RESEARCH STATEMENT

LINQUAN MA

My research is primarily focused on studying singularities arising in commutative algebra and algebraic geometry. I specialized in using positive characteristic methods, local cohomology, multiplicity, and techniques from the theory of perfectoid spaces to study singularities. Recently I have made some important progress.

Multiplicities over local rings and Lech’s conjecture. One way to study singularities is to assign numerical invariants that allows us to compare them with one another. For instance, the degree of the smallest order term in the equation of a plane curve is an example of multiplicity: a smooth curve such as the parabola $y = x^2$ has multiplicity one at the origin, while the cusp $y^2 = x^3$ has multiplicity two at the origin. More generally, let $(R, m)$ be a Noetherian local ring (e.g., the ring of germs of functions at a point of an algebraic variety). Then the Hilbert-Samuel multiplicity of $R$ can be defined as:

$$e(R) = \lim_{t \to \infty} d! \cdot \frac{l(R/m^t)}{t^d}$$

where $d = \dim R$. This is a classical invariant that measures the singularity of the ring $R$. Morally speaking, the larger the multiplicity, the worse the singularity. Around 1960, Lech made a remarkable conjecture in [Lec60]:

Conjecture (Lech’s conjecture). Let $(R, m) \to (S, n)$ be a flat local extension of Noetherian local rings. Then $e(R) \leq e(S)$.

This conjecture has now stood for sixty years and remains open in most cases (despite it is simple to state and natural to expect). The best previous result was the case when $\dim R \leq 2$ [Lec60]. Other partial results were obtained in [Lec64, HUB91, Her94, Han99, Han05]. It is one of my long term goals to attack this conjecture. In 2017 I resolved the dimension three equal characteristic case, and obtained partial estimates in all higher dimensions.

Theorem A ([Ma17]). Let $(R, m) \to (S, n)$ be a flat local extension of Noetherian local rings of equal characteristic. If $\dim R = d$, then we have

$$e(R) \leq \max\{1, \frac{d!}{2^d}\} \cdot e(S).$$

In particular, if $\dim R = 3$, then $e(R) \leq e(S)$.

Very recently, I have made substantial progress, proving Lech’s conjecture in all dimensions provided the base ring admits a standard grading over a perfect field.

Theorem B ([Ma20]). Let $(R, m) \to (S, n)$ be a flat local extension of Noetherian local rings. Suppose $(R, m)$ is a standard graded\footnote{That is, $\mathbb{N}$-graded and generated by degree one forms.} ring over a perfect field (localized at the homogeneous maximal ideal). Then $e(R) \leq e(S)$. 
The main new idea in the proof, which I hope might be useful to attack Lech’s conjecture in general in future, is a notion called lim Ulrich sequence. This is inspired from my work in progress with Bhatt and Hochster [BHM]: we are developing a theory of lim Cohen–Macaulay sequence to attack Serre’s conjecture on intersection multiplicities. The lim Ulrich sequence is a special type of lim Cohen–Macaulay sequence, and roughly speaking, it is a sequence of finitely generated modules that are not necessarily Cohen–Macaulay, but asymptotically behave like Ulrich modules. In [Ma20] I proved that the existence of a weak form of such sequence implies Lech’s conjecture, and I used positive characteristic methods along with other techniques to construct these weak lim Ulrich sequences for standard graded rings.

I have made some other important contributions to the theory of multiplicity. In [KMQ+19], we proved the existence of a uniform upper bound on the ratio between colength and multiplicity of m-primary ideals, settling a conjecture of Stückrad–Vogel from 1996 [SV96].

**Theorem C** ([KMQ+19]). Let (R, m) be a Noetherian local ring and let

\[ n(R) = \sup_{\sqrt{I} = m} \left\{ \frac{l(R/I)}{e(I, R)} \right\}. \]

Then \( n(R) < \infty \) if and only if the \( m \)-adic completion of \( R \) is equidimensional.

**Singularities in mixed characteristic via perfectoid algebras.** In characteristic zero, singularities are usually studied via resolutions accompanied with the vanishing theorems. In positive characteristic, the Frobenius or the \( p \)-th power map plays a crucial role. Quite surprisingly, singularities arising in zero and positive characteristics have deep connections. This was pioneered by the work of Smith [Sm97], who proved that a variety defined over \( \mathbb{C} \) has rational singularities if its mod \( p \) reductions are \( F \)-rational for almost all primes \( p > 0 \).

However, most of these techniques do not translate well to the mixed characteristic setting (e.g., for varieties defined over the integers or the \( p \)-adic numbers), and many fundamental questions have remained open in this case.

