Journal of Functional Analysis **167**, 69–78 (1999)
Article ID jfan.1999.3436, available online at http://www.idealibrary.com on IDEAL®

On the Approximation of Quasidiagonal C*-Algebras

Marius Dadarlat*

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

Communicated by D. Sarason

Received March 9, 1998; accepted March 9, 1999

Let A be a separable exact quasidiagonal C*-algebra. Suppose that $\pi: A \to L(H)$ is a faithful representation whose image does not contain nonzero compact operators. Then there exists a sequence $\varphi_n: A \to L(H)$ of completely positive contractions such that $\|\pi(a) - \varphi_n(a)\| \to 0$ for all $a \in A$, and the C*-algebra generated by $\varphi_n(A)$ is finite dimensional for each n. As an application it is shown that if the C*-algebra generated by a quasidiagonal operator T is exact and does not contain any nontrivial compact operator, then T is norm-limit of block-diagonal operators $D = D_1 \oplus D_2 \oplus \cdots$ with $\sup_i$ *rank*$(D_i) < \infty$. © 1999 Academic Press

1. INTRODUCTION

Let $\pi: A \to L(H)$ be a representation of a separable exact C*-algebra. Then π is nuclear [Ki_2], hence it is a point-norm limit of completely positive contractions $\varphi_n: A \to L(H)$ such that $\varphi_n(A)$ are finite dimensional subspaces of $L(H)$. In general one cannot arrange that the C*-algebra generated by $\varphi_n(A)$ is finite dimensional for that would imply that $\pi(A)$ is a quasidiagonal set of operators. In this note we show that if A is exact and $\pi(A) \cap K(H) = \{0\}$, then the quasidiagonality of $\pi(A)$ is the only obstruction to such an approximation.

In this introductory part, we give some background and discuss some motivation. An open question will be formulated at the end of the paper.

Let H be an infinite dimensional complex separable Hilbert space. We denote by $L(H)$ the linear bounded operators on H and by $K(H)$ the compact operators. If E is a subset of $L(H)$ we denote by $C^*\{E\}$ the C*-subalgebra of $L(H)$ generated by E. The quasidiagonal operators were introduced in [H]. A set of operators $B \subset L(H)$ is quasidiagonal if there is an increasing sequence (p_n) of finite dimensional selfadjoint projections converging to one, such that $\|bp_n - p_n b\| \to 0$ for all $b \in B$. A separable

* Partially supported by NSF Grant DMS-9622434.

0022-1236/99 $30.00

C*-algebra A is quasidiagonal if there is a faithful representation $\pi: A \to L(H)$ such that $\pi(A) \subset L(H)$ is a quasidiagonal set of operators. By Voiculescu's Theorem [Vo_1], if a separable C*-algebra A is quasidiagonal, then $\pi(A)$ is a quasidiagonal set of operators for any faithful, essential representation of A. By an *essential* representation we mean a representation whose image does not contain nonzero compact operators.

It is easy to see that an operator is quasidiagonal if and only if it is norm-limit of block-diagonal operators [H]. An operator $T \in L(H)$ is called block-diagonal if $T = T_1 \oplus T_2 \oplus \cdots$ for some decomposition $H = H_1 \oplus H_2 \oplus \cdots$ with all H_i finite dimensional. If in addition one could arrange that $\sup\{\dim(H_i): i = 1, 2, ...\} = k < \infty$, we write that $T \in BD(k)$ and say that T is block-diagonal with blocks of bounded dimension ($\leqslant k$). Answering a question of Herrero, Szarek [Sz] has shown that the union of $BD(k)$ with $k \geqslant 1$ is not dense in the set of block-diagonal operators (or equivalently, in the set of quasidiagonal operators). Two other proofs of this result were given in [Vo_2, Vo_3]. Voiculescu makes the remark that T is approximable in norm by operators in the union of $BD(k)$ if and only if T is the norm-limit of a sequence of operators (T_n) such that $C^*\{T_n\}$ is finite dimensional for all n. That implies that the inclusion $C^*\{T\} \hookrightarrow L(H)$ must be a nuclear map [Vo_2]. Therefore, the existence of non-approximable block-diagonal operators is reduced, essentially, to the existence of separable quasidiagonal C*-algebras that are not nuclearly embeddable. Voiculescu [Vo_3] uses discrete residually finite dimensional groups with the property T of Kazhdan, such as $SL_3(\mathbb{Z})$, to exhibit concrete example of such algebras and hence of block-diagonal operators which are not approximable by operators in the union of $BD(k)$. Subsequently Wassermann [Wa_1] shows that nuclearly embeddable C*-algebras are exact and shows that the $C^*(\mathbb{F}_2) \otimes M_4(\mathbb{C})$ has a generator which can be represented as a block-diagonal operator without the approximability property discussed above.

As opposed to these examples, we show that the left regular representation of finitely generated, discrete, amenable, residually finite groups can be approximated in the point-norm topology by representations with finite dimensional image (see Corollary 4).

Davidson, Herrero, and Salinas [DHS] have shown that in fact the union of $BD(k)$ is nowhere dense in the set of block-diagonal operators. In view of the results of [Vo_2] they ask whether $T \in \overline{\bigcup_k BD(k)}$ provided that T is a quasidiagonal operator with $C^*\{T\}$ nuclear. We offer a positive answer to this question under the additional assumption that the $C^*\{T\}$ does not contain any nonzero compact operators (see Corollary 8). The study of nuclear quasidiagonal C*-algebras goes back to Salinas [Sa]. A new approach to this class is developed by Blackadar and Kirchberg [BK_1, BK_2].

For C*-algebras A, B we let CP(A, B) denote the set of all linear completely positive contractions from A to B. Such a map will be called a CP-contraction. A C*-algebra A is called *nuclearly embeddable* if there is a nuclear $*$-monomorphism $\iota: A \to B$ for some C*-algebra B. If A is separable, B can be chosen to be separable. As an easy application of Arveson's extension theorem one checks that any CP-contraction from a nuclearly embeddable C*-algebra to $L(H)$ is nuclear. Any nuclearly embeddable C*-algebra is exact by a result of Wassermann [Wa_1]. Conversely, any exact C*-algebra is nuclearly embeddable as proved by Kirchberg [Ki_1]. In particular exactness passes to subalgebras.

Much more is true. Quotients of a separable exact C*-algebras are exact. In fact the class of separable exact C*-algebras coincides with the class of separable subnuclear C*-algebras [Ki_2]. A rather selfcontained proof that any separable exact C*-algebra is embeddable as a C*-subalgebra of the Cuntz algebra $\mathcal{O}_2$ is given in [KiPh]. We refer the reader to [Pa, Wa_2, Ki_2] for basic facts on complete positivity, nuclearity and exactness.

Let us emphasize, that in this paper we only need to use the equivalence between nuclear embeddability and exactness. A concise proof of this equivalence is presented in Chapter 7 of [Wa_2].

2. RESULTS

Let H be an infinite dimensional separable Hilbert space and let (H_n) be an increasing chain of finite dimensional linear subspaces of H whose union is dense in H. If B is a C*-algebra, let H_B denote the Hilbert B-module $H \otimes B$. As in [Ka], $L(H)$ is regarded as the subalgebra of scalar operators in $L(H_B)$.

THEOREM 1 [Ka]. *Let A be a separable C*-algebra and let $\rho: A \to L(H) \subset L(H_B)$ be a faithful representation such that $\rho(A) \cap K(H) = \{0\}$. Let $\varphi: A \to B$ be a nuclear CP-contraction. Then there is a sequence (ξ_n) in H_B such that*

(i) $\lim_{n\to\infty} \|\varphi(a) - \langle \xi_n, \rho(a)\, \xi_n \rangle\| = 0$ *for all* $a \in A$.

If A, B are unital, $\rho(1) = 1$ and $\varphi(1) = 1$, then we may arrange that

(ii) $\langle \xi_n, \xi_n \rangle = 1_B$ *and* $\xi_n \in H_{j(n)} \otimes B$ *for all n, where $(j(n))$ is some increasing sequence.*

Proof. Part (i) is a consequence of [Ka, Theorem 4]. For part (ii), assuming that (ξ_n) is as in (i), we see that $\langle \xi_n, \xi_n \rangle \to 1_B$ since both φ and ρ are unital. Next, since the union of $H_n \otimes B$ is dense in H_B we find $h_n \in$

$H_{j(n)} \otimes B$ with $\|\xi_n - h_n\| \to 0$, hence $\langle h_n, h_n \rangle \to 1_B$. Then for n large $\eta_n := h_n \langle h_n, h_n \rangle^{-1/2}$ is a well defined vector in $H_{j(n)} \otimes B$ with $\langle \eta_n, \eta_n \rangle = 1_B$ and $\|\xi_n - \eta_n\| \to 0$. Together with (i) this gives

$$\lim_{n \to \infty} \|\varphi(a) - \langle \eta_n, \rho(a)\, \eta_n \rangle\| = 0$$

for all $a \in A$. ∎

A C*-algebra A is called residually finite dimensional (abbreviated RFD) if for any nonzero element $a \in A$, there is a finite dimensional representation π of A such that $\pi(a) \neq 0$. The following proposition is a key tool for our approach.

PROPOSITION 2. *Let A be a separable RFD C*-algebra and let $\varphi\colon A \to B$ be a nuclear $*$-homomorphism to a unital C*-algebra B. Then there is a sequence $\tau_n\colon A \to M_{r(n)-1}(B)$ of CP-contractions and there is a sequence $\mu_n\colon A \to M_{r(n)}(B)$ of $*$-homomorphisms with finite dimensional image such that*

$$\lim_{n \to \infty} \|\mathrm{diag}(\varphi(a), \tau_n(a)) - \mu_n(a)\| = 0$$

for all $a \in A$. The $*$-homomorphisms μ_n are of the form $\mu_n(a) = u_n(\rho_n(a) \otimes 1_B)\, u_n^*$ where $\rho_n\colon A \to M_{r(n)}(\mathbb{C})$ are $*$-representations and $u_n \in M_{r(n)}(B)$ are unitaries. If A is unital and $\varphi(1_A) = 1_B$, then we may arrange that τ_n and μ_n are unital.

Proof. After replacing φ by its unital extension $\tilde{\varphi}\colon \tilde{A} \to B$ (which is also a nuclear CP-contraction by [ChE]), we may assume that A is unital and $\varphi(1) = 1$. Let $\pi_n\colon A \to L(K_n)$ be a separating sequence of unital finite dimensional $*$-representations such that each π_n repeats infinitely many times in the sequence. Let $H_n = K_1 \oplus \cdots \oplus K_n$. Let (ξ_n) be given by Theorem 1(ii), applied for φ and $\rho = \bigoplus_{n=1}^{\infty} \pi_n$. Define $\rho_n\colon A \to L(H_{j(n)})$ by $\rho_n = \pi_1 \oplus \cