Approximate homogeneity is not a local property

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Abstract

It is shown that the AH algebras satisfy a certain splitting property at the level of K-theory with torsion coefficients. The splitting property is used to prove the following:

(i) There are locally homogeneous C^*-algebras which are not AH algebras.

(ii) The class of AH algebras is not closed under countable inductive limits.

(iii) There are real rank zero split quasidiagonal extensions of AH algebras which are not AH algebras

1 Introduction

Classes of C^*-algebras defined from inductive limit descriptions have been a fertile area of research in recent years. A fundamental example of such a class is of course the AF algebras, defined as those C^*-algebras which can be written as countable inductive limits of finite-dimensional C^*-algebras. As discovered already in the foundational paper by Bratteli ([5]), this class — defined extrinsically by the inductive limit description — can be characterized intrinsically by a local criterion:

(i) A is an AF algebra if and only if for every $\varepsilon > 0$ and every finite set of elements $\mathcal{F} \subseteq A$ there exists a finite-dimensional C^*-algebra B and a $*$-homomorphism $\varphi : B \to A$ with $\operatorname{dist}(\mathcal{F}, \varphi(B)) < \varepsilon$.

This basic result plays an important role in establishing the following two closure properties of the class of AF algebras:

(ii) A countable inductive limit of AF algebras is again an AF algebra.

(iii) An extension
$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$
of AF algebras A, B is again an AF algebra. This result is due to Brown [6].

It is the purpose of this note to demonstrate that the expected local criterion <u>fails</u> for the class of AH *algebras*, defined as those C^*-algebras that can be written as countable inductive limits of algebras of the form

$$\bigoplus_{i=1}^{k} p_i \mathbf{M}_{n_i}(C(X_i))p_i$$

for some compact metrizable spaces X_i. This is achieved by proving that all AH algebras must satisfy a certain splitting property involving K-groups with torsion coefficients. Applying this necessary condition it is easy to construct a C^*-algebra which satisfies the local criterion but is not AH.

This answers the question, mentioned in Remark 6.1.2 [3] and in [23], of whether every locally homogeneous C^*-algebra is an AH algebra. The same example also shows that there are no reasonable closure properties along the lines of (ii) and (iii) for this class.

Much effort has gone into such generalizations of (i)–(iii). Local criteria for the so called AT and AD algebras were given by Elliott in [18], in the second case based on a stability result for the dimension drop intervals by Loring ([25]). A local criterion for classes defined by more general subhomogeneous algebras with 1-dimensional spectrum can also be found in [17]. However, since a fundamental technical tool in establishing all of these results is *semiprojectivity* (or, rather, weak semiprojectivity, cf. [26]) for the building blocks, and by Loring's work this property is known to fail for $C(X)$ with $\dim(X) \geq 2$, it is maybe not so surprising that the result does not generalize.

Before discussing generalizations of (iii) one must realize that the closure property of AF algebras given here is deceptively simple. In fact, as was made clear in [7], one needs the extension to be *quasidiagonal* to expect good closure properties. This property implies that the corresponding K-theoretical six term exact sequence

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(E) & \longrightarrow & K_0(B) \\ \uparrow & & & & \downarrow \\ K_1(B) & \longrightarrow & K_1(E) & \longrightarrow & K_1(A) \end{array}$$

comes apart into two pure exact sequences, and in fact it has been shown for the class of AT algebras of real rank zero ([24]) and for certain subclasses of AD algebras of real rank zero ([11]) that this K-theoretical obstruction is the only one. This, we shall prove, is not the case for AH algebras; as we will give an example of a quasidiagonal (and split!) extension of AH algebras which is not itself AH.

The results in the present paper are closely related to the observation in [10], obtained by comparing classification results, that not every ASH algebra of real rank zero can be written as an AH algebra with slow dimension growth. We show here that not every ASH algebra can be written as an AH algebra, with or without slow dimension growth. Another proof of this result can be obtained by a suitable modification of the argument in [10].

