factorization $R \longrightarrow T \twoheadrightarrow S = T/J$ so that*

(a) the generic fiber of $R \longrightarrow T$ is regular;
(b) J is a perfect ideal, and T, S are both equidimensional;
(c) each minimal prime of J contains exactly one minimal prime of T and each minimal prime of T is contained in some minimal prime of J.

Proof. By the Cohen–Gabber theorem, there exists a Noether–Cohen normalization $A \longrightarrow R$ that is generically étale. Let $A \longrightarrow T_0 \longrightarrow S$ be a Cohen-factorization of $A \longrightarrow S$ (by [AFH94]) and set $T := T_0 \otimes_A R$. We check that T satisfies all the desired conditions. First of all, $R \longrightarrow T$ is flat and $T/\mathfrak{m}T \cong T_0/\mathfrak{m}_A T_0$ is regular (which in particular implies T is complete local as T_0 is so) and thus $R \longrightarrow T \longrightarrow S$ is a Cohen-factorization. Let L and K denote the fraction fields of R and A respectively. The generic fiber of $R \longrightarrow T$ is

$$T \otimes_R L \cong (T_0 \otimes_A K) \otimes_K L.$$

Since T_0 is a regular local ring, $T_0 \otimes_A K$ and $(T_0 \otimes_A K) \otimes_K L$ are both regular (the latter is a finite étale extension of the former). The facts that J is perfect and T is equidimensional are well-known (see [Ma17, Lemma 3.5] and [MQS26, Lemma 3]). Let P be a minimal prime of J. Since $J \subseteq T$ is perfect, so is $J_P \subseteq T_P$ and thus

$$\mathrm{ht}(P) = \mathrm{pd}_{T_P} S_P \leq \mathrm{pd}_T S = \mathrm{ht}(J) \leq \mathrm{ht}(P)$$

which implies that $\mathrm{ht}(P) = \mathrm{ht}(J)$ and thus $S = T/J$ is also equidimensional.

Finally, since $R \longrightarrow S = T/J$ is faithfully flat, every minimal prime P of J contracts to (0) in R and thus T_P is a regular local ring. It follows that each such P contains exactly one minimal prime of T. On the other hand, for each minimal prime Q of T, let P be a minimal prime of $J + Q$. The perfect complex $T_P/JT_P \otimes^{\mathbb{L}}_{T_P} T_P/QT_P$ over T_P/QT_P is only supported at $\{PT_P/QT_P\}$ and thus its length is at least $\dim(T_P/QT_P) = \mathrm{ht}(P)$ (since T is equidimensional) by the new intersection theorem. But we also know its length is bounded by $\mathrm{pd}_{T_P} S_P \leq \mathrm{pd}_T S = \mathrm{ht}(J)$. It follows that $\mathrm{ht}(P) \leq \mathrm{ht}(J)$ and thus P is also a minimal prime of J. Thus every minimal prime of T is contained in some minimal prime of J. □

Let $z_1, \ldots, z_n$ be a minimal reduction of $\mathfrak{n}$ and let $F^N_\bullet := T/J \otimes^{\mathbb{L}}_T \mathrm{Kos}(z_1^N, \ldots, z_n^N; T)$. Since J is perfect, each $F^N_\bullet$ is a perfect complex over T of length $\dim(T) =: d$. By Claim 2.13, we let $Q_1, \ldots, Q_m$ be the minimal primes of T and let $P_1, \ldots, P_h$ be the minimal primes of J. Then for each Q_i we have

$$\chi^x_{\mathrm{perfd}}(F^N_\bullet \otimes^{\mathbb{L}}_T T/Q_i)^4 = N^n \chi^x_{\mathrm{perfd}}(F^1_\bullet \otimes^{\mathbb{L}}_T T/Q_i) \geq N^n e(T/Q_i) \tag{2.13.1}$$

[4]Note that here we are adapting Notation 1.3 to each T/Q_i, which is a Noetherian complete local domain of residue characteristic p and dimension d with algebraically closed residue field.

where the first equality follows by filtering $\mathrm{Kos}(z_1^N, \dots, z_n^N; T)$ by $\mathrm{Kos}(z_1, \dots, z_n; T)$ and the inequality follows from Theorem 2.7. On the other hand, by Corollary 2.5, we have

$$\chi^{\underline{x}}_{\mathrm{perfd}}(F_\bullet^N \otimes^{\mathbb{L}}_T T/Q_i) = \lambda_\infty\big((T/J + (z_1^N, \dots, z_n^N)) \otimes_T (T/Q_i)^{A_\infty}_{\mathrm{perfd}}\big). \tag{2.13.2}$$

Fixing a prime filtration of T/J by T/P for various P's containing J and noting that in such a prime filtration, each minimal prime P_j of J appears exactly $\ell(T_{P_j}/JT_{P_j})$ many times, we have that

$$\begin{aligned}
&\lambda_\infty\big((T/J + (z_1^N, \dots, z_n^N)) \otimes_T (T/Q_i)^{A_\infty}_{\mathrm{perfd}}\big) \\
\leq &\sum_P \lambda_\infty\big((T/P + (z_1^N, \dots, z_n^N)) \otimes_T (T/Q_i)^{A_\infty}_{\mathrm{perfd}}\big) \\
\leq &\sum_{P_j \supseteq Q_i} \ell(T_{P_j}/JT_{P_j}) \cdot \lambda_\infty\big((T/P_j + (z_1^N, \dots, z_n^N)) \otimes_T (T/Q_i)^{A_\infty}_{\mathrm{perfd}}\big) \\
&+ \sum_{\mathrm{other} P's} \lambda_\infty\big((T/P + (z_1^N, \dots, z_n^N)) \otimes_T (T/Q_i)^{A_\infty}_{\mathrm{perfd}}\big) \\
\leq &\sum_{P_j \supseteq Q_i} \ell(T_{P_j}/JT_{P_j}) \cdot e(P_j + (z_1^N, \dots, z_n^N); T/Q_i) \qquad (2.13.3) \\
&+ \sum_{\mathrm{other} P's} e(P + (z_1^N, \dots, z_n^N); T/Q_i).
\end{aligned}$$

where the last inequality follows from [CLM+26, Corollary 4.0.18].

At this point, we choose and fix $y_1, \dots, y_{d-n}$ in P_j whose images in $P_j/\mathfrak{n}P_j$ are sufficiently general. By the behavior of Hilbert–Samuel multiplicities modulo parameters and the associativity formula, we have

$$\begin{aligned}
e(P_j + (z_1^N, \dots, z_n^N); T/Q_i) &\leq e\big(P_j + (z_1^N, \dots, z_n^N); T/Q_i + (y_1, \dots, y_{d-n})\big) \\
&= \ell\big((T/Q_i + (y_1, \dots, y_{d-n}))_{P_j}\big) \cdot e(P_j + (z_1^N, \dots, z_n^N); T/P_j) \\
&\quad + \sum_{P \neq P_j} \ell\big((T/Q_i + (y_1, \dots, y_{d-n}))_P\big) \cdot e(P_j + (z_1^N, \dots, z_n^N); T/P) \\
&= e(z_1^N, \dots, z_n^N; T/P_j) \\
&\quad + \sum_{P \neq P_j} \ell\big((T/Q_i + (y_1, \dots, y_{d-n}))_P\big) \cdot e(P_j + (z_1^N, \dots, z_n^N); T/P) \qquad (2.13.4)
\end{aligned}$$

where the sum runs over all minimal primes of $T/Q_i + (y_1, \dots, y_{d-n})$ that are different from P_j, and the last equality follows from $(y_1, \dots, y_{d-n})T_{P_j} = P_jT_{P_j}$ by our choice of $y_1, \dots, y_{d-n}$ and the fact that T_{P_j} is regular local (and Q_i is the unique minimal prime of T that is contained in P_j). We now estimate the other terms in (2.13.3) and (2.13.4) as $N \longrightarrow \infty$. In (2.13.4), since $\dim(T/P) = n$ and $P_j \nsubseteq P$, $\dim(T/P + P_j) < n$ and thus

$$e(P_j + (z_1^N, \dots, z_n^N); T/P) = o(N^n).$$

In (2.13.3), for each of the other P's in the second part of the summation, $\mathrm{ht}(P \cdot (T/Q_i)) > d - n$ and thus $\dim(T/Q_i + P) < n$, and again it follows that

$$e(P + (z_1^N, \dots, z_n^N); T/Q_i) = o(N^n).$$

Putting these into (2.13.3) and (2.13.4) yields that

$$\sum_{P_j \supseteq Q_i} \ell(T_{P_j}/JT_{P_j}) \cdot e(z_1^N, \dots, z_n^N; T/P_j) + o(N^n) \geq \lambda_\infty\big((T/J + (z_1^N, \dots, z_n^N)) \otimes_T (T/Q_i)^{A_\infty}_{\text{perfd}}\big)$$

for each i. Combining this together with (2.13.2) and (2.13.1) together gives

$$\sum_{P_j \supseteq Q_i} \ell(T_{P_j}/JT_{P_j}) \cdot e(z_1^N, \dots, z_n^N; T/P_j) + o(N^n) \geq N^n e(T/Q_i) \tag{2.13.5}$$

for each i. Taking the sum over all i we obtain that

$$\begin{aligned} e(z_1^N, \dots, z_n^N; T/J) + o(N^n) &= \sum_j \ell(T_{P_j}/JT_{P_j}) \cdot e(z_1^N, \dots, z_n^N; T/P_j) + o(N^n) \\ &\geq N^n \sum_i e(T/Q_i) = N^n e(T) \end{aligned}$$

where the first and last equalities use the associativity formula for multiplicities and the fact that T_{Q_i} is a field as it is a regular local ring of dimension zero by Claim 2.13; and the inequality uses (2.13.5) and Claim 2.13 (c). Finally,

$$e(z_1^N, \dots, z_n^N; T/J) = N^n e(z_1, \dots, z_n; S) = N^n e(S)$$

as $z_1, \dots, z_n$ is a reduction of $\mathfrak{n}$. Thus dividing the above by N^n and letting $N \longrightarrow \infty$, we obtain that $e(S) \geq e(T) = e(R)$. □

3. Proof of Theorem 1.5

In this section we will prove Theorem 1.5. We start by setting up some notations.

Setting 3.1. We continue to use Notation 1.3. For the ease of the presentation below, we set $S := R^{A_\infty}_{\mathrm{perfd}}$ throughout this section. We will fix a system of parameters $z_1, \dots, z_d$ of R as in the statement of Theorem 1.5 and let $I = (z_1, \dots, z_d)S \subseteq S$. Let Y be the blowup of S along I and let E be the exceptional locus. We summarize these in the following diagram:

$$\begin{array}{ccccccc} Y = \mathrm{Bl}_I(S) & \hookrightarrow & \mathbb{P}^{d-1}_S & \twoheadrightarrow & \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \\ \uparrow & & \uparrow & & \uparrow & & \\ E = \mathrm{Proj}\,\mathrm{gr}_I(S) & \hookrightarrow & \mathbb{P}^{d-1}_{S/I} & \twoheadrightarrow & \mathrm{Spec}(S/I). & & \end{array}$$

For a vector bundle V on $\mathbb{P}^{d-1}_{\mathbb{Z}}$, we will abuse notation and still use V to denote its pullback to $\mathbb{P}^{d-1}_S, \mathbb{P}^{d-1}_{S/I}, Y$, and E respectively.

Lemma 3.2. *With notations as in Setting 3.1. We have:*

(a) $S \to \mathbf{R}\Gamma(Y, \mathcal{O}_Y)$ is a g-almost isomorphism.

(b) for any vector bundle V on $\mathbb{P}^{d-1}_{\mathbb{Z}}$, the natural map

$$\mathbf{R}\Gamma(\mathbb{P}^{d-1}_{S/I}, V) \longrightarrow \mathbf{R}\Gamma(E, V)$$

is a g-almost isomorphism.

Proof. The key point is that $z_1, \dots, z_d$ is a g-almost Koszul regular sequence on S, that is, $H_i(\mathrm{Kos}(z_1, \dots, z_d; S))$ is g-almost zero for all $i > 0$, see [CLM+26, Corollary 4.0.4] (which essentially follows from the almost purity theorem [BS22, Theorem 10.9]). Given this, both parts of the lemma are g-almost versions of standard facts on blowups of regular sequences. Thus we will be brief in the presentation below.

We first prove (b), by taking a finite resolution of V on $\mathbb{P}^{d-1}_{\mathbb{Z}}$ by vector bundles of the form $\oplus_i \mathcal{O}_{\mathbb{P}^{d-1}_{\mathbb{Z}}}(a_i)$, we reduce to the case $V = \mathcal{O}(j)$. The point is that the natural map

$$(S/I)[Z_1, \dots, Z_d] \longrightarrow \mathrm{gr}_I(S)$$

is surjective and g-almost injective (i.e., the kernel is annihilated by $(g)_{\mathrm{perfd}}$). This follows from the g-almost version of the same argument as in [Sta26, Tag 062I] (or simply applying that to $(g)_{\mathrm{perfd}}$ to obtain that $I^n(g)_{\mathrm{perfd}}/I^{n+1}(g)_{\mathrm{perfd}}$ is a direct sum of $\nu_S(I^n)$ many copies of $(g)_{\mathrm{perfd}}/I(g)_{\mathrm{perfd}}$, which then implies that $(g)_{\mathrm{perfd}}$ annihilates the degree n piece of the kernel). The result then follows from the Čech complex computations of $\mathbf{R}\Gamma(\mathbb{P}^{d-1}_{S/I}, \mathcal{O}_{\mathbb{P}^{d-1}_{S/I}}(j))$ and $\mathbf{R}\Gamma(E, \mathcal{O}_E(j))$.

For (a), by standard Čech complex computation, it is enough to show that $H^{>0}_{It}(S[It])_0 \cong H^{>0}_{It}(S[It, t^{-1}])_0$ is annihilated by $(g)_{\mathrm{perfd}}$. Consider the long exact sequence on local cohomology induced by

$$0 \longrightarrow S[It, t^{-1}] \xrightarrow{\cdot t^{-1}} S[It, t^{-1}] \longrightarrow \mathrm{gr}_I(S) \longrightarrow 0$$

and using the already established part (b), we obtain that multiplication by t^{-1} is g-almost injective on $H^{>0}_{It}(S[It, t^{-1}])$. This implies that

$$H^i_{It}(S[It, t^{-1}]) \longrightarrow H^i_{It}(S[t, t^{-1}])$$

is g-almost injective for all $i > 0$. The latter is g-almost zero for $i < d$ since $z_1, \dots, z_d$ is a g-almost Koszul regular sequence on S. For $i = d$, it is easy to check that $H^d_{It}(S[It, t^{-1}])_0 = 0$ for degree reasons. □

Lemma 3.3. *With notations as in Setting 3.1. Let $F_\bullet$ be a perfect complex over R supported only at $\{\mathfrak{m}\}$ and let $F_{\bullet,Y}$ be its pullback to Y. Then for any vector bundle V on $\mathbb{P}^{d-1}_{\mathbb{Z}}$, we have that*

$$\lambda_\infty\big(H^i\mathbf{R}\Gamma(Y, F_{\bullet,Y} \otimes V)\big) < \infty$$

for all i. In particular, $\chi^{A_\infty}_\infty\big(\mathbf{R}\Gamma(Y, F_{\bullet,Y} \otimes V)\big) < \infty$ is well-defined and we have

$$\chi^{A_\infty}_\infty\big(\mathbf{R}\Gamma(Y, F_{\bullet,Y} \otimes V)\big) = (\operatorname{rank} V) \cdot \chi^{A_\infty}_\infty\big(\mathbf{R}\Gamma(Y, F_{\bullet,Y})\big)$$

Proof. If $V = \mathcal{O}_{\mathbb{P}^{d-1}_{\mathbb{Z}}}$, then by the projection formula we have

$$\mathbf{R}\Gamma(Y, F_{\bullet,Y}) \cong F_\bullet \otimes^{\mathbb{L}}_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y)$$

and the latter is g-almost isomorphic to $F_\bullet \otimes^{\mathbb{L}}_R S$ by Lemma 3.2. It follows that

$$\lambda_\infty\big(H^i\mathbf{R}\Gamma(Y, F_{\bullet,Y})\big) = \lambda_\infty\big(H^i(F_\bullet \otimes^{\mathbb{L}}_R S)\big) < \infty$$

for all i by Lemma 2.3. In general, there is a finite resolution of V by vector bundles on $\mathbb{P}^{d-1}_{\mathbb{Z}}$ of the form $\oplus_i \mathcal{O}_{\mathbb{P}^{d-1}_{\mathbb{Z}}}(a_i)$. Thus it suffices to prove that

$$\lambda_\infty\big(H^i\mathbf{R}\Gamma(Y, F_{\bullet,Y} \otimes \mathcal{O}_Y(j))\big) < \infty$$

for all i and j. Taking the obvious filtration connecting $\mathcal{O}_Y(j)$ and $\mathcal{O}_Y$ and examining the successive quotients, it is enough to show that

$$\lambda_\infty\big(H^i\mathbf{R}\Gamma(Y, F_{\bullet,Y} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \mathcal{O}_E(j))\big) < \infty.$$

By the projection formula, we have

$$\mathbf{R}\Gamma(Y, F_{\bullet,Y} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \mathcal{O}_E(j)) \cong (F_\bullet \otimes^{\mathbb{L}}_R S/I) \otimes^{\mathbb{L}}_{S/I} \mathbf{R}\Gamma(E, \mathcal{O}_E(j)). \tag{3.3.1}$$

By Lemma 3.2, $\mathbf{R}\Gamma(E, \mathcal{O}_E(j))$ is g-almost isomorphic to $\mathbf{R}\Gamma(\mathbb{P}^{d-1}_{S/I}, \mathcal{O}_{\mathbb{P}^{d-1}_{S/I}}(j))$ and it is easy to see that the latter is a free S/I-module up to a shift. Thus it is enough to observe that $\lambda_\infty\big(H^i(F_\bullet \otimes^{\mathbb{L}}_R S/I)\big) < \infty$ for all i, which follows as $F_\bullet \otimes^{\mathbb{L}}_R S/I$ is a perfect complex over S/I and $\lambda_\infty(S/I) < \infty$ by [CLM+26, Proposition 4.0.14].

Finally, for the last assertion, again by resolving V by vector bundles of the form $\oplus_i \mathcal{O}_{\mathbb{P}^{d-1}_{\mathbb{Z}}}(a_i)$, it is enough to prove that for all j,

$$\chi^{A_\infty}_\infty\big(\mathbf{R}\Gamma(Y, F_{\bullet,Y} \otimes \mathcal{O}_Y(j))\big) = \chi^{A_\infty}_\infty\big(\mathbf{R}\Gamma(Y, F_{\bullet,Y})\big).$$

Taking the obvious filtration connecting $\mathcal{O}_Y(j)$ and $\mathcal{O}_Y$ and examining the successive quotients, it comes down to proving that for all j,

$$\chi^{A_\infty}_\infty\big(\mathbf{R}\Gamma(Y, F_{\bullet,Y} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \mathcal{O}_E(j))\big) = 0.$$

By (3.3.1) and the discussion that follows, it remains to observe that $\chi^{A_\infty}_\infty(F_\bullet \otimes^{\mathbb{L}}_R S/I) = 0$, which follows as $F_\bullet \otimes^{\mathbb{L}}_R S/I$ is a perfect complex over S/I whose alternating sum of the Betti numbers is 0. □

The following lemma will be crucial, the idea of the proof is similar in spirit to [MMRGS26, Theorem 2.2].

Lemma 3.4. *With notations as in Setting 3.1. Suppose $G_\bullet$ is a perfect complex on Y such that each $H^i\mathbf{R}\Gamma(Y, G_\bullet)$ is $\mathfrak{m}$-power torsion for all i. Then there exists $m_0 > 0$ (depending on $G_\bullet$) so that for all $m \geq m_0$, the natural maps*

$$\mathbf{R}\Gamma(Y, G_\bullet \otimes \mathcal{O}_Y(m)) \longrightarrow \mathbf{R}\Gamma(Y, G_\bullet) \quad \text{and} \quad \mathbf{R}\Gamma(Y, G_\bullet) \longrightarrow \mathbf{R}\Gamma(Y, G_\bullet \otimes \mathcal{O}_Y(-m))$$

are both zero in $D(S)$.

Proof. Since $\mathbf{R}\Gamma(Y, G_\bullet)$ is bounded and each $H^i\mathbf{R}\Gamma(Y, G_\bullet)$ is annihilated by a power of $\mathfrak{m}$, there exists N such that

$$\mathfrak{m}^N \subseteq \ker\left(S \longrightarrow \operatorname{Hom}_{D(S)}(\mathbf{R}\Gamma(Y, G_\bullet), \mathbf{R}\Gamma(Y, G_\bullet)) \cong H^0\big(\mathbf{R}\operatorname{Hom}_S(\mathbf{R}\Gamma(Y, G_\bullet), \mathbf{R}\Gamma(Y, G_\bullet))\big)\right).$$

It follows that the two canonical maps

$$\mathbf{R}\Gamma(Y, G_\bullet) \longrightarrow \operatorname{Kos}\big(z_1^N, \dots, z_d^N; \mathbf{R}\Gamma(Y, G_\bullet)\big) \tag{3.4.1}$$

$$\operatorname{Kos}\big(z_1^N, \dots, z_d^N; \mathbf{R}\Gamma(Y, G_\bullet)\big) \longrightarrow \mathbf{R}\Gamma(Y, G_\bullet)[d] \tag{3.4.2}$$

are both split in $D(S)$. Next, we notice that on Y, the surjection of locally free sheaves $\mathcal{O}_Y^{\oplus d} \twoheadrightarrow (z_1^N, \dots, z_d^N)\mathcal{O}_Y = \mathcal{O}_Y(N)$ induces an exact Koszul complex:

$$0 \longrightarrow \mathcal{O}_Y(-dN) \longrightarrow \cdots \longrightarrow \mathcal{O}_Y(-N)^{\oplus d} \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

The following diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y^{\oplus d} & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}_Y^{\oplus d} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_Y(-dN) & \longrightarrow & \mathcal{O}_Y(-(d-1)N)^{\oplus d} & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}_Y(-N)^{\oplus d} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0
\end{array}$$

then shows that the natural map

$$\operatorname{Kos}(z_1^N, \dots, z_d^N; \mathcal{O}_Y) \longrightarrow \mathcal{O}_Y(-m)[d] \tag{3.4.3}$$

is zero in $D(Y)$ for $m \geq dN$, as it factors through an exact complex. Applying $\mathbf{R}\operatorname{Hom}_Y(-, \mathcal{O}_Y)[d]$ to (3.4.3) we obtain that the natural map

$$\mathcal{O}_Y(m) \longrightarrow \operatorname{Kos}(z_1^N, \dots, z_d^N; \mathcal{O}_Y) \tag{3.4.4}$$

is zero in $D(Y)$ for all $m \geq dN$. Now applying $\mathbf{R}\Gamma(Y, - \otimes G_\bullet)$ to (3.4.4) and using (3.4.1), we obtain that

$$\mathbf{R}\Gamma(Y, G_\bullet \otimes \mathcal{O}_Y(m)) \longrightarrow \mathbf{R}\Gamma(Y, G_\bullet)$$

is zero in $D(S)$. Similarly, applying $\mathbf{R}\Gamma(Y, - \otimes G_\bullet)$ to (3.4.3) and using (3.4.2), we obtain

$$\mathbf{R}\Gamma(Y, G_\bullet) \longrightarrow \mathbf{R}\Gamma(Y, G_\bullet \otimes \mathcal{O}_Y(-m))$$

is zero in $D(S)$. □

We next recall a construction due to Eisenbud–Schreyer [ES09, Section 6]. Fix an integer $d \geq 2$. Consider the symmetrization map

$$f : \ (\mathbb{P}^1_{\mathbb{Z}})^{d-1} = \mathbb{P}^1_{\mathbb{Z}} \times \cdots \times \mathbb{P}^1_{\mathbb{Z}} \longrightarrow \mathbb{P}^{d-1}_{\mathbb{Z}}.$$

This map is finite of degree $(d-1)!$, and is linear in the sense that $f^*\mathcal{O}_{\mathbb{P}^{d-1}_{\mathbb{Z}}}(1) = \boxtimes_{1 \leq j \leq d-1}\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)$. For any sequence of integers

$$r_1 > r_2 > \cdots > r_{d-1},$$

consider the following vector bundle on $\mathbb{P}^{d-1}_{\mathbb{Z}}$:

$$V := f_*\big(\boxtimes_{1\le j\le d-1}\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(-1-r_j)\big). \tag{3.4.5}$$

This is a vector bundle of rank $(d-1)!$ on $\mathbb{P}^{d-1}_{\mathbb{Z}}$. By the Künneth formula, it is straightforward to check that for each integer n, $V(n)$ has cohomology in at most one degree, namely

$$q = \#\{j \mid r_j > n\} \tag{3.4.6}$$

Furthermore, we have $H^q(\mathbb{P}^{d-1}_{\mathbb{Z}}, V(n))$ a free $\mathbb{Z}$-module of rank $(-1)^q\prod_{j=1}^{d-1}(n-r_j)$.

Base-change from $\mathbb{Z}$ to S/I, it is easy to see that

$$\mathbf{R}\Gamma(\mathbb{P}^{d-1}_{S/I}, V(n)) = H^q(\mathbb{P}^{d-1}_{S/I}, V(n))[-q]$$

and the latter is a free S/I-module of rank $(-1)^q\prod_{j=1}^{d-1}(n-r_j)$ up to shift. By Lemma 3.2, we know that $\mathbf{R}\Gamma(E, V(n))$ is g-almost isomorphic to $H^q(\mathbb{P}^{d-1}_{S/I}, V(n))[-q]$ and thus for all integers n, we have

$$\chi^{A_\infty}_\infty\big(\mathbf{R}\Gamma(E, V(n))\big) = \lambda_\infty(S/I)\cdot\prod_{j=1}^{d-1}(n-r_j). \tag{3.4.7}$$

We are now ready to prove the key technical result Theorem 1.5.

Proof of Theorem 1.5. We continue to use the notations as in Setting 3.1. If $d=1$, then we have $F_\bullet = 0 \longrightarrow R^{n_0} \xrightarrow{X} R^{n_0} \longrightarrow 0$ and it is easy to see that

$$\lambda_\infty\big(H_0(F_\bullet\otimes^{\mathbb{L}}_R S)\big) = \chi^{A_\infty}_\infty\big(H_0(F_\bullet\otimes^{\mathbb{L}}_R S)\big) = \chi(F_\bullet) = \ell_R(R/\det(X)).$$

Since the assumption says that all entries of X lie in (z_1^N), we thus have

$$\ell_R(R/\det(X)) \ge \ell_R(R/z_1^{Nn_0}) = e(z_1; R)\cdot Nn_0.$$

Thus we can simply take $C=0$ in this case.

Henceforth we assume $d\ge 2$. Consider the function $H(x)=\prod_{s=1}^{d}(x+s)$ and let

$$s_1 > s_2 > \cdots > s_{d-1}$$

be the roots of $H'(x)$. Since $H(x)$ has zeros at $-1,\dots,-d$, we have $-(j+1)<s_j<-j$ for all $1\le j\le d-1$. Let N be the integer in the statement of Theorem 1.5 and we may assume that $N\ge 2$.[5] Then we can find integers r_j so that

$$-(j+1)N < r_j < -jN, \quad \text{and} \quad |r_j - Ns_j| < 1. \tag{3.4.8}$$

Let V be the vector bundle on $\mathbb{P}^{d-1}_{\mathbb{Z}}$ constructed in (3.4.5) with respect to $r_1>\cdots>r_{d-1}$. We shall estimate $\chi^{A_\infty}_\infty\big(\mathbf{R}\Gamma(Y, F_{\bullet,Y}\otimes V)\big)$, which will give us estimates on $\chi^{A_\infty}_\infty\big(\mathbf{R}\Gamma(Y, F_{\bullet,Y})\big) = \chi^{A_\infty}_\infty(F_\bullet\otimes_R S)$ by Lemma 3.3. In order to do this, we need an auxiliary perfect complex on Y constructed below.

Recall that $F_{\bullet,Y}$ is the pullback of $F_\bullet$ to Y. Since $\partial(F_\bullet)\subseteq(z_1,\dots,z_d)^N F_\bullet$, each differential $\mathcal{O}_Y^{n_{j+1}}\longrightarrow\mathcal{O}_Y^{n_j}$ in $F_{\bullet,Y}$ actually induces a map $\mathcal{O}_Y(-N)^{n_{j+1}}\longrightarrow\mathcal{O}_Y^{n_j}$ and hence a map $\mathcal{O}_Y(-(j+1)N)^{n_{j+1}}\longrightarrow\mathcal{O}_Y(-jN)^{n_j}$. We set $G_\bullet$ to be the following perfect complex on Y:

$$G_\bullet := 0\longrightarrow\mathcal{O}_Y(-dN)^{n_d}\longrightarrow\mathcal{O}_Y(-(d-1)N)^{n_{d-1}}\longrightarrow\cdots\longrightarrow\mathcal{O}_Y(-N)^{n_1}\longrightarrow\mathcal{O}_Y^{n_0}\longrightarrow 0.$$

[5] We can easily enlarge the constant C so that the right hand side of the desired inequality in Theorem 1.5 is negative when $N=1$.

Claim 3.5. *We have* $\lambda_\infty\big(H^i\mathbf{R}\Gamma(Y, G_\bullet \otimes V)\big) < \infty$ *for all* i.

Proof. Resolving V by vector bundles of the form $\oplus_i \mathcal{O}_{\mathbb{P}^{d-1}_{\mathbb{Z}}}(a_i)$, we may assume $V = \mathcal{O}_{\mathbb{P}^{d-1}_{\mathbb{Z}}}(a)$ for some integer a. Note that $F_{\bullet,Y}$ and $G_\bullet$ can be connected by the following sequence of maps of perfect complexes on Y:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{O}_Y^{n_d} & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}_Y^{n_2} & \longrightarrow & \mathcal{O}_Y^{\oplus n_1} & \longrightarrow & \mathcal{O}_Y^{n_0} \longrightarrow 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \mathcal{O}_Y(-N)^{n_d} & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}_Y(-N)^{n_2} & \longrightarrow & \mathcal{O}_Y(-N)^{\oplus n_1} & \longrightarrow & \mathcal{O}_Y^{n_0} \longrightarrow 0 \\
 & & \downarrow & & & & \downarrow & & \| & & \| \\
0 & \longrightarrow & \mathcal{O}_Y(-2N)^{n_d} & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}_Y(-2N)^{n_2} & \longrightarrow & \mathcal{O}_Y(-N)^{\oplus n_1} & \longrightarrow & \mathcal{O}_Y^{n_0} \longrightarrow 0 \\
 & & \downarrow & & & & \| & & \| & & \| \\
 & & \vdots & & & & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & \mathcal{O}_Y(-dN)^{n_d} & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}_Y(-2N)^{n_2} & \longrightarrow & \mathcal{O}_Y(-N)^{\oplus n_1} & \longrightarrow & \mathcal{O}_Y^{n_0} \longrightarrow 0
\end{array}$$

Tensoring with V and examining the quotient of each two successive complexes, it is enough to show that $\lambda_\infty\big(H^i\mathbf{R}\Gamma(Y, \mathcal{O}_{NE}(-j))\big) < \infty$ for all i and j. Filtering further, it suffices to show that $\lambda_\infty\big(H^i\mathbf{R}\Gamma(E, \mathcal{O}_E(-j))\big) < \infty$ for all i and j. But this is clear as $\mathbf{R}\Gamma(E, \mathcal{O}_E(-j))$ is g-almost isomorphic to a finite free S/I-module up to a shift by Lemma 3.2 and $\lambda_\infty(S/I) < \infty$ by [CLM+26, Proposition 4.0.14]. □

The following is the key computation.

Claim 3.6. *We have* $\chi_\infty^{A_\infty}\big(\mathbf{R}\Gamma(Y, G_\bullet \otimes V)\big) \geq 0$.

Proof. First of all, by Claim 3.5 we have $\chi_\infty^{A_\infty}\big(\mathbf{R}\Gamma(Y, G_\bullet \otimes V)\big) < \infty$ is well-defined. We fix an integer m as in Lemma 3.4 so that the following natural maps

$$\mathbf{R}\Gamma(Y, G_\bullet \otimes V \otimes \mathcal{O}_Y(m)) \to \mathbf{R}\Gamma(Y, G_\bullet \otimes V) \tag{3.6.1}$$

$$\mathbf{R}\Gamma(Y, G_\bullet \otimes V) \to \mathbf{R}\Gamma(Y, G_\bullet \otimes V \otimes \mathcal{O}_Y(-m)) \tag{3.6.2}$$

are zero in $D(S)$. We will use these to show that $\mathbf{R}\Gamma(Y, G_\bullet \otimes V)$ is g-almost concentrated in degree 0. Consider the following sequence of maps of perfect complexes on Y:

$$G_\bullet \otimes V \otimes \mathcal{O}_Y(m) \to G_\bullet \otimes V \otimes \mathcal{O}_Y(m-1) \to \cdots \to G_\bullet \otimes V \otimes \mathcal{O}_Y(1) \to G_\bullet \otimes V. \tag{3.6.3}$$

Note that for each integer $0 \leq r \leq m-1$, the quotient complex at the r-th stage is

$$0 \to V(r-dN)|_E^{n_d} \to \cdots \to V(r-iN)|_E^{n_i} \to \cdots \to V(r)|_E^{n_0} \to 0.$$

The i-th term $V(r-iN)|_E^{n_i}$ in this complex sits in cohomological degree $-i$. By Lemma 3.2, $\mathbf{R}\Gamma(E, V(r-iN)|_E)$ is g-almost isomorphic to $H^q(\mathbb{P}^{d-1}_{S/I}, V(r-iN))[-q]$, where by (3.4.6)

$$q = \#\{j \mid r_j > r - iN\} \leq i$$

where the critical inequality follows from our choice of r_j, see (3.4.8) (in fact, we have strict inequality when $i \neq 0$). It follows that for each $0 \leq r \leq m-1$,

$$\operatorname{cone}\big(\mathbf{R}\Gamma(Y, G_\bullet \otimes V \otimes \mathcal{O}_Y(r+1)) \longrightarrow \mathbf{R}\Gamma(Y, G_\bullet \otimes V \otimes \mathcal{O}_Y(r))\big)$$

is g-almost concentrated in $D^{\leq 0}(S)$. Thus by (3.6.3) and a simple induction, we have

$$\operatorname{cone}\big(\mathbf{R}\Gamma(Y, G_\bullet \otimes V \otimes \mathcal{O}_Y(m)) \longrightarrow \mathbf{R}\Gamma(Y, G_\bullet \otimes V)\big)$$

is g-almost concentrated in $D^{\leq 0}(S)$. This combined with (3.6.1) shows that

$$H^{>0}\mathbf{R}\Gamma(Y, G_\bullet \otimes V) \text{ is } g\text{-almost zero.}$$

An essentially similar argument will handle $H^{<0}\mathbf{R}\Gamma(Y, G_\bullet \otimes V)$ and we give some details. Consider the following sequence of maps of perfect complexes on Y:

$$G_\bullet \otimes V \longrightarrow G_\bullet \otimes V \otimes \mathcal{O}_Y(-1) \longrightarrow \cdots \longrightarrow G_\bullet \otimes V \otimes \mathcal{O}_Y(-(m-1)) \longrightarrow G_\bullet \otimes V \otimes \mathcal{O}_Y(-m). \tag{3.6.4}$$

For each integer $1 \leq r \leq m$, the quotient complex at the $(-r)$-th stage is

$$0 \longrightarrow V(-r-dN)|_E^{n_d} \longrightarrow \cdots \longrightarrow V(-r-iN)|_E^{n_i} \longrightarrow \cdots \longrightarrow V(-r)|_E^{n_0} \longrightarrow 0.$$

The i-th term $V(-r-iN)|_E^{n_i}$ sits in cohomological degree $-i$, and by Lemma 3.2 we have $\mathbf{R}\Gamma(E, V(-r-iN)|_E)$ is g-almost isomorphic to $H^q(\mathbb{P}^{d-1}_{S/I}, V(-r-iN))[-q]$, where by (3.4.6)

$$q = \#\{j \mid r_j > -r-iN\} \geq i-1$$

where again the critical inequality follows from our choice of r_j, see (3.4.8). It follows that for each $1 \leq r \leq m$,

$$\operatorname{cone}\big(\mathbf{R}\Gamma(Y, G_\bullet \otimes V \otimes \mathcal{O}_Y(-r+1)) \longrightarrow \mathbf{R}\Gamma(Y, G_\bullet \otimes V \otimes \mathcal{O}_Y(-r))\big)$$

is g-almost concentrated in $D^{\geq -1}(S)$. Thus by (3.6.4) and a simple induction, we have

$$\operatorname{cone}\big(\mathbf{R}\Gamma(Y, G_\bullet \otimes V) \longrightarrow \mathbf{R}\Gamma(Y, G_\bullet \otimes V \otimes \mathcal{O}_Y(-m))\big)$$

is g-almost concentrated in $D^{\geq -1}(S)$. This combined with (3.6.2) shows that

$$H^{<0}\mathbf{R}\Gamma(Y, G_\bullet \otimes V) \text{ is } g\text{-almost zero.}$$

We have proved that $\mathbf{R}\Gamma(Y, G_\bullet \otimes V)$ is g-almost concentrated in degree 0, thus

$$\chi_\infty^{A_\infty}\big(\mathbf{R}\Gamma(Y, G_\bullet \otimes V)\big) = \lambda_\infty\big(H^0\mathbf{R}\Gamma(Y, G_\bullet \otimes V)\big) \geq 0. \qquad \square$$

We now finish the proof of the theorem. Consider the natural inclusion of perfect complexes $F_{\bullet,Y} \otimes V \longrightarrow G_\bullet \otimes V$. The quotient complex in cohomological degree $-i$ has a filtration by $V(t)|_E^{n_i}$ where $t = -1, \ldots, -iN$. By Claim 3.6, we have

$$\begin{aligned}
0 &\leq \chi_\infty^{A_\infty}\big(\mathbf{R}\Gamma(Y, F_{\bullet,Y} \otimes V)\big) + \sum_{i=0}^{d}(-1)^i n_i \sum_{t=-iN}^{-1} \chi_\infty^{A_\infty}\big(\mathbf{R}\Gamma(E, V(t)|_E)\big) \\
&= (d-1)! \cdot \chi_\infty^{A_\infty}\big(\mathbf{R}\Gamma(Y, F_{\bullet,Y})\big) + \lambda_\infty(S/I) \cdot \sum_{i=0}^{d}(-1)^i n_i \sum_{t=-iN}^{-1} \prod_{j=1}^{d-1}(t-r_j) \\
&= (d-1)! \cdot \lambda_\infty\big(H_0(F_\bullet \otimes_R^{\mathbb{L}} S)\big) + \lambda_\infty(S/I) \cdot \sum_{i=0}^{d}(-1)^i n_i \sum_{t=-iN}^{-1} \prod_{j=1}^{d-1}(t-r_j). \qquad (3.6.5)
\end{aligned}$$

Here the equality on the second line follows from Lemma 3.3 and (3.4.7), while the equality on the third line follows from the projection formula $\mathbf{R}\Gamma(Y, F_{\bullet,Y}) \cong F_\bullet \otimes^{\mathbb{L}}_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y)$, the almost isomorphism $F_\bullet \otimes^{\mathbb{L}}_R S \to F_\bullet \otimes^{\mathbb{L}}_R \mathbf{R}\Gamma(Y, \mathcal{O}_Y)$ by Lemma 3.2, as well as Lemma 2.3.

It remains to control the last term in (3.6.5). The strategy is to compare

$$\sum_{i=0}^{d}(-1)^i n_i \sum_{t=-iN}^{-1}\prod_{j=1}^{d-1}(t-r_j); \quad \sum_{i=0}^{d}(-1)^i n_i \sum_{t=-iN}^{-1}\prod_{j=1}^{d-1}(t-Ns_j); \quad \sum_{i=0}^{d}(-1)^i n_i N^d \int_{-i}^{0}\prod_{j=1}^{d-1}(x-s_j)dx.$$

Since $|r_j - Ns_j| < 1$, one can check that

$$\prod_{j=1}^{d-1}(t-r_j) - \prod_{j=1}^{d-1}(t-Ns_j) = a_0 t^{d-2} + a_1 t^{d-3} + \cdots + a_{d-2}$$

where $|a_k| \leq A_k N^k$ for some constant A_k depending only on d. It follows that

$$\Big|\sum_{i=0}^{d}(-1)^i n_i \sum_{t=-iN}^{-1}\prod_{j=1}^{d-1}(t-r_j) - \sum_{i=0}^{d}(-1)^i n_i \sum_{t=-iN}^{-1}\prod_{j=1}^{d-1}(t-Ns_j)\Big| \leq C_1 N^{d-1}\cdot\sum n_i \tag{3.6.6}$$

for some constant C_1 depending only on d.

By a similar analysis, we have

$$\Big|\prod_{j=1}^{d-1}(t-Ns_j) - \prod_{j=1}^{d-1}(\lfloor t\rfloor - Ns_j)\Big| \leq b_0|t|^{d-2} + b_1|t|^{d-3} + \cdots + b_{d-2}$$

where $|b_k| \leq B_k N^k$ for some constant B_k depending only on d. It follows that

$$\Big|\sum_{i=0}^{d}(-1)^i n_i \sum_{t=-iN}^{-1}\prod_{j=1}^{d-1}(t-Ns_j) - \sum_{i=0}^{d}(-1)^i n_i \int_{-iN}^{0}\prod_{j=1}^{d-1}(t-Ns_j)dt\Big| \leq C_2 N^{d-1}\cdot\sum n_i \tag{3.6.7}$$

for some constant C_2 depending only on d.

Finally, we compute

$$\begin{aligned}
\sum_{i=0}^{d}(-1)^i n_i \int_{-iN}^{0}\prod_{j=1}^{d-1}(t-Ns_j)dt &= N^d \sum_{i=0}^{d}(-1)^i n_i \int_{-i}^{0}\prod_{j=1}^{d-1}(x-s_j)dx \\
&= N^d \sum_{i=0}^{d}(-1)^i n_i \frac{\prod_{s=1}^{d}(x+s)}{d}\Big|_{x=-i}^{x=0} \\
&= N^d\cdot(d-1)!\cdot\sum_{i=1}^{d}(-1)^i n_i \\
&= -N^d n_0\cdot(d-1)!
\end{aligned}$$

where the second equality follows from our definition of s_j at the beginning of the proof and the last equality follows from the fact that $\sum_i (-1)^i n_i = 0$. Putting this together with (3.6.7), (3.6.6), and (3.6.5), we find that

$$\lambda_\infty\big(H_0(F_\bullet \otimes^{\mathbb{L}}_R S)\big) \geq \lambda_\infty(S/I)\cdot\big(N^d n_0 - (C_1+C_2)N^{d-1}\cdot\sum_i n_i\big).$$

Let $C = C_1 + C_2$ (which is a constant depending only on d) and recall that under our notation (see Setting 3.1 and Notation 1.3), $S = R^{A_\infty}_{\mathrm{perfd}}$ and $I = (z_1, \dots, z_d)$. In particular, $\lambda_\infty(S/I) = e(z_1, \dots, z_d; R)$ by [CLM+26, Proposition 4.0.14]. Thus rewriting the above we obtain that

$$\lambda_\infty\big(H_0(F_\bullet \otimes^{\mathbb{L}}_R R^{A_\infty}_{\mathrm{perfd}})\big) \geq e(z_1, \dots, z_d; R) \cdot N^d \left(n_0 - \frac{C \sum_i n_i}{N}\right).$$

This completes the proof. □

Appendix A. Galois descent for rational K-theory with support

The goal is to explain the following theorem. Note that in [Ope26b, Section 3], a stronger result was claimed. We do not need the result in that level of generality, and the approach here is also different.

Theorem A.1. *Let $(A, \mathfrak{m}_A)$ be a Noetherian complete normal local domain of residue characteristic p with fraction field K. Let L/K be a finite Galois extension with Galois group G and let $(R, \mathfrak{m})$ be the integral closure of A in L. Then pullback induces an isomorphism*

$$\mathrm{K}_i^{\mathfrak{m}_A}(A)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{K}_i^{\mathfrak{m}}(R)_{\mathbb{Q}}^G \tag{A.1.1}$$

for all i.

The proof of Theorem A.1 relies on two facts: the first is a comparison between rational K-theory and rational homotopy K-theory, and the second is a descent result for the latter.

To minimize the use of K-theory, we first recall that rational K-theory admits a functorial cochain-complex model $\mathrm{K}_{\mathbb{Q}}(X)$ over $\mathbb{Q}$, with $H^{-i}(\mathrm{K}_{\mathbb{Q}}(X)) = \mathrm{K}_i(X)_{\mathbb{Q}}$ for all $i \in \mathbb{Z}$; this fact is well-known to experts, for example see [Pri17, Definition 1.18 and the discussion that follows] (see also [Pri16, Definitions 1.16 and 1.17]). For $Z \subseteq X$ closed and $U = X \setminus Z$, localization identifies

$$\mathrm{K}_{\mathbb{Q}}^Z(X) := \mathrm{cone}(\mathrm{K}_{\mathbb{Q}}(X) \to \mathrm{K}_{\mathbb{Q}}(U))[-1]$$

so that $H^{-i}(\mathrm{K}_{\mathbb{Q}}^Z(X)) = \mathrm{K}_i^Z(X)_{\mathbb{Q}}$, and we have an exact sequence of $\mathbb{Q}$-vector spaces:

$$\cdots \longrightarrow \mathrm{K}_{i+1}(U)_{\mathbb{Q}} \longrightarrow \mathrm{K}_i^Z(X)_{\mathbb{Q}} \longrightarrow \mathrm{K}_i(X)_{\mathbb{Q}} \longrightarrow \mathrm{K}_i(U)_{\mathbb{Q}} \longrightarrow \cdots .$$

Note that over a Noetherian local ring $(R, \mathfrak{m})$, $\mathrm{K}_0^{\mathfrak{m}}(R)_{\mathbb{Q}} := \mathrm{K}_0^{V(\mathfrak{m})}(\mathrm{Spec}(R))_{\mathbb{Q}}$ is precisely the rationalized Grothendieck group of perfect complexes over R supported only at $\{\mathfrak{m}\}$, see [Wei13, V §7].

Next we briefly recall rational homotopy K-theory. Set

$$\Delta^r = \mathrm{Spec}(\mathbb{Z}[t_0, \ldots, t_r]/(\sum t_j - 1)) \cong \mathbb{A}^r_{\mathbb{Z}}.$$

The face maps from $\mathrm{K}_{\mathbb{Q}}(X \times_{\mathbb{Z}} \Delta^r)$ to $\mathrm{K}_{\mathbb{Q}}(X \times_{\mathbb{Z}} \Delta^{r-1})$ set one coordinate equal to zero. The rational homotopy K-theory is $\mathrm{KH}_{\mathbb{Q}}(X) := \mathrm{colim}_{\Delta^{op}} \mathrm{K}_{\mathbb{Q}}(X \times_{\mathbb{Z}} \Delta^\bullet)$. More concretely,

$$\mathrm{KH}_{\mathbb{Q}}(X) = \mathrm{Tot}\big(\mathrm{K}_{\mathbb{Q}}(X \times_{\mathbb{Z}} \Delta^\bullet)\big), \qquad H^{-i}(\mathrm{KH}_{\mathbb{Q}}(X)) = \mathrm{KH}_i(X)_{\mathbb{Q}}.$$

