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A Note on Asymptotic Homomorphisms

MARIUS DADARLAT
Department of Mathematics, Purdue University, West Lafayette, IN 47907, U.S.A.
e-mail: mdd@math.purdue.edu

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Abstract. Using the notion of asymptotic homomorphism due to Connes and Higson we construct bivariant homology-cohomology theories for separable C^*-algebras, which satisfy general excision axioms and are nonperiodic.

Key words: Asymptotic homomorphisms, C^*-algebras, homotopy, homology, excision.

1. Introduction

The aim of this note is to introduce some *nonperiodic* homology and cohomology theories satisfying the excision property and defined on the category of separable C^*algebras. These theories are relevant even for commutative C^*-algebras in which case they correspond to flexible extensions of stable homotopy theory. The (co)cycles of these theories are based on asymptotic homomorphisms, a notion due to Connes and Higson [2]. All the excision (co)homology theories that have been previously known were derived from KK-theory or E-theory and therefore were periodic. E-theory is due to Connes and Higson [2]. Its abstract existence was proven earlier by Higson [6]. The E-theory group $E(A, B)$ is defined in terms of homotopy classes of asymptotic homomorphisms from $SA \otimes \mathcal{K}$ to $SB \otimes \mathcal{K}$. E-theory coincides with Kasparov's KK-theory on K-nuclear C^*-algebras. However in general they are distinct as shown by Skandalis [12].

Consider the following exact sequence of separable C^*-algebras:

$$0 \to J \to A \xrightarrow{p} A/J \to 0$$

A central result in E-theory is the homotopy equivalence via asymptotic homomorphisms between $SJ \otimes \mathcal{K}$ and $SC_p \otimes \mathcal{K}$. Here C_p is the mapping cone of p, $\mathcal{K}$ is the C^*-algebra of compact operators and S stands for the suspension functor. In this note we refine this result by removing the compact operators i.e. we show that SJ is homotopy equivalent to SC_p via asymptotic homomorphisms. Then, in analogy with [10], we provide Puppe sequences for the homotopy classes of asymptotic homomorphisms. By putting these two facts together we get a bivariant functor from separable C^*-algebras to abelian groups which is exact in each variable. This functor is a flexible extension of stable homotopy to separable C^*-algebras. Moreover, its restriction to stable C^*-algebras coincides with E-theory.

The commutative case is also of special interest because the homotopy theory based on asymptotic homomorphisms gives rise to a nice extension of classical homotopy theory to singular spaces. For instance, the Warshaw circle is homotopy equivalent to the standard circle within this theory, whereas there are no interesting continuous maps from the standard circle to the Warshaw circle. While very flexible the homotopy theory based on asymptotic homomorphisms is stronger than shape theory which is the classical substitute for the homotopy theory of singular spaces. In particular an asymptotic homomorphism between compact metrizable spaces induces a map on Čech cohomology or more generally on any continuous cohomology theory.

After this paper was completed the author learned that independently A. Connes and N. Higson proved the unstable excision $SC_p \sim SJ$ and developed the general theory that follows from this [13], [14]. Following the ramification of these results in the commutative case A. Connes and J. Kaminker showed how to extend a generalized homology theory defined by a spectrum to a generalized Steenrod homology theory, by using a homotopy theory based on asymptotic homomorphisms [13], [15]. This gives a short proof for a theorem of Kahn, Kaminker and Schochet whose original approach was based on duality [17].

In a forthcoming paper [16], we prove that the homotopy theory based on asymptotic homomorphisms is equivalent to a strong shape theory. In particular, its restriction to commutative C^*-algebras is equivalent to the approaching homotopy category of Quigley [18].

2. Asymptotic Homomorphisms

In this section we shall review the construction of the category of separable C^*-algebras and homotopy classes of asymptotic homomorphisms due to Connes and Higson [2].

DEFINITION 1. Let A, B be two C^*-algebras. An asymptotic homomorphism from A to B is a family of maps $\varphi_t\colon A \to B$, indexed by $t \in T = [1, +\infty)$, subject to the following conditions:

(1) For all $a \in A$ the map $t \to \varphi_t(a)$ from T to B is continuous.
(2) For all a, $b \in A$ and $\lambda \in \mathbb{C}$ one has

$$\lim_{t\to\infty} \|\varphi_t(a+b) - \varphi_t(a) - \varphi_t(b)\| = 0,$$
$$\lim_{t\to\infty} \|\varphi_t(\lambda a) - \lambda\varphi_t(a)\| = 0,$$
$$\lim_{t\to\infty} \|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| = 0,$$
$$\lim_{t\to\infty} \|\varphi_t(a^*) - \varphi_t(a)^*\| = 0.$$

The standard notation for an asymptotic homomorphism from A to B will be $(\varphi_t)\colon A \to B$.

LEMMA 2. ([2]) *For any* $a \in A$, $\limsup_t \|\varphi_t(a)\| \leqslant \|a\|$.

The boundedness property of asymptotic homomorphisms given in Lemma 2 is reminiscent of automatic contractivity of $*$-homomorphisms. Actually the proof that shows that for any $*$-homomorphism $\pi \in \mathrm{Hom}(A,\ B)$ one has $\|\pi(a)\| \leqslant \|a\|$ for all $a \in A$, [9], can be easily modified to derive the statement of Lemma 2. One uses the semicontinuity of the spectrum of $\varphi_t(a)$ with respect t.

Denote by $C_b(T,\ B)$ the C^*-algebra of all continuous bounded functions from T to B. Let $C_0(T,\ B)$ be the closed ideal of $C_b(T,\ B)$ consisting of functions vanishing at infinity. The quotient C^*-algebra $C_b(T,\ B)/C_0(T,\ B)$ is denoted by B_∞. Then one has the following exact sequence

$$0 \to C_0(T,\ B) \to C_b(T,\ B) \xrightarrow{q} B_\infty \to 0.$$

Lemma 2 enables us to identify any asymptotic homomorphism $(\varphi_t)\colon A \to B$ with a map $\varphi\colon A \to C_b(T,\ B)$ given by $\varphi(a)(t) = \varphi_t(a)$ for all $a \in A$ and $t \in T$. It is clear that φ satisfies the following conditions:

For all $a,\ b \in A$ and $\lambda \in \mathbb{C}$, one has

$$\begin{aligned}
&\varphi(a+b) - \varphi(a) - \varphi(b) \in C_0(T,\ B),\\
&\varphi(\lambda a) - \lambda\varphi(a) \in C_0(T,\ B),\\
&\varphi(ab) - \varphi(a)\varphi(b) \in C_0(T,\ B),\\
&\varphi(a^*) - \varphi(a)^* \in C_0(T,\ B).
\end{aligned}$$

In the sequel, we are going to use freely both notation $(\varphi_t)\colon A \to B$ and $\varphi\colon A \to C_b(T,\ B)$ for asymptotic homomorphisms. With any asymptotic homomorphism $(\varphi_t)\colon A \to B$ one associates a $*$-homomorphism $\hat\varphi \in \mathrm{Hom}(A,\ B_\infty)$ given by $\hat\varphi(a) = q(\varphi(a))$, where q is the quotient map of $C_b(T,\ B)$ onto B_∞.