2 Notation

An AH algebra is a C^*-algebra that can be written as a countable inductive limit of C^*-algebras of the form

$$\bigoplus_{i=1}^{k} p_i \mathbf{M}_{n_i}(C(X_i)) p_i$$

where X_i is a compact metrizable space, and p_i is a projection in $\mathbf{M}_{n_i}(C(X_i))$. This is the standard definition, but two comments are in order here: First, one must note that the class is actually bigger than the class of C^*-algebras which can be written as countable inductive limit of C^*-algebras of the form

$$\bigoplus_{i=1}^{k} \mathbf{M}_{n_i}(C(X_i))$$

as seen in Proposition 4.23 [19]. Second, the class is smaller than the class of C^*-algebras which can be written as countable inductive limits of C^*-algebras of the form

$$\bigoplus_{i=1}^{k} A_i$$

where A_i is *homogeneous* in the sense that all irreducible representations of A_i are of the same finite dimension d_i, because we exclude algebras with nontrivial Dixmier-Douady class. Hence the 'H' in the acronym only stands for 'homogeneous' in a rather limited sense. By results of Blackadar [2] one may assume that all the spaces X_i in the definition of AH algebras are finite connected CW complexes. The same goes for the definition of ASH algebras below.

The class of ASH algebras consists of those C^*-algebras that are given as inductive limits over

$$\left[\bigoplus_{i=1}^{k} p_i \mathbf{M}_{n_i}(C(X_i)) p_i\right] \oplus \left[\bigoplus_{j=1}^{l} \mathbf{M}_{m_i}(\mathbb{I}_{\tilde{d}_i})\right]$$

where $\mathbb{I}_{\tilde{d}} = \{f : [0,1] \to \mathbf{M}_d \mid f(0), f(1) \in \mathbb{C}1\}$. The subclass of the AH algebras obtained by requiring that every X_i is S^1 is called AT. The subclass of the ASH algebras obtained by $X_i = S^1$ is called AD. K-theoretically, these classes are distinguished in their superclasses by having torsion-free K_0. We also use the version of the *slow dimension growth* condition defined in [9].

When (G, G^+) is any preordered group, we denote by $S(G)$ the set of order preserving homomorphisms $f : G \to \mathbb{R}$. We then define the *infinitesimals* of G as the set of group elements vanishing on $S(G)$, i.e.

$$\mathrm{Inf}(G) = \{g \in G \mid \forall f \in S(G) : f(g) = 0\}.$$

Note that every order-preserving group homomorphism will map infinitesimals to infinitesimals. Hence, since $K_0(-)$ is a functor sending $*$-homomorphisms to order homomorphisms, we achieve by the definition

$$K_0^{\mathrm{Inf}}(A) = K_0(A)/\,\mathrm{Inf}(K_0(A))$$

a functor. This is *not* continuous, as AF examples will show, but there is a natural map

$$k : \varinjlim K_0^{\mathrm{Inf}}(A_n) \to K_0^{\mathrm{Inf}}(\varinjlim A_n)$$

defined by the fact that $\mathrm{Inf}(K_0(A_n))$ is sent to $\mathrm{Inf}(K_0(\varinjlim A_n))$ by the canonical map.

3 A partial splitting

The invariant $\underline{\mathbf{K}}(-)$ — which was proven in [9] to be complete for the class of ASH algebras of real rank zero with slow dimension growth — consists of doubly graded ordered groups

$$K_0(A) \oplus K_1(A) \oplus K_0(A; \mathbb{Z}/m) \oplus K_1(A; \mathbb{Z}/m),$$

and natural maps

$$\begin{aligned} \rho^i_m &: K_i(A) \to K_i(A; \mathbb{Z}/m), \\ \beta^i_m &: K_i(A; \mathbb{Z}/m) \to K_{i+1}(A), \\ \kappa^i_{s,m} &: K_i(A; \mathbb{Z}/m) \to K_i(A; \mathbb{Z}/s). \end{aligned}$$

There are two six term exact sequences involving ρ, β and κ (see [28]), and one of these may be unspliced to yield group homomorphisms $\tilde{\rho}_m, \tilde{\beta}_m$ fitting into a short exact sequence

$$0 \longrightarrow K_0(A) \otimes \mathbb{Z}/m \xrightarrow{\tilde{\rho}_m} K_0(A; \mathbb{Z}/m) \xrightarrow{\tilde{\beta}_m} \mathrm{Tor}(K_1(A), \mathbb{Z}/m) \longrightarrow 0.$$

It is a nontrivial fact due to Bödigheimer ([4]) that all of these sequences split.