Here the simplicial complex has r-th term $\mathrm{K}_{\mathbb{Q}}(X \times_{\mathbb{Z}} \Delta^r)$ and Tot denotes direct sum totalization, with the internal and alternating face differentials (simplicial degree r has cohomological degree $-r$), see [Wei13, Definitions IV 12.1 and IV 12.7]. Examining the $r = 0$ term gives the comparison $\mathrm{K}_{\mathbb{Q}}(X) \to \mathrm{KH}_{\mathbb{Q}}(X)$. We define $\mathrm{KH}_{\mathbb{Q}}^Z(X)$ by the same shifted cone as above. Since totalization commutes with taking cones,

$$\mathrm{KH}_{\mathbb{Q}}^Z(X) \cong \mathrm{Tot}\left(\mathrm{K}_{\mathbb{Q}}^{Z \times_{\mathbb{Z}} \Delta^\bullet}(X \times_{\mathbb{Z}} \Delta^\bullet)\right) \tag{A.1.2}$$

Now let X be a separated Noetherian scheme of finite dimension. Suppose either X has characteristic p or p is a nonzerodivisor on $\mathcal{O}_X$. Let $Z \subseteq V(p)$ be a closed subscheme of X. We claim that

$$\mathrm{K}_i^Z(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{KH}_i^Z(X)_{\mathbb{Q}} \tag{A.1.3}$$

for all i. To see this, we need to use Bass' NK-groups. For any ring B, we write $\mathrm{NK}_i(B) := \ker\big(\mathrm{K}_i(B[t]) \xrightarrow{t=0} \mathrm{K}_i(B)\big)$. Note that the evaluation map $B[t] \xrightarrow{t \mapsto 0} B$ is split (by the natural

inclusion $B \longrightarrow B[t]$). If B has characteristic p, then we have $\mathrm{NK}_i(B)[1/p] = 0$ by [Wei13, Corollary IV 6.7.2]. If B is p-torsion free, then by [Wei81, Corollary 6.4 on page 489] (applied with base ring $\mathbb{Z}$ and multiplicative set $\{1, p, p^2, \ldots\}$), we have

$$\mathrm{NK}_i(B) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \;\cong\; \mathrm{NK}_i(B[1/p]). \tag{A.1.4}$$

In particular, in either case $\mathrm{NK}_i(B)_{\mathbb{Q}} \cong \mathrm{NK}_i(B[1/p])_{\mathbb{Q}}$ for all i. The definition of NK_i and (A.1.4) extend to Noetherian schemes by using finite affine covers and the Mayer–Vietoris sequence [Wei13, Chapter V Corollary 7.10]. Now to prove (A.1.3), note that since $Z \subseteq V(p)$, (A.1.4) implies $\mathrm{NK}_i(X)_{\mathbb{Q}} \cong \mathrm{NK}_i(U)_{\mathbb{Q}}$ where $U = X \setminus Z$. Consider the following commutative diagram of $\mathbb{Q}$-vector spaces:

$$\begin{array}{ccccccccc}
 & & & & & & \mathrm{NK}_i(X)_{\mathbb{Q}} & \xrightarrow{\cong} & \mathrm{NK}_i(U)_{\mathbb{Q}} \\
 & & & & & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathrm{K}_{i+1}(U \times \Delta^1)_{\mathbb{Q}} & \longrightarrow & \mathrm{K}_i^{Z\times\Delta^1}(X \times \Delta^1)_{\mathbb{Q}} & \longrightarrow & \mathrm{K}_i(X \times \Delta^1)_{\mathbb{Q}} & \longrightarrow & \mathrm{K}_i(U \times \Delta^1)_{\mathbb{Q}} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathrm{K}_{i+1}(U)_{\mathbb{Q}} & \longrightarrow & \mathrm{K}_i^{Z}(X)_{\mathbb{Q}} & \longrightarrow & \mathrm{K}_i(X)_{\mathbb{Q}} & \longrightarrow & \mathrm{K}_i(U)_{\mathbb{Q}}
\end{array}$$

Chasing this diagram (noting that the last two columns are exact sequences), we find that $\mathrm{K}_i^{Z\times\Delta^1}(X \times \Delta^1)_{\mathbb{Q}} \cong \mathrm{K}_i^Z(X)_{\mathbb{Q}}$. Repeating the argument after adjoining variables shows that $\mathrm{K}_{\mathbb{Q}}^{Z\times\Delta^r}(X \times \Delta^r) \cong \mathrm{K}_{\mathbb{Q}}^Z(X)$. Now (A.1.3) follows from (A.1.2): the constant simplicial complex maps by a quasi-isomorphism to every term, and totalization preserves these maps.

Now let X be an excellent, normal, integral, separated Noetherian scheme of finite dimension. Let $f : Y \longrightarrow X$ be its normalization in a finite Galois extension of the function field of X, with Galois group G. By [Kha22, Theorem 5.3 and Remark 5.5], the presheaf $\mathrm{KH}_{\mathbb{Q}}(-)$ satisfies h-hyperdescent and hence qfh-hyperdescent[6] (see [CD19, 3.3.c]: the h-topology is finer than the qfh-topology). Now by the first paragraph of the proof of [CD19, Corollary 3.3.39] (applied to $C = \mathrm{KH}_{\mathbb{Q}}(-)$, $X' = X$ and $Y' = Y$), we have

$$\mathrm{KH}_{\mathbb{Q}}(X) \cong \mathbf{R}\Gamma_{\mathrm{qfh}}(X, \mathrm{KH}_{\mathbb{Q}}) \cong \mathbf{R}\Gamma_{\mathrm{qfh}}(Y, \mathrm{KH}_{\mathbb{Q}})^G \cong \mathrm{KH}_{\mathbb{Q}}(Y)^G. \tag{A.1.5}$$

Since taking G-invariants is exact over $\mathbb{Q}$, taking cohomology in (A.1.5) yields

$$\mathrm{KH}_i(X)_{\mathbb{Q}} \;\xrightarrow{\sim}\; \mathrm{KH}_i(Y)_{\mathbb{Q}}^G. \tag{A.1.6}$$

For a closed subscheme $Z \subseteq X$, let $W = f^{-1}(Z)$ and apply (A.1.6) to $X \setminus Z$ and $Y \setminus W$. The localization sequences for $\mathrm{KH}_{\mathbb{Q}}$ and the exactness of taking G-invariants over $\mathbb{Q}$ then give

$$\mathrm{KH}_i^Z(X)_{\mathbb{Q}} \;\xrightarrow{\sim}\; \mathrm{KH}_i^W(Y)_{\mathbb{Q}}^G. \tag{A.1.7}$$

for all i.

Proof of Theorem A.1. Consider f: $Y = \mathrm{Spec}(R) \longrightarrow X = \mathrm{Spec}(A)$, with $Z = \{\mathfrak{m}_A\}$ and $W = f^{-1}(Z) = \{\mathfrak{m}\}$. The theorem follows from (A.1.7) and (A.1.3). □

[6] [Kha22] uses Čech descent. To upgrade to hyperdescent, we work on the qfh-site of schemes quasi-finite and of finite type over X. This is a finitary site whose objects have dimension at most $\dim(X)$. Its rational cohomological dimension is at most $\dim(X)$ by [Par19, Theorem 6.21] and thus Čech descent for complexes of $\mathbb{Q}$-vector spaces implies hyperdescent by [CM21, Proposition 2.31].

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