Two asymptotic homomorphisms $(\varphi_t),\ (\psi_t)\colon A \to B$ are *equivalent*, written, $(\varphi_t) \cong (\psi_t)$ if for any $a \in A$ one has $\lim_{t\to\infty}\|\varphi_t(a) - \psi_t(a)\| = 0$. Equivalently, $\varphi \cong \psi$ iff $\varphi(a) - \psi(a) \in C_0(T,\ B)$ for all $a \in A$. The correspondence $\varphi \mapsto \hat\varphi$ induces a bijection between the equivalence classes of asymptotic homomorphisms from A to B and $\mathrm{Hom}(A,\ B_\infty)$. Any map $\varphi\colon A \to C_b(T,\ B)$ that lifts a given $\hat\varphi \in \mathrm{Hom}(A,\ B_\infty)$ is automatically an asymptotic homomorphism. Any two liftings are equivalent.

Remarks. (1) Using the selection theorem of Bartle and Graves one can find a continuous (in general nonadditive, but homogeneous) cross-section for the quotient map q. This shows that any asymptotic homomorphism is equivalent to an asymptotic homomorphism given by a continuous map $\varphi\colon A \to C_b(T,\ B)$.

(2) Suppose that the C^*-algebra A is separable and nuclear. Then, by Choi-Effros theorem [1], any homomorphism $\hat\varphi\colon A \to B_\infty$ has a linear completely positive lifting $\varphi\colon A \to C_b(T,\ B)$. One concludes that any asymptotic homomorphism from A to B is equivalent to a linear completely positive map $\varphi\colon A \to C_b(T,\ B)$ satisfying $\varphi(ab) - \varphi(a)\varphi(b) \in C_0(T,\ B)$.

For a C^*-algebra B denote by $B[0, 1]$ the C^*-algebra of continuous functions from the unit interval to B. For $s \in [0, 1]$ let $e_s: B[0, 1] \to B$ denote the evaluation map at s. We still use e_s to denote the map $C_b(T, B[0, 1]) \to C_b(T, B)$ induced by e_s. Suppose that $\Phi: A \to C_b(T, B[0, 1])$ is an asymptotic homomorphism. Then the composition $e_s\Phi: A \to C_b(T, B)$ is an asymptotic homomorphism.

DEFINITION 3. Two asymptotic homomorphisms (φ_t), $(\psi_t): A \to B$ are said to be homotopy equivalent, written $(\varphi_t) \sim (\psi_t)$, if there is an asymptotic homomorphism $(\Phi_t): A \to B[0, 1]$ such that the restrictions of (Φ_t) at 0 and 1 are equal with (φ_t) and (ψ_t), respectively. That is $e_0\Phi = \varphi$ and $e_1\Phi = \psi$.

Note that equivalent asymptotic homomorphisms are homotopy equivalent. Indeed if $\varphi_t \cong \psi_t$ then $\Phi_t(a)(s) = (1 - s)\varphi_t(a) + s\psi_t(a)$ is a homotopy between the two asymptotic homomorphisms. Since $B_\infty[0, 1]$ is not the same with $B[0, 1]_\infty$, a homotopy equivalence $\varphi \sim \psi$ does not correspond to a homotopy of $*$-homomorphisms, between $\hat{\varphi}$ and $\hat{\psi}$. However, if $\hat{\varphi}$ is homotopy equivalent to $\hat{\psi}$, then $\varphi \sim \psi$.

The homotopy classes of asymptotic homomorphisms from A to B are denoted by $[[A, B]]$. The homotopy class of an asymptotic homomorphism $(\varphi_t): A \to B$ is denoted by $[[\varphi_t]]$ or $[[\varphi]]$. The notation for the homotopy classes of $*$-homomorphisms from A to B is $[A, B]$. From now on, we restrict our considerations to separable C^*-algebras. There is a composition law of homotopy classes of asymptotic homomorphisms. We are going to describe briefly this composition. One starts with arbitrary asymptotic homomorphisms $(\varphi_t): A \to B$ and $(\psi_t): B \to C$. As noticed above, after replacing these asymptotic homomorphisms by equivalent asymptotic homomorphisms if necessary, one may assume that the maps $\varphi: A \to C_b(T, B)$ and $\psi: B \to C_b(T, C)$ are continuous. Let A' be a dense $*$-subalgebra of A which is a countable union of compacts. It is proven in [2] that there is an increasing continuous function $r: T \to T$ such that for any increasing continuous function $s: T \to T$ with $s(t) \geqslant r(t)$, the composition $\theta'_t = \psi_{s(t)} \circ \varphi_t$ is an asymptotic homomorphism from A' to C. Moreover, $\lim\sup\|\theta'_t(a)\| \leqslant \|a\|$ for all $a \in A'$. Thus, the associated map $\theta': A' \to C_b(T, C)$ is well defined and $\hat{\theta}': A' \to C_\infty$ is a bounded $*$-homomorphism that extends to a $*$-homomorphism $\hat{\theta}$ on A. Let (θ_t) be any lifting of $\hat{\theta}$. By definition $[[\psi_t]] \circ [[\varphi_t]] = [[\theta_t]]$.

PROPOSITION 4. ([2]) *Any extension (θ_t) of $(\psi_{s(t)} \circ \varphi_t)$ is an asymptotic homomorphism. The homotopy class of $[[\theta_t]]$ in $[[A, C]]$ depends only on the homotopy classes of $[[\varphi_t]] \in [[A, B]]$ and $[[\psi_t]] \in [[B, C]]$. Moreover the composition of homotopy classes $[[\psi_t]] \circ [[\varphi_t]]$ is associative.*

Proposition 4 shows that there is a well defined associative composition law

$$[[A, B]] \times [[B, C]] \to [[A, C]].$$

It is natural then to consider the category denoted here by $\mathcal{A}$, *whose objects are all the separable C^*-algebras and whose morphisms are homotopy classes of* *asymptotic homomorphisms.* It turns out that this category is interesting even for commutative C^*-algebras as a flexible substitute for the homotopy category of singular spaces (see Section 6).