We shall mainly be interested in the group homomorphisms in the diagram

$$\begin{tikzcd} K_0(A) \arrow[r] \arrow[dr, "q_m"'] & K_0(A)\otimes\mathbb{Z}/m \arrow[r, "\tilde{\rho}_m"] \arrow[d, "\tilde{q}_m"'] & K_0(A;\mathbb{Z}/m) \arrow[dl, dashed, "\alpha_m"] \\ & K_0^{\mathrm{Inf}}(A)\otimes\mathbb{Z}/m & \end{tikzcd}$$

The composition of the two horizontal maps is equal to ρ_m^0, and $\tilde{q}_m$ is the canonical quotient map. Using the splitting map for $\tilde{\rho}_m$ we easily get a homomorphism α_m filling out the triangle above in a commuting fashion. However, as splitting maps of this type are never natural, we can not expect a map thus defined to have any extra properties reflecting the internal structure of A. The technical vehicle of this paper is the non-trivial observation that α can always be chosen to be *ideal-preserving* in a restricted sense, when A is AH. To make this more precise we give the following definition — where, as in [8], $F(A\|I)$ denotes the image of

$$F(I) \xrightarrow{F(\iota)} F(A)$$

whenever $F(-)$ is a functor defined on C^*-algebras and $*$-homomorphisms:

Definition 3.1 We say that a C^*-algebra A has an IMI_m*-splitting* (ideal-preserving splitting modulo infinitesimals) when there exists a group homomorphism

$$\alpha_m : K_0(A;\mathbb{Z}/m) \to K_0^{\mathrm{Inf}}(A)\otimes\mathbb{Z}/m$$

with the properties:

(i) $\alpha_m\rho_m^0 = q_m$,

(ii) whenever I is an ideal of A which is generated by projections,

$$\alpha_m(K_0(A\|I;\mathbb{Z}/m)) \subseteq K_0^{\mathrm{Inf}}(A\|I)\otimes\mathbb{Z}/m.$$

Our examples involve C^*-algebras A of real rank zero, in which case every ideal is generated by projections. In the definition we restrict ourselves to ideals generated by projections in order to obtain a splitting property valid for all AH algebras.

Recall from [28] that any group homomorphism

$$\kappa_{s,m} : \mathbb{Z}/m \to \mathbb{Z}/s$$

induces natural transformations $\kappa_{s,m}^0 : K_0(A;\mathbb{Z}/m) \to K_0(A;\mathbb{Z}/s)$.

A sequence (α_m) of IMI_m-splittings is said to be *coherent* if for any $m, s \in \mathbb{N}$ the following diagram is commutative

$$\begin{tikzcd} K_0(A;\mathbb{Z}/m) \arrow[r, "\alpha_m"] \arrow[d, "\kappa^0_{s,m}"'] & K_0^{\mathrm{Inf}}(A)\otimes\mathbb{Z}/m \arrow[d, "\mathrm{id}\otimes\kappa_{s,m}"] \\ K_0(A;\mathbb{Z}/s) \arrow[r, "\alpha_s"'] & K_0^{\mathrm{Inf}}(A)\otimes\mathbb{Z}/s \end{tikzcd}$$

We shall prove the following:

Theorem 3.2 *Every AH algebra has a coherent sequence (α_m) of IMI_m-splittings.*

We postpone the proof of this to Section 5 below, and we shall see in Section 4 how this necessary condition for A being an AH algebra allows us to prove the statement in the title of the paper.

In the remainder of this section we shall show how to phrase the IMI-property using the more economical invariant $\mathbf{K}_\infty(-)$ defined in [8]. This is particularly interesting for applications, since it is clear from Definition 3.1 that in the case where $\mathrm{Inf}\, K_0(A) = 0$, the partial splitting maps α_m are in fact actual splitting maps for $\tilde{\rho}_m$. We note that the condition $\mathrm{Inf}\, K_0(A) = 0$ implies that $\mathrm{tor}\, K_0(A) = 0$ and hence that the condition [8, 1] is met.

Unsplicing a six term exact sequence from [8] we get a short exact sequence

$$0 \longrightarrow K_0(A)\otimes\mathbb{Q}/\mathbb{Z} \xrightarrow{\tilde{\rho}_\infty} K_0(A;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\tilde{\beta}_\infty} \mathrm{tor}\, K_1(A) \longrightarrow 0,$$

which clearly splits because the group on the left is divisible. Also, a splitting map can be found by taking limits of a coherent family as given above. We get a diagram

$$\begin{tikzcd} K_0(A)\otimes\mathbb{Q} \arrow[r] \arrow[dr, "q"'] & K_0(A)\otimes\mathbb{Q}/\mathbb{Z} \arrow[r, "\tilde{\rho}_\infty"] \arrow[d, "\tilde{q}"] & K_0(A;\mathbb{Q}/\mathbb{Z}) \arrow[dl, dashed, "\alpha_\infty"] \\ & K_0^{\mathrm{Inf}}(A)\otimes\mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

and define as above:

Definition 3.3 We say that a C^*-algebra A has an *IMI_∞-splitting* when there exists a group homomorphism

$$\alpha_\infty : K_0(A;\mathbb{Q}/\mathbb{Z}) \to K_0^{\mathrm{Inf}}(A)\otimes\mathbb{Q}/\mathbb{Z}$$

with the properties:

(i) $\alpha_\infty \rho^0_\infty = q$,

(ii) whenever I is an ideal of A which is generated by projections,

$$\alpha_\infty(K_0(A\|I;\mathbb{Q}/\mathbb{Z})) \subseteq K_0^{\mathrm{Inf}}(A\|I) \otimes \mathbb{Q}/\mathbb{Z}.$$

Lemma 3.4 *Suppose that a C^*-algebra A has a coherent sequence (α_m) of IMI_m-splittings. Then A has an IMI_∞-splitting.*

Proof: Recall from [8] that $K_0(A,\mathbb{Q}/\mathbb{Z})$ can be defined as the inductive limit of the system $(K_0(A;\mathbb{Z}/m),\kappa_{sm,m})$. It is then clear that $\alpha_\infty = \varinjlim \alpha_m$ is an IMI_∞-splitting. ■

Remark 3.5 The existence of IMI_m- and IMI_∞-splittings is closely related to the property defining *ideally split* C^*-algebras in [16]. This will be explained elsewhere.

4 Local homogeneity

Definition 4.1 We say that a C^*-algebra A is *locally homogeneous* when for every $\varepsilon > 0$ and every finite set of elements $\mathcal{F} \subseteq A$ there exists a C^*-algebra

$$\bigoplus_{i=1}^k p_i \mathbf{M}_{n_i}(C(X_i))p_i,$$

where X_i is a compact metrizable space, p_i is a projection in $\mathbf{M}_{n_i}(C(X_i))$, and there is a $*$-homomorphism $\varphi : B \to A$ with $\mathrm{dist}(\mathcal{F},\varphi(B)) < \varepsilon$.

We show below that local homogeneity is not enough to ensure approximate homogeneity, not even for real rank zero and stable rank one C^*-algebras.

Theorem 4.2 *There exists a separable unital C^*-algebra E, which has real rank zero and stable rank one and is not an AH algebra but has the following properties:*

(i) E is locally homogeneous C^*-algebra.

(ii) E is an inductive limit of AH algebras.

(iii) E fits into a split and quasidiagonal extension

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

of AH algebras A and B.

Proof: Our example is only superficially different from one given in [12]. It will be clear from the proof that it satisfies a stronger version of local homogeneity in that the local approximation is done by algebras of the form

$$B = \bigoplus_{i=1}^{k} \mathbf{M}_{n_i}(C(X_i)),$$

with the X_i finite CW-complexes of dimension three.

Denote by D the two by two matrix algebra over the 3^∞ UHF algebra and by C the unique simple unital AD algebra of real rank zero with $K_0(C) = \mathbb{Z}[\frac{1}{3}]$ and $K_1(C) = \mathbb{Z}/2$ ordered by

$$K_\bullet(C)^+ = \{(x, y) \mid x > 0 \text{ or } (x = 0, y = 0)\}.$$

Recall that in the case

$$\operatorname{tor} K_0(-) = 0 \qquad m \operatorname{tor} K_1(-) = 0,$$

as described in [15] and [9], the invariant

$$\mathbf{K}(-; m): \qquad K_0(-) \xrightarrow{\rho_m^0} K_0(-; \mathbb{Z}/m) \xrightarrow{\beta_m^0} K_1(-)$$

is complete for AD algebras with real rank zero and slow dimension growth. There is by (a one-sided version of) Theorem 3.6 in [13] a unital $*$-homomorphism φ, unique up to approximately inner equivalence, having $\mathbf{K}(\varphi; 2)$ given by

$$\begin{array}{lccccc}
\mathbf{K}(C;2): & \mathbb{Z}[\frac{1}{3}] & \xrightarrow{\rho} & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \xrightarrow{\beta} & \mathbb{Z}/2 \\
 & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle [0\ 1]} & & \downarrow{\scriptstyle 0} \\
\mathbf{K}(D;2): & \mathbb{Z}[\frac{1}{3}] & \xrightarrow[\rho]{} & \mathbb{Z}/2 & \xrightarrow[\beta]{} & 0.
\end{array}$$

We are going to study the continuous field over the one point compactification of $\mathbb{N}$ defined as

$$E = \left\{ (c, (d_n)) \in C \oplus \prod_{n=1}^{\infty} D \,\middle|\, \|\varphi(c) - d_n\| \to 0 \right\}.$$

Note that E is an inductive limit of C^*-algebras of the form

$$E_k = C \oplus \bigoplus_{1}^{k} D$$

Since C and D are both AD algebras of real rank zero and stable rank one, so is E_k, and the same can be said of E.

Since one immediately finds that

$$K_0(E) = \left\{ (x,(y_n)) \in \mathbb{Z}[\tfrac{1}{3}] \oplus \prod_{n=1}^{\infty} \mathbb{Z}[\tfrac{1}{3}] \,\middle|\, y_n \to 2x \right\}$$

equipped with the positive cone

$$K_0(E)_+ = \{(x,(y_n)) \mid x \geq 0, y_n \geq 0\}$$

it follows that $\operatorname{Inf} K_0(E) = 0$. We have $K_1(E) = \mathbb{Z}/2$ and choosing generators appropriately, we get

$$K_0(E;\mathbb{Z}/2) = \left\{ (v,z,(w_n)) \in \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \prod_{1}^{\infty} \mathbb{Z}/2 \,\middle|\, w_n \to z \right\}$$

with

$$\rho_2^0(x,(y_n)) = (\overline{x},0,(\overline{y_n})) \qquad \beta_2^0(v,z,w_n) = z,$$

where $\overline{x} = x' + 2\mathbb{Z}$ for $x = x'/3^r$.

Assume that E had an IMI_2-splitting α_2, and consider the ideals $I_k \lhd E$ given by

$$I_k = \{(c,(d_n)) \mid d_k = 0\}.$$

We get

$$K_0(E\|I_k) \otimes \mathbb{Z}/2 = \{(\overline{x},(\overline{y}_n)) \in K_0(E) \otimes \mathbb{Z}/2 \mid \overline{y}_k = 0\}$$
$$K_0(E\|I_k;\mathbb{Z}/2) = \{(v,z,(w_n)) \in K_0(E;\mathbb{Z}/2) \mid w_k = 0\}$$

and since E has real rank zero, so that I_k is generated by projections, α_2 must preserve these sets. In particular, if

$$\alpha_2(v,z,(w_n)) = (\overline{x},(\overline{y}_n)) \text{ and } \alpha_2(v,z,(w_n')) = (\overline{x'},(\overline{y'}_n)),$$

then $\overline{y}_k = \overline{y}_k'$ whenever $w_k = w_k'$ for some k. On the other hand, as $\operatorname{Inf} K_0(E) = 0$ the map α_2 would be a genuine splitting map, and hence if $(\overline{x},0,(\overline{y}_n)) \in \operatorname{Im} \rho_2^0 = \mathbb{Z}/2 \oplus 0 \oplus \bigoplus \mathbb{Z}/2$, then

$$\alpha_2(\overline{x},0,(\overline{y}_n)) = (\overline{x},(\overline{y}_n)).$$

Let (w_n) be the sequence with all elements equal to $\overline{1}$. Then

$$\alpha_2(0,1,(w_n)) = (\overline{x},(\overline{y}_n)) \in K_0(E) \otimes \mathbb{Z}/2,$$

and we note that the observations above show that $\overline{y_n} = \overline{1}$ for all n. This contradicts the condition $\overline{y_n} \to 0$ which follows from the definition of $K_0(E)$.

We conclude from Theorem 3.2 that E is not an AH-algebra. On the other hand by 4.11 and 4.12 of [19] $K_\bullet(C)$ is a graded ordered group satisfying the Riesz interpolation property as well as the weak unperforation property. By Theorem 4.18 in [19] there is then an AH algebra C' of real rank zero with $K_\bullet(C) \simeq K_\bullet(C')$. In fact, one may use only 3-dimensional finite CW-complexes in the construction of C'. By [9] or [20] C' is isomorphic to C hence C is AH. Since D is UHF it follows that E_k is AH. This proves (ii). Note that (i) is a straightforward consequence of (ii). Finally, to prove (iii), note that the ideal of E corresponding to requiring $c = 0$ gives rise to an extension

$$0 \longrightarrow \bigoplus_1^\infty D \longrightarrow E \longrightarrow C \longrightarrow 0,$$

which splits by $c \mapsto (c, \varphi(c), \varphi(c), \dots)$ and in which the projections

$$e_n = (\overbrace{1, \dots, 1}^{n}, 0, \dots)$$

are central in E and form an approximate unit for $\bigoplus_1^\infty D$. ∎

5 Construction of IMI-splittings

This section is devoted to proving Theorem 3.2. We begin with a lemma related to the work of Pasnicu ([27]):

Lemma 5.1 *Let I be an ideal of an AH algebra $A = \varinjlim (A_n, \psi_{s,n})$. Then the following are equivalent:*

(i) I can be written as $I = \varinjlim I_n$ with I_n consisting of direct sum of full blocks of A_n.

(ii) I is generated, as an ideal, by its projections.

Proof: Let $\psi_n : A_n \to A$ denote the natural maps. For (i)$\Longrightarrow$(ii), note that if e_n is the unit of I_n, then $(\psi_n(e_n))$ will generate I as an ideal. In the other direction, let (p_r) be a generating sequence of projections for I. We allow the possibility that the sequence p_r has constant tail. By induction we find a sequence $i(1) < i(2) < \cdots$ and projections $e(1,r), e(2,r), ..., e(r,r)$ in $A_{i(r)}$ such that

$$\psi_{i(r+1),i(r)}(e(j,r)) = e(j, r+1), \qquad 1 \le j \le r \tag{1}$$

and

$$\psi_{i(r)}(e(r,r)) = p_r \tag{2}$$

for all r.

Let $I_{i(r)}$ be the ideal of $A_{i(r)}$ generated by the projections $e(1,r), e(2,r), \ldots, e(r,r)$. It is clear that $I_{i(r)}$ is equal to direct sum of full blocks of $A_{i(r)}$. Moreover $I_{i(r)}$ is taken into $I_{i(r+1)}$ by $\psi_{i(r+1),i(r)}$ because of (1). And by (2), the image in A contains every p_r and hence must be all of I. Finally for $i(r) < n < i(r+1)$ we let I_n be the ideal of A_n generated by $\psi_{n,i(r)}(e(j,r))$ with $1 \leq j \leq r$. ■

Proposition 5.2 *Let $A = \bigoplus_{i=1}^{k} p_i \mathbf{M}_{n_i}(C(X_i))p_i$ with each X_i a finite, connected CW-complex. There exists a sequence (α_m) of group homomorphisms*

$$\alpha_m = \alpha_{A;m} : K_0(A; \mathbb{Z}/m) \to K_0^{\mathrm{Inf}}(A) \otimes \mathbb{Z}/m$$

such that

(i) $\alpha_m \circ \rho_m^0 = q_m$.

(ii) $\alpha_s \circ \kappa_{s,m}^0 = (\mathrm{id}_{K_0^{\mathrm{Inf}}(A)} \otimes \kappa_{s,m}) \circ \alpha_m$ *for any* $s \in \mathbb{N}$.

(iii) $\varphi_* \circ \alpha_{A;m} = \alpha_{B;m} \circ \varphi_*$ *whenever* $\varphi : A \to B$ *is a* $*$*-homomorphism to a* C^**-algebra* B *of the same form.*

In other words, (iii) says that α_m is a natural transformation from $K_0(-; \mathbb{Z}/m)$ to $K_0^{\mathrm{Inf}}(-) \otimes \mathbb{Z}/m$ on the category whose objects are C^*-algebras of this special form, and whose morphisms all $*$-homomorphisms between them.

Proof: We introduce notation from [19]. When A is a C^*-algebra of the form

$$\bigoplus_{i=1}^{k} p_i \mathbf{M}_{n_i}(C(X_i))p_i$$

as above, we choose in each X_i a base point and define $M_{n_i}(C_0(X_i))$ as the C^*-algebra of matrix-valued functions vanishing at that point. We write

$$A^0 = \bigoplus_{i=1}^{k} p_i M_{n_i}(C_0(X_i))p_i \qquad rA = A/A^0.$$

The short exact sequences of C^*-algebras

$$0 \longrightarrow A^0 \xrightarrow{\iota} A \xrightarrow{\pi} rA \longrightarrow 0$$

induces a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & K_0(A^0) \arrow[r, "\times m"] \arrow[d, "\iota_*"] & K_0(A^0) \arrow[r, "\rho_m^0"] \arrow[d, "\iota_*"] & K_0(A^0;\mathbb{Z}/m) \arrow[r] \arrow[d, "\iota_*"] & K_1(A^0) \arrow[d, "\iota_*"] \\
 & K_0(A) \arrow[r, "\times m"] \arrow[d, "\pi_*"] & K_0(A) \arrow[r, "\rho_m^0"] \arrow[d, "\pi_*"] & K_0(A;\mathbb{Z}/m) \arrow[r] \arrow[d, "\pi_*"] & K_1(A) \arrow[d, "\pi_*"] \\
0 \arrow[r] & K_0(rA) \arrow[r, "\times m"] \arrow[d] & K_0(rA) \arrow[r, "\rho_m^0"] \arrow[d] & K_0(rA;\mathbb{Z}/m) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where the zeros in the last row result from the fact that $\dim rA < \infty$. The map $\pi_* : K_0(A) \to K_0(rA)$ is surjective since if $q \geq \dim(X_i)$ then $p_i \otimes 1_q$ has subprojections of rank one, by stability properties of vector bundles.

Now ρ_m^0 induces an isomorphism $\tilde{\rho}_m : K_0(rA) \otimes \mathbb{Z}/m \to K_0(rA;\mathbb{Z}/m)$ and π_* induces an isomorphism $\tilde{\pi} : K_0(A)/\iota_*(K_0(A^0)) \to K_0(rA)$. Note also that $\iota_*(K_0(A^0)) = \mathrm{Inf}(K_0(A))$ because the only positive group homomorphisms $f : K_0(p\mathbf{M}_n(C(X))p) \to \mathbb{R}$ are of the form

$$f(x) = \gamma \,\mathrm{rank}(x)$$

with $\gamma \in (0,\infty)$ and $\mathrm{rank}([e]-[f]) = \mathrm{rank}(e) - \mathrm{rank}(f)$ as sketched in Problem 6.3.10 of [1]. We define α_m as the composite homomorphism

$$K_0(A;\mathbb{Z}/m) \xrightarrow{\pi_*} K_0(rA;\mathbb{Z}/m) \xrightarrow{\tilde{\rho}_m^{-1}} K_0(rA) \otimes \mathbb{Z}/m \xrightarrow{\tilde{\pi}^{-1}\otimes \mathrm{id}} K_0^{\mathrm{Inf}}(A) \otimes \mathbb{Z}/m.$$

Using the naturality of ρ_m^0 one sees that α_m satisfies (i). Property (ii) is a consequence of the definition of α_m and the commutativity of the diagram

$$\begin{tikzcd}
K_0(A;\mathbb{Z}/m) \arrow[r, "\pi_*"] \arrow[d, "\kappa_{s,m}"] & K_0(rA;\mathbb{Z}/m) \arrow[r, "\tilde{\rho}_m^{-1}"] \arrow[d, "\kappa_{s,m}"] & K_0(rA) \otimes \mathbb{Z}/m \arrow[d, "\mathrm{id}\otimes\kappa_{s,m}"] \\
K_0(A;\mathbb{Z}/s) \arrow[r, "\pi_*"'] & K_0(rA;\mathbb{Z}/s) \arrow[r, "\tilde{\rho}_s^{-1}"'] & K_0(rA) \otimes \mathbb{Z}/s
\end{tikzcd}$$

The key observation for proving (iii), found in Corollary 3.15 of [19] (and implicit in the case $p_i = \mathbf{1}$ in 4.2.8 and 6.4.5 of [14]), is now that any $*$-homomorphism $\varphi : A \to B$ for A and B of this form is homotopic to $\psi : A \to B$ with $\psi(A^0) \subseteq B^0$. Because K-theory is a homotopy invariant, we may thus work with ψ_*, for which the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^0 \arrow[r, "\iota"] \arrow[d, "\psi"'] & A \arrow[r, "\pi"] \arrow[d, "\psi"'] & rA \arrow[r] \arrow[d, "\chi"'] & 0 \\
0 \arrow[r] & B^0 \arrow[r, "\iota"] & B \arrow[r, "\pi"] & rB \arrow[r] & 0
\end{tikzcd}$$

is available. This induces the following commutative diagram

$$\begin{CD}
K_0(A;\mathbb{Z}/m) @>{\pi_*}>> K_0(rA;\mathbb{Z}/m) @>{(\tilde{\rho}_m)_*^{-1}}>> K_0(rA)\otimes\mathbb{Z}/m @>{(\tilde{\pi})_*^{-1}\otimes\mathrm{id}}>> K_0^{\mathrm{Inf}}(A)\otimes\mathbb{Z}/m\\
@V{\psi_*}VV @V{\chi_*}VV @V{\chi_*}VV @V{\psi_*}VV\\
K_0(B;\mathbb{Z}/m) @>>{\pi_*}> K_0(rB;\mathbb{Z}/m) @>>{(\tilde{\rho}_m)_*^{-1}}> K_0(rB)\otimes\mathbb{Z}/m @>>{(\tilde{\pi})_*^{-1}\otimes\mathrm{id}}> K_0^{\mathrm{Inf}}(A)\otimes\mathbb{Z}/m
\end{CD}$$

which in view of the definition of α_m clearly implies (iii). ■

Proof of Theorem 3.2: We write $A = \varinjlim (A_n, \varphi_{s,n})$, where

$$A_n = \bigoplus_{i=1}^{k_n} p_{i,n}\mathbf{M}_{[i,n]}(C(X_{i,n}))p_{i,n}$$

for $[i,n] \in \mathbb{N}$, finite connected CW-complexes $X_{i,n}$, and projections $p_{i,n} \in M_{[i,n]}(C(X_{i,n}))$. Using Proposition 5.2 we get a commutative diagram

$$\begin{CD}
@>>> K_0(A_n;\mathbb{Z}/m) @>{(\psi_{n+1,n})_*}>> K_0(A_{n+1};\mathbb{Z}/m) @>>> \dots @>>> K_0(A;\mathbb{Z}/m)\\
@. @V{\alpha_m}VV @V{\alpha_m}VV @. @V{\alpha'_{A;m}}VV\\
@>>> K_0^{\mathrm{Inf}}(A_n)\otimes\mathbb{Z}/m @>>{(\psi_{n+1,n})_*}> K_0^{\mathrm{Inf}}(A_{n+1})\otimes\mathbb{Z}/m @>>> \dots @>>> (\varinjlim K_0^{\mathrm{Inf}}(A_n))\otimes\mathbb{Z}/m
\end{CD}$$

We set $\alpha_{A;m} = (k \otimes \mathrm{id}_{\mathbb{Z}/m}) \circ \alpha'_{A;m}$, where

$$k : \varinjlim K_0^{\mathrm{Inf}}(A_n) \to K_0^{\mathrm{Inf}}(\varinjlim A_n).$$

Using (i) of 5.2 one checks that $\alpha_m \circ \rho_m^0 = q_m$, so that property (i) of 3.1 is met. For property (ii), we let an ideal I of A generated by projections be given and apply Lemma 5.1 to get a diagram

$$\begin{CD}
\dots @>>> I_n @>{(\psi_{n+1,n})_*}>> I_{n+1} @>>> \dots @>>> I\\
@. @V{\iota}VV @V{\iota}VV @. @V{\iota}VV\\
\dots @>>> A_n @>>{(\psi_{n+1,n})_*}> A_{n+1} @>>> \dots @>>> A
\end{CD}$$

where every I_n is of the form required in Proposition 5.2. By repeating the construction for $\alpha'_{A;m}$ we get a group homomorphism $\alpha'_{I;m}$, such that the diagram

$$\begin{CD}
K_0(I;\mathbb{Z}/m) @>{\alpha'_{I;m}}>> \varinjlim K_0^{\mathrm{Inf}}(I_n)\otimes\mathbb{Z}/m @>{k}>> K_0^{\mathrm{Inf}}(I)\otimes\mathbb{Z}/m\\
@V{\iota_*}VV @V{\iota_*}VV @V{\iota_*}VV\\
K_0(A;\mathbb{Z}/m) @>>{\alpha'_{A;m}}> \varinjlim K_0^{\mathrm{Inf}}(A_n)\otimes\mathbb{Z}/m @>>{k}> K_0^{\mathrm{Inf}}(A)\otimes\mathbb{Z}/m
\end{CD}$$

commutes. It follows that $\alpha_m(K_0(A\|I;\mathbb{Z}/m)) \subseteq K_0^{\mathrm{Inf}}(A\|I) \otimes \mathbb{Z}/m$.

Therefore α_m is an IMI_m splitting. Finally using (ii) of 5.2 one checks that these splittings are compatible in the sense that $\alpha_s \circ \kappa_{s,m} = (\mathrm{id}_{K_0^{\mathrm{Inf}}(A)} \otimes \kappa_{s,m}) \circ \alpha_m$. ■

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