In my joint works with Schwede and others [MS18a, MS18b, MST+19], we have begun to build singularity theory in mixed characteristic using perfectoid algebras and spaces. We introduced perfectoid multiplier/test ideals, and produced mixed characteristic analogs of singularities arising from the minimal model program and tight closure theory. Our works are largely inspired from recent breakthroughs on the direct summand conjecture and the existence of big Cohen–Macaulay algebras by André [And18, And20], and further refined by Bhatt and Gabber [Bha18, Gab18]. Roughly speaking, our strategy was to use perfectoid big Cohen–Macaulay algebras to replace resolution of singularities in characteristic zero (or the Frobenius map in positive characteristic), and desired properties about singularities translate to questions about perfectoid algebras, where we can apply various tools from [Sch12, And20].

For instance, following these ideas, we were able to generalize the aforementioned result of Smith [Sm97] and Hara–Watanabe [HW02] by proving that a variety defined over \( \mathbb{C} \) has rational/log terminal singularities provided its mod \( p \) reduction is \( F \)-rational/strongly \( F \)-regular for one single prime.

**Theorem D** ([MS18a, MST+19]). Let \( (R, m) \) be a local ring essentially of finite type over \( \mathbb{C} \). Then \( R \) has rational (resp. log terminal) singularities if its mod \( p \) reduction \( (R_p, m_p) \) has \( F \)-rational (resp. strongly \( F \)-regular) singularities for one prime \( p > 0 \).

\[^2\text{In fact, the converse is also true by results of Hara and Mehta–Srinivas [Har98, MS97].}\]
Among other things, this result has an interesting algorithmic consequence for checking rational and log terminal singularities using a computer [MS18b].

Our investigation on the perfectoid multiplier/test ideal leads to many interesting applications in commutative algebra. Perhaps the most important one is the following, which we extend the celebrated result of Ein–Lazarsfeld–Smith [ELS01] and Hochster–Huneke [HH02] on the uniform growth of symbolic powers in equal characteristic to mixed characteristic (hence the result holds in all characteristic).

**Theorem E** ([MS18a]). Suppose $R$ is an excellent regular ring and $Q \subseteq R$ is a prime ideal of height $h$. Then for all $n > 0$ we have $Q^{(nh)} \subseteq Q^n$, where $Q^{(nh)}$ denotes the $nh$ symbolic power of $Q$, i.e., elements of $R$ which vanish generically to order $nh$ at $Q$.

Another application is a uniform version of the Briançon–Skoda theorem, which, to the best of our knowledge, was not known before in mixed characteristic.

**Theorem F** ([MST+19]). Let $(R, \mathfrak{m})$ be a Noetherian complete normal local domain of dimension $d$. Let $J$ be the defining ideal of the singular locus of $R$. Then there exists an integer $N$ such that $J^N \bar{I}^d \subseteq I$ for all $I \subseteq R$, where $\bar{I}^d$ denotes the integral closure of $I^d$.

The keys behind all these results are certain comparisons between our perfectoid versions of singularities and multiplier/test ideals with their existing equal characteristic counterparts. In our latest work [MST+19] in this series, we have obtained quite general adjunction and inversion of adjunction type results. Some ideas in this approach originate from my earlier work [Ma18].

Over the next several years, I plan to apply perfectoid methods to study birational geometry in mixed characteristic. The results and techniques in [MS18b MST+19] are just the start of such investigation. In fact, with Schwede and others, we think we can prove the existence of certain graded big Cohen–Macaulay algebras, which will allow us to globalize our definitions by passing to affine cones. We hope to develop relative versions of our singularity theory that might have applications to the minimal model program in mixed characteristic, at least in low dimensions. For example, one of our goals in this direction is to establish the existence of flips for threefolds in mixed characteristic using the singularity theory we built and the framework of Hacon–Xu [HX15].

It has become apparent to me that perfectoid algebras and spaces are very powerful tools to study singularities in mixed characteristic. Besides the work discussed above, I have also proved a $p$-adic version of Kunz’s theorem [Kun69] characterizing regular local rings, and an analog of Boutot’s theorem [Bou87] that direct summand of regular rings are pseudo-rational, in mixed characteristic.

**Theorem G** ([BIM19]). Let $(R, \mathfrak{m})$ be a Noetherian local ring of mixed characteristic. Then $R$ is regular if and only there exists a perfectoid algebra $R_\infty$ faithfully flat over $R$.

**Theorem H** ([HM18]). Suppose $R$ is a direct summand of a regular ring $S$ (i.e., $R \to S$ is split). Then $R$ is pseudo–rational (in particular, $R$ is Cohen–Macaulay).
References


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Department of Mathematics, Purdue University, West Lafayette, IN 47907

Email address: ma326@purdue.edu