There is a natural notion of tensor product for asymptotic homomorphisms that is defined up to an equivalence. Suppose that $(\varphi_t)\colon A \to C$ and $(\psi_t)\colon B \to C$ are asymptotic homomorphisms such that the commutator $[\varphi_t(a),\ \psi_t(b)]$ converges in norm to 0 for all $a \in A$ and $b \in B$. Then there exists an asymptotic homomorphism $(\theta_t)\colon A \otimes_{\max} B \to C$, uniquely defined up to an equivalence by the condition $\lim_t \|\theta_t(a \otimes b) - \varphi_t(a)\psi_t(b)\| = 0$. Indeed $\hat{\varphi}(a)$ commutes with $\hat{\psi}(b)$ in C_∞ and, therefore, there is a unique $*$-homomorphism $\hat{\theta}\colon A \otimes_{\max} B \to C_\infty$ such that $\hat{\theta}(ab) = \hat{\varphi}(a)\hat{\psi}(b)$ for all $a \in A$ and $b \in B$. One takes (θ_t) to be any lifting of $\hat{\theta}$. In particular for given asymptotic homomorphisms $(\varphi_t)\colon A \to C$ and $\psi_t\colon B \to D$ one gets an asymptotic homomorphism $(\varphi_t \otimes \psi_t)\colon A \otimes_{\max} B \to C \otimes_{\max} D$, defined up to an equivalence. This construction allows us to construct the suspension functor S. S takes a C^*-algebra A to its suspension $SA = C_0(0,\ 1) \otimes A$, and takes an asymptotic homomorphism $(\varphi_t)\colon A \to B$ to $(\mathrm{id}_{C_0(0,1)} \otimes \varphi_t)\colon SA \to SB$.

3. Puppe Sequences

In this section we introduce the additive category of separable C^*-algebras and stable homotopy classes of asymptotic homomorphisms denoted by $\mathcal{A}^s$. This category is a very flexible generalization of the stable homotopy category of polyhedra.

The *objects* of $\mathcal{A}^s$ are separable C^*-algebras. The *set of morphisms* from A to B is by definition

$$\{\{A,\ B\}\} = \lim_k [[S^k A,\ S^k B]],$$

where $[[S^k A,\ S^k B]]$ maps to $[[S^{k+1} A,\ S^{k+1} B]]$ via the suspension functor. As in the case of homomorphisms, one has a group structure on $[[A,\ SB]]$ induced by the loop composition. To be specific, given the asymptotic homomorphisms $(\varphi_t),\ (\psi_t)\colon A \to SB$ we define $[[\varphi_t]]\ [[\psi_t]$ to be the homotopy class of the asymptotic homomorphism $(\theta_t)\colon A \to SB$ given by

$$\theta_t(a)(s) = \begin{cases} \varphi_t(a)(2s), & \text{if } 0 \leqslant s \leqslant 1/2, \\ \psi_t(a)(2s-1), & \text{if } 1/2 \leqslant s \leqslant 1, \end{cases}$$

where s is the suspension parameter. The class of the null-homotopic asymptotic homomorphisms acts as the identity element. Like for ordinary homotopy the group $[[A,\ S^2 B]]$ is Abelian. It is convenient to introduce the higher-order groups

$$\{\{A,\ B\}\}_q = \lim_k \{\{s^{k+q} A,\ S^k B\}\},$$

where $q \in \mathbb{Z}$. It follows that $\{\{A,\ B\}\}_q$ is an Abelian group.

We shall write $B[0, 1]$ for $C[0, 1] \otimes B$ and $CB = B[0, 1)$ for $C_0[0, 1) \otimes B$. Let $p: A \to B$ be a $*$-homomorphism. The mapping cone of p is the C^*-algebra $C_p = \{(f, a) \in B[0, 1) \oplus A: f(0) = p(a)\}$. There is a short exact sequence of C^*-algebras

$$0 \to SB \xrightarrow{\delta} C_p \xrightarrow{k} A \to 0,$$

where $\delta(f) = (f, 0)$ for $f \in SB$ and $k(f, a) = a$ for $(f, a) \in C_p$.

Let J denote the kernel of p and let $i: J \to C_p$ denote the map $i(x) = (0, x)$.

Next we are going to exhibit Puppe exact sequences in both variables. Everything is quite similar to the corresponding exact sequences for homotopy classes of $*$-homomorphisms given in [10].

PROPOSITION 5. *Let A, B, C be C^*-algebras and $p: A \to B$ a $*$-homomorphism. Then the following sequence of pointed sets is exact in the middle.*

$$[[C, C_p]] \xrightarrow{k_*} [[C, A]] \xrightarrow{p_*} [[C, B]].$$

Proof. Let $(\varphi_t): C \to A$ such that $(p \circ \varphi_t): C \to B$ is null homotopic. Thus there is an asymptotic homomorphism $(h_t): C \to B[0, 1]$ such that $h_t(c)(0) = p(\varphi_t(c))$ and $h_t(c)(1) = 0$ for all $t \in T$ and $c \in C$. It is clear that (h_t) takes values in $B[0, 1)$. Define $(\psi_t): C \to C_p$ by $\psi_t(c) = (h_t(c), \varphi_t(c))$. It is easily seen that (ψ_t) is a well defined asymptotic homomorphism. To finish the proof we check that (ψ_t) is a lifting of (φ_t):

$$k(\psi_t(c)) = k(h_t(c), \varphi_t(c)) = \varphi_t(c).$$ □

PROPOSITION 6. *Let A, B, C be C^*-algebras and $p: A \to B$ a $*$-homomorphism. Then the following sequence of groups and pointed sets is exact.*

$$\cdots \to [[C, SA]] \xrightarrow{(Sp)_*} [[C, SB]] \xrightarrow{\delta_*} [[C, C_p]] \xrightarrow{k_*} [[C, A]] \xrightarrow{p_*} [[C, B]].$$

Proof. Using the proof of Theorem 3.8 in [10], we derive the following commutative diagram of C^*-algebras and $*$-homomorphisms

$$\begin{tikzcd}
& C_k \arrow[r] & C_p \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, "p"] & B \\
C_\delta \arrow[r] & SB \arrow[u] \arrow[r, "\delta"] \arrow[d, equal] & C_p & & \\
SA \arrow[u] \arrow[r, "Sp"] & SB & & &
\end{tikzcd}$$

with all the vertical arrows inducing homotopy equivalences. Passing to homotopy classes of asymptotic homomorphisms, we get a commutative diagram with all the vertical arrows isomorphisms.

$$\begin{tikzcd}
 & {[[C,\, C_k]]} \arrow[r] & {[[C,\, C_p]]} \arrow[r, "k_*"] & {[[C,\, A]]} \arrow[r, "p_*"] & {[[C,\, B]]} \\
{[[C,\, C_\delta]]} \arrow[r] & {[[C,\, SB]]} \arrow[u] \arrow[r, "\delta_*"] & {[[C,\, C_p]]} \arrow[u, equal] & & \\
{[[C,\, SA]]} \arrow[u] \arrow[r, "Sp_*"] & {[[C,\, SB]]} \arrow[u, equal] & & &
\end{tikzcd}$$

By Proposition 5 the first two upper rows are exact. This concludes the proof. □

THEOREM 7. *Let $A,\ B,\ C$ be C^*-algebras and $p\colon A \to B$ a $*$-homomorphism. Then there is a long exact sequences of Abelian groups*

$$\cdots \to \{\{C,\, B\}\}_{q-1} \xrightarrow{\delta_*} \{\{C,\, C_p\}\}_q \xrightarrow{k_*} \{\{C,\, A\}\}_q \xrightarrow{p_*} \{\{C,\, B\}\}_q \to$$
$$\to \{\{C,\, C_p\}\}_{q+1} \to \cdots$$

Proof. The exact sequence of Proposition 6 is natural with respect suspensions and inductive limits of exact sequences are exact sequences. Thus, Theorem 7 is a straightforward consequence of Proposition 6.

Let $A,\ B,\ D$ be C^*-algebras and $p\colon A \to B$ a $*$-homomorphism. Consider the following diagram

$$\begin{tikzcd}
 & {[[B,\, D]]} \arrow[r, "p^*"] \arrow[d, "\mathrm{Susp}"'] & {[[A,\, D]]} \\
{[[C_p,\, SD]]} \arrow[r, "\delta^*"] & {[[SB,\, SD]]} &
\end{tikzcd}$$

where the vertical map is induced by the suspension functor.

PROPOSITION 8. *The suspension map maps the kernel of p^* into the image of δ^*.*

Proof. Suppose that $(\varphi_t)\colon B \to D$ is an asymptotic homomorphism such that $(\varphi_t \circ p)\colon A \to D$ is null homotopic. Therefore there is an asymptotic homomorphism $(g_t)\colon A \to D[0,\, 1]$ such that $g_t(a)(1) = \varphi_t(p(a))$ and $g_t(a)(0) = 0$. Note that $\varphi_t(0) \in C_0(T, D)$ since φ_t is asymptotically additive. Thus by replacing φ_t by $\varphi_t(.) - \varphi_t(0)$ we may assume that $\varphi_t(0) = 0$. Similarly we may assume that $g_t(0) = 0$. A lifting of $(S\varphi_t)$ is produced as follows. Define $(\sigma_t)\colon C_p \to SD$ by

$$\sigma_t(f, a)(s) = \begin{cases} g_t(a)(2s), & \text{if } 0 \leqslant s \leqslant 1/2, \\ \varphi_t(f(2s-1)) & \text{if } 1/2 \leqslant s \leqslant 1. \end{cases}$$

It is not hard to check that (σ_t) is a well defined asymptotic homomorphism. We claim that the restriction of (σ_t) to SB is homotopic to $(S\varphi_t)$. We have $\delta^*[[\sigma_t]] = [[\sigma_t \circ \delta]]$ and

$$(\sigma_t \circ \delta)(f)(s) = \begin{cases} 0, & \text{if } 0 \leqslant s \leqslant 1/2, \\ \varphi_t(f(2s-1)) & \text{if } 1/2 \leqslant s \leqslant 1. \end{cases}$$

The equality $[[\sigma_t \circ \delta]] = [[S\varphi_t]]$ follows since the above formula shows that $[[\sigma_t \circ \delta]]$ is equal to the sum of $[[0]]$ with $[[S\varphi_t]]$ in $[[SB, SD]]$. □

THEOREM 9. *Let A, B, D be C^*-algebras and $p\colon A \to B$ a $*$-homomorphism. Then there is a long exact sequence of Abelian groups*

$$\cdots \longrightarrow \{\{C_p, D\}\}_{q-1} \xrightarrow{\delta^*} \{\{B, D\}\}_q \xrightarrow{p^*} \{\{A, D\}\}_q \xrightarrow{k^*} \{\{C_p, D\}\}_q \xrightarrow{\delta^*} \{\{B, D\}\}_{q+1} \longrightarrow \cdots$$

Proof. The $*$-homomorphism $\delta \circ Sp\colon SA \to C_p$ is null homotopic. Combining this with Proposition 8, it follows that the $*$-homomorphism $p\colon A \to B$ induces a sequence

$$\{\{C_p, D\}\}_{q-1} \xrightarrow{\delta^*} \{\{B, D\}\}_q \xrightarrow{p^*} \{\{A, D\}\}_q$$

exact in the middle. Recall that the map δ appears in the mapping cone extension

$$0 \to SB \xrightarrow{\delta} C_p \xrightarrow{k} A \longrightarrow 0.$$

We are going to apply this result for the $*$-homomorphism $k\colon C_p \to A$. We get that the sequence

$$\{\{C_k, D\}\}_{q-1} \xrightarrow{\delta_1^*} \{\{A, D\}\}_q \xrightarrow{k^*} \{\{C_p, D\}\}_q$$

is exact in the middle. The map δ_1 appears in the mapping cone extension

$$0 \to SA \xrightarrow{\delta_1} C_k \xrightarrow{k_1} C_p \to 0.$$

Like in the commutative case δ_1 can be identified with Sp, modulo the homotopy identification of C_k with SB. This is again contained in the proof of Theorem 3.8 in [10]. Thus the above sequence becomes

$$\{\{SB, D\}\}_{q-1} \xrightarrow{p^*} \{\{A, D\}\}_q \xrightarrow{k^*} \{\{C_p, D\}\}_q$$

or

$$\{\{B, D\}\}_q \xrightarrow{p^*} \{\{A, D\}\}_q \xrightarrow{k*} \{\{C_p, D\}\}_q.$$

The final part of the proof is similar to the proof of Theorem 7. We just apply the functor $\{\{., D\}\}_q$ to the first commutative diagram from the proof of Proposition 6.

4. Excision

All the C^*-algebras to be considered in this section are separable. In [2] Connes and Higson associated an asymptotic homomorphism $(\gamma_t)\colon SB \to J$ with any short exact sequence

$$0 \to J \to A \xrightarrow{p} B \to 0 \qquad (\gamma)$$

They use this construction to show that $SC_p \otimes \mathcal{K}$ is isomorphic to $SJ \otimes \mathcal{K}$ in the category $\mathcal{A}$ (see Section 2). In this section we show that the isomorphism still holds after removing the compacts operators.

We begin by reviewing the construction of (γ_t). Let $0 \leqslant u_t \leqslant 1$ be an approximate unit of J which is quasicentral in A. Thus $\lim_{t\to\infty}\|x - u_t x\| = 0$ for all $x \in J$ and $\lim_{t\to\infty}\|au_t - u_t a\| = 0$ for all $a \in A$. We may assume that the map $t \to u_t$ is continuous. An obvious way to construct (u_t) is by linear interpolation of a discrete approximative unit (u_n) with the same properties [9].

Define an asymptotic homomorphism $(\mu_t)\colon SA \to J$ such that $\mu_t(f \otimes a) = f(u_t)a$ for $f \in C_0(0, 1)$ and $a \in A$. This is well defined by the discussion on tensor products of asymptotic homomorphisms. Since u_t is an approximate unit in J one has that $\lim_{t\to\infty}\mu_t(g) = 0$ for all $g \in SJ$. Thus (μ_t) descends to an asymptotic homomorphism $(\gamma_t)\colon SB \to J$. Let $\sigma\colon B \to A$ be any right inverse for p. Then up to an equivalence $\gamma_t(f \otimes b) = f(u_t)\sigma(b)$ for all $f \in C_0(0, 1)$ and $b \in B$. No continuity assumption on σ is necessary. The homotopy class of γ_t depends only on the exact sequence (γ). Indeed another choice (u_t') of an approximate unit in J which is quasicentral in A gives an asymptotic homomorphism (γ_t') homotopic to (γ_t) via the homotopy $(H_t)\colon SB \to J[0, 1]$, defined by $H_t(f \otimes b)(s) = f((1 - s)u_t + su_t')\sigma(b)$.

Next we give some functorial properties of $[[\gamma_t]]$.

PROPOSITION 10. *Suppose that the diagram*

$$\begin{tikzcd}
(\gamma) & 0 \arrow[r] & J \arrow[r] & A \arrow[r, "p"] & B \arrow[r] & 0 \\
(\gamma') & 0 \arrow[r] & J \arrow[u, equal] \arrow[r] & A' \arrow[u] \arrow[r, "p'"] & B' \arrow[u, "i"] \arrow[r] & 0
\end{tikzcd}$$

has exact rows, is commutative and i is injective. Then $[[\gamma_t \circ Si]] = [[\gamma'_t]]$.

Proof. We will think of A' as being a subalgebra of A. Let σ' be a right inverse of p' and let (u_t) be an approximate unit of J which is quasicentral in A. Of course (u_t) is quasicentral in A' too. Thus we may put $\gamma'_t(f \otimes b') = f(u_t)\sigma'(b')$, for $f \in C_0(0,\ 1)$ and $b' \in B'$. Via the embedding of A' in A, $\sigma'(b')$ is a lifting of $i(b')$. It follows that

$$(\gamma_t \circ Si)(f \otimes b') = \gamma_t(f \otimes i(b')) = f(u_t)\sigma'(b')$$

Thus $(\gamma_t \circ Si)(f \otimes b') = \gamma'_t(f \otimes b')$. □

PROPOSITION 11. *Suppose that the diagram*

$$\begin{tikzcd}
(\gamma') & 0 \arrow[r] & J' \arrow[d, "i"] \arrow[r] & A' \arrow[d] \arrow[r, "p'"] & B \arrow[d, equal] \arrow[r] & 0 \\
(\gamma) & 0 \arrow[r] & J \arrow[r] & A \arrow[r, "p"] & B \arrow[r] & 0
\end{tikzcd}$$

has exact rows, is commutative and i is an injection. Then $[[\gamma_t]] = [[i \circ \gamma'_t]]$.

Proof. We will identify A' with its image in A. Let σ' be a right inverse of p'. Let (u_t) be an approximate unit of J which is quasicentral in A and let (u'_t) be an approximate unit of J' which is quasicentral in A'. Consider the corresponding asymptotic homomorphisms $(\gamma_t)\colon SB \to J$ and $(\gamma'_t)\colon SB \to J'$. One has $\gamma_t(f \otimes b) = f(u_t)\sigma'(b)$ and $i \circ \gamma'_t(f \otimes b) = f(i(u'_t))\sigma'(b)$. Define a homotopy $(H_t)\colon SB \to J[0,\ 1]$ by $H_t(f \otimes b)(s) = f((1-s)i(u'_t) + su_t)\sigma'(b)$. It is clear that (H_t) is a homotopy from $(i \circ \gamma'_t)$ to (γ_t). □

The following remark stated here as a Lemma is made in [2]. A proof is included for the sake of completeness.

LEMMA 12. [2] *Let A be a C^*-algebra. The homotopy class of the asymptotic homomorphism defined by the exact sequence*

$$0 \to SA \to CA \to A \to 0$$

is equal to $[[\mathrm{id}_{SA}]]$.

Proof. Let (v_t) and (u_t) be approximate units of $C_0(0,\ 1)$ and A respectively. Then $(v_t \otimes u_t)$ is an approximate unit of SA, quasicentral in CA. For $a \in A$, the map $s \mapsto (1-s)a$ is a lifting of a in CA. With these choices the asymptotic homomorphism defined by the given exact sequence in $\gamma_t(f \otimes a) = f(v_t \otimes u_t)(1-s)a$, for all $f \in C_0(0,\ 1)$ and $a \in A$. Note that

$$\lim_{t \to \infty} \|f(v_t \otimes u_t)(1-s)a - f(v_t)(1-s)a\| = 0$$

since (u_t) is an approximate unit for A. This is easily seen by approximating f with polynomials. Hence, we may take $\gamma_t(f \otimes a) = f(v_t)(1-s)a$. This shows that $\gamma_t = \gamma_t^0 \otimes \mathrm{id}_A$, where $(\gamma_t^0)\colon C_0(0,\ 1) \to C_0(0,\ 1)$ is the asymptotic homomorphism corresponding to $A = \mathbb{C}$. Thus it suffices to show that $[[\gamma_t^0]] = [[\mathrm{id}_{C_0(0,1)}]]$. After adjoining a unit to $C_0(0,\ 1)$ and extending γ_t^0 to an unital asymptotic homomorphism,

$$\gamma_t^0(\mathrm{e}^{2\pi i s}) = \mathrm{e}^{2\pi i v_t(s)}(1-s) + s.$$

Let

$$v_t(s) = \begin{cases} ts, & \text{if } 0 \leqslant s \leqslant 1/t, \\ 1, & \text{if } 1/t \leqslant s \leqslant 1 - 1/t, \\ t(1-s), & \text{if } 1 - 1/t \leqslant s \leqslant 1. \end{cases}$$

With this choice of v_t, it is easy to check that for fixed t, $\gamma_t^0(\mathrm{e}^{2\pi i s}) \in C(S^1)$ is a degree 1 function. This implies that $[[\gamma_t^0]] = [[\mathrm{id}_{C_0(0,1)}]]$, (see Corollary 17 below).

THEOREM 13. *The map $Si\colon SJ \to SC_p$ is an isomorphism in the category $\mathcal{A}$.*

Proof. We show that the asymptotic homomorphism γ_t defined by the extension

$$0 \to SJ \to CA \to C_p \to 0$$

is an inverse in $\mathcal{A}$ for Si. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & SJ \arrow[r] & CA \arrow[r] & C_p \arrow[r] & 0 \\ 0 \arrow[r] & SJ \arrow[r] \arrow[u, equal] & CJ \arrow[r] \arrow[u] & J \arrow[r] \arrow[u, "i"'] & 0 \end{tikzcd}$$

It follows from Proposition 10 and Lemma 12 that

$$[[\gamma_t \circ Si]] = [[\mathrm{id}_{SJ}]].$$

This shows that $[[\gamma_t]]$ is a left inverse for Si. To show that $[[\gamma_t]]$ is a right inverse for Si, consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & SJ \arrow[r] \arrow[d, "Si"'] & CA \arrow[r] \arrow[d] & C_p \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & SC_p \arrow[r] & CC_p \arrow[r] & C_p \arrow[r] & 0
\end{tikzcd}$$

The map $CA \to CC_p$ takes $a(s)$ to $(p(a(s+t-st)), a(s))$.

It follows from Proposition 11 and Lemma 12 that $[[Si \circ \gamma_t]] = [[\mathrm{id}_{SC_p}]]$. □

5. Homology and Cohomology Theories for C*-algebras

We shall use the results of the previous sections to obtain nonperiodic homology and cohomology theories satisfying general excision properties. These theories are defined on the category of separable C^*-algebras.

A general discussion on axiomatic homology for C^*-algebras is provided in [11]. Recall that a *homology theory* on the category of separable C^*-algebras is a sequence of covariant functors $\{h_q\}$ from separable C^*-algebras to Abelian groups which satisfy the following axioms:

(1) *Homotopy axiom:* Suppose that $\varphi,\ \psi\colon A \to B$ are homotopy equivalent $*$-homomorphisms. Then $\varphi_* = \psi_*\colon h_q(A) \to h_q(B)$ for all q.

(2) *Exactness axiom:* Let

$$0 \to J \xrightarrow{i} A \xrightarrow{p} B \to 0$$

be an exact sequence of separable C^*-algebras. Then there is a boundary map $h_q(B) \to h_{q-1}(J)$ and a long exact sequence

$$\cdots \to h_q(J) \xrightarrow{i_*} h_q(A) \xrightarrow{p_*} h_q(B) \to h_{q-1}(J) \to \cdots$$

The boundary map is natural with respect to morphisms of short exact sequences.

The axioms for cohomology are quite similar [11].

THEOREM 14. *Let*

$$0 \to J \xrightarrow{i} A \xrightarrow{p} B \to 0$$

be an exact sequence of separable C^*-algebras. Then for any separable C^*-algebra C there are long exact sequences of Abelian groups

$$\cdots \to \{\{C, B\}\}_{q-1} \to \{\{C, J\}\}_q \xrightarrow{i_*} \{\{C, A\}\}_q \xrightarrow{p_*} \{\{C, B\}\}_q \to \{\{C, J\}\}_{q+1} \to \cdots$$

$$\cdots \to \{\{J, C\}\}_{q-1} \xrightarrow{i^*} \{\{B, C\}\}_q \xrightarrow{p^*} \{\{A, C\}\}_q \to \{\{J, C\}\}_q \to \{\{B, C\}\}_{q+1} \to \cdots$$

The connecting maps are given by appropriate compositions with (γ_t), the asymptotic homomorphism associated with the given short exact sequence.

Proof. The statement follows from Theorems 7, 9 and 13. □

COROLLARY 15.
(a) *For any separable C^*-algebra A, the functors $\{\{A, -\}\}_*$ form a homology theory.*
(b) *For any separable C^*-algebra B, the functors $\{\{-, B\}\}_*$ form a cohomology theory.*

The restrictions of these theories to compact pointed ANR spaces coincide with stable homotopy theory and, respectively, stable cohomotopy theory. That is,

$$\{\{\mathbb{C}, C_0(X)\}\}_q = \pi_s^q(X) \quad \text{and} \quad \{\{C_0(X), \mathbb{C}\}\}_{-q} = \pi_q^s(X).$$

This will be proved in Section 5 (see Corollary 17). Thus we are dealing with nontrivial nonperiodic theories. If $B \cong B \otimes \mathcal{K}$ then, according to [2], the suspension map

$$[[SA, SB]] \to [[S^2A, S^2B]]$$

is an isomorphism. Thus $\{\{A, B\}\}$ is isomorphic to the E-theory group $E(A, B)$ for any stable C^*-algebra B.

6. Commutative C^*-Algebras

In this section we consider asymptotic homomorphisms between commutative C^*-algebras. However the first result addresses a slightly more general case in view of future applications to subhomogeneous C^*-algebras.

PROPOSITION 16. *Suppose that A is a separable nuclear C^*-algebra such that* $\mathrm{Hom}(A, M_n)$ *is ANR. Then the canonical map*

$$c\colon [A, C_0(Y) \otimes M_n] \to [[A, C_0(Y) \otimes M_n]]$$

is an isomorphism for any locally compact metrizable space Y.

Proof. For C^*-algebras A, B we let $\mathrm{CP}(A, B)$ denote the space of all linear, contractive, completely positive maps from A to B, endowed with the point-norm topology.

In the first part of the proof we show that c is onto. More precisely we prove that any asymptotic homomorphism $(\varphi_t): A \to C_0(Y) \otimes M_n$ is equivalent to a $*$-homomorphism. Since A is nuclear we may assume that (φ_t) is given by some linear, contractive, completely positive map $\varphi: A \to C_b(T, C_0(Y) \otimes M_n)$. Define $f: T \times Y \to \mathrm{CP}(A, M_n)$ by $f(t, y)(a) = \varphi(a)(t)(y)$. Note that $\mathrm{CP}(A, M_n)$ is a convex *weak**-compact subset of $(A^*)^{n^2}$, where A^* denotes the dual Banach space of A. $\mathrm{Hom}(A, M_n)$ is a closed subset of $\mathrm{CP}(A, M_n)$. Since $\mathrm{Hom}(A, M_n)$ is ANR, there is a continuous retraction r of some neighborhood U of $\mathrm{Hom}(A, M_n)$ onto $\mathrm{Hom}(A, M_n)$. The next step of the proof is based on the following claim.

CLAIM. *For any open neighborhood V of* $\mathrm{Hom}(A, M_n)$ *in* $\mathrm{CP}(A, M_n)$ *there is t_0 such that $f(t, y) \in V$ for all $t \geqslant t_0$ and $y \in Y$.*

Proof of the Claim. To get a contradiction assume that there is a sequence (t_i, y_i) with $t_i \mapsto \infty$ such that $f(t_i, y_i)$ is not in V for all i. Since the complement of V in $\mathrm{CP}(A, M_n)$ is compact by passing to a subsequence of (y_i) we may assume that $f(t_i, y_i)$ is convergent to $h \in \mathrm{CP}(A, M_n) \backslash \mathrm{Hom}(A, M_n)$. However, since φ is an asymptotic homomorphism, it follows that h must be a $*$-homomorphism. □

According to the Claim there is $t_0 \in T$ such that $f(t, y) \in U$ for all $t \geqslant t_0$ and $y \in Y$. Let $g: T \times Y \to \mathrm{Hom}(A, M_n)$ be defined by $g(t, y) = r(f(t, y))$ for all $t \geqslant t_0$ and $y \in Y$. The map g defines a $*$-homomorphism $\psi \in \mathrm{Hom}(A, C_b(T, C_0(Y) \otimes M_n))$ given by $\psi(a)(t)(y) = g(t, y)(a)$. We show that φ is equivalent to ψ. That is $\varphi(a) - \psi(a) \in C_0(T, C_0(Y) \otimes M_n)$, or equivalently

$$\lim_{t \to \infty} \sup_{y \in Y} \|f(t, y)(a) - g(t, y)(a)\| = 0,$$

for all $a \in A$. Since A is separable, the topology of $\mathrm{CP}(A, M_n)$ is metrizable. If (a_k) is a dense sequence in the unit ball of A then

$$d(\alpha, \beta) = \sum_{k=1}^{\infty} 2^{-k} \|\alpha(a_k) - \beta(a_k)\|$$

is a metric on $\mathrm{CP}(A, M_n)$. By using this metric all we have to show is that

$$\lim_{t \to \infty} \sup_{y \in Y} d(f(t, y), rf(t, y)) = 0.$$

Find a sequence (U_i) of neighborhoods of $\mathrm{Hom}(A, M_n)$ contained in U such that $d(\alpha, r\alpha) \leqslant 1/i$ for all $\alpha \in U_i$. According to the above Claim, there is a sequence (t_i) converging to infinity such that $f(t, y) \in U_i$ for all $t \geqslant t_i$ and $y \in Y$. By the choice of U_i it follows that $d(f(t, y), rf(t, y)) \leqslant 1/i$ for all $t \geqslant t_i$ and $y \in Y$.

Thus $\lim_{t\to\infty}\sup_{y\in Y} d(f(t, y), rf(t, y)) = 0$, and this is easily seen to imply that $\varphi(a) - \psi(a) \in C_0(T, C_0(Y) \otimes M_n)$ for all $a \in A$.

So far we have shown that any asymptotic homomorphism $(\varphi_t): A \to C_0(Y) \otimes M_n$ is equivalent to $(\psi_t): A \to C_0(Y) \otimes M_n$, where each ψ_t is a $*$-homomorphism. To complete the proof of the surjectivity of c note that $(H_t): A \to C_0(Y) \otimes M_n[0, 1]$, $H_t(a)(s) = \psi_{(1-s)+ts}(a)$ is a homotopy from (ψ_t) to ψ_1.

Next we show the injectivity of c. Suppose that $\varphi, \psi \in \mathrm{Hom}(A, B)$ are homotopic as asymptotic homomorphisms. Thus, there is an asymptotic homomorphism $(\Phi_t): A \to C_0(Y) \otimes M_n[0, 1]$ joining φ with ψ. It follows from the first part of the proof that (Φ_t) is equivalent to some asymptotic homomorphism (Ψ_t) such that each individual map Ψ_t is a $*$-homomorphism. Since the construction of (Ψ_t) is based on some retract r onto $\mathrm{Hom}(A, M_n)$ it follows that (Ψ_t) has the same ends as (Φ_t). We conclude that for some big enough t, $s \to \Psi_t(.)(s)$ is a homotopy of $*$-homomorphisms from φ to ψ. This shows that the map c is injective. $\square$

COROLLARY 17. *Let X, Y, be locally compact metrizable spaces. Suppose that X is ANR. Then the natural map*

$$[C_0(X), C_0(Y)] \to [[C_0(X), C_0(Y)]]$$

is a bijection.

The assumption that X is ANR is essential. This follows from Corollary 19 below.

Suppose that X and Y are compact metrizable spaces. The proof of Proposition 16 provides a good description of asymptotic homomorphisms $(\varphi_t): C(X) \to C(Y)$ in terms of continuous maps between topological spaces. Up to an equivalence (φ_t) corresponds to a unital positive linear map $\varphi: C(X) \to C_b(T, C(Y))$ such that $\varphi(ab) - \varphi(a)\varphi(b) \in C_0(T, C(Y))$ for all a, b in $C(X)$. This defines a continuous map $f: T \times Y \to C(X)^*$, given by $f(t, y)(a) = \varphi_t(a)(y)$ for all $a \in C(X)$. Here $C(X)^*$ is endowed with the *weak**-topology. Let $M(X)$ denote the closed convex subset of $C(X)^*$ consisting of positive measures of total mass one. X is embedded in $M(X)$ as the set of extreme points. Each $x \in X$ corresponds to a Dirac measure δ_x. According to the Claim in the proof of Proposition 16, for any neighborhood V of X in $M(X)$ there is some $t_0 \geqslant 1$ such that $f(t, y) \in V$ for all $t \geqslant t_0$ and $y \in Y$. Conversely given any continuous map $f: T \times Y \to M(X)$ having this property, the formula $\varphi_t(a)(y) = f(t, y)(a)$ defines an asymptotic homomorphism $(\varphi_t): C(X) \to C(Y)$. To prove this one has only to check the asymptotic multiplicativity of (φ_t). This goes as follows. Fix a, $b \in C(X)$. For each positive integer n let

$$V_n = \{\mu \in M(X): |\mu(ab) - \mu(a)\mu(b)| < 1/n\}$$

V_n is an open neighborhood of X in $M(X)$. By hypothesis there is some $t_n \geqslant 1$ such that $f(t, y) \in V_n$ whenever $t \geqslant t_n$ and $y \in Y$. This clearly implies that

$$\lim_{t\to\infty} \sup_{y\in Y} \|f(t, y)(ab) - f(t, y)(a)f(t, y)(b)\| = 0$$

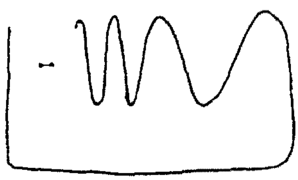
or

$$\lim_{t\to\infty} \|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| = 0.$$

The above description gives a clear picture of asymptotic homomorphisms between spaces. A $*$-homomorphism from $C(X)$ to $C(Y)$ corresponds to a continuous map from Y to X. An asymptotic homomorphism corresponds to a family of continuous maps (f_t) from Y to the space of probability measures on X, such that the image of f_t approaches X in the *weak**-topology as t goes to infinity.

Let $M_0(X)$ be the convex closure of X in $M(X)$. By Krein–Milman theorem $M_0(X)$ is dense in $M(X)$. Thus one can perturb the maps (f_t) to take values in $M_0(X)$. If X is ANR then one can retract some neighborhood of X in $M(X)$ onto X in order to get an equivalent family of maps with values in X. This cannot be done for general compact metrisable spaces. The following discussion is relevant into this respect.

Let W denote the Warshaw circle. W can be described as the following closure of the $\sin(1/x)$ curve:



and let A be the left vertical side. Note that the quotient space W/A is homeomorphic to a standard circle S^1. There is a retraction $r: W \to A$ given by the orthogonal projection onto A. It follows that there is a split extension

$$0 \to C_0(S^1\backslash pt) \xrightarrow{i} C_0(W\backslash pt) \to C_0(A\backslash pt) \to 0.$$

PROPOSITION 18. *Suppose that*

$$0 \to J \xrightarrow{i} A \xrightarrow{p} B \to 0$$

is a split extension of separable C^*-algebras and B is contractable. Then there is an asymptotic homomorphism $(\varphi_t): A \to J$ such that $[[i \circ \varphi_t]] = [[\mathrm{id}_A]]$.

Proof. By [3] for any split extension:

$$0 \to J \xrightarrow{i} A \xrightarrow{p} B \to 0$$

the mapping cone C_p is isomorphic to J in $\mathcal{A}$. By combining this excision result with Proposition 6, for any C^*-algebra C we get the following exact sequence of pointed sets:

$$0 \to [[C, J]] \xrightarrow{i_*} [[C, A]] \xrightarrow{p_*} [[C, B]].$$

Since B is contractable it follows that $i_*: [[C, J]] \to [[C, A]]$ is onto for all separable C^*-algebras C. Finally with $C = A$ we find $[[\varphi_t]] \in [[A, J]]$ such that $i_*[[\varphi_t]] = \mathrm{id}_A$. □

COROLLARY 19. *The Warshaw circle is isomorphic to the standard circle in the category $\mathcal{A}$.*

Proof. Using Proposition 18 we find $\varphi_t : C_0(W\backslash pt) \to C_0(S^1\backslash pt)$ such that $[[i \circ \varphi_t]] = [[\mathrm{id}_{C_0(W\backslash pt)}]]$. In particular φ_t induces an isomorphism on K-theory. By Corollary 17 $\varphi_t \circ i$ is homotopic to a $*$-homomorphism. This is necessarily induced by a degree one map $S^1 \to S^1$ and therefore is homotopic to $\mathrm{id}_{C_0(S^1\backslash pt)}$. It follows that $C_0(W\backslash pt)$ is isomorphic to $C_0(S^1\backslash pt)$ in $\mathcal{A}$. This remains true after we add units. □

Although W and S^1 have similar global properties they are not homotopy equivalent. However according to Corollary 19 they are homotopy equivalent through asymptotic homomorphisms. In particular this implies that Corollary 17 does not hold for $X = W$, so that it is essential to assume that X is ANR. Another consequence of Corollary 19 is the existence of an asymptotic homomorphism $C(W) \to C(S^1)$ inducing an isomorphism on Čech cohomology. There are no $*$-homomorphisms $C(W) \to C(S^1)$ having this property.

Spaces like the Warshaw circle indicate that ordinary homotopy is not well suited for the study of singular spaces. For this reason Borsuk developed the shape theory as a substitute for the homotopy theory of singular spaces [8]. The Warshaw circle is shape equivalent to the standard circle. Corollary 19 suggests possible connections between shape theory and the homotopy theory based on asymptotic homomorphisms. Also the description of asymptotic homomorphisms between commutative C^*-algebras after Corollary 17 reveals similarities of $\mathcal{A}$ with the approaching homotopy category of [18]. We clarify these matters in a forthcoming paper [16] where the category $\mathcal{A}$ is identified with a strong shape theory for separable C^*-algebras that refines the shape theories of [5] and [19]. It turns out that for compact metrisable spaces $C(X)$ is isomorphic to $C(Y)$ in $\mathcal{A}$ if and only if X is shape equivalent to Y. Moreover this statement generalizes to a noncommutative setting [16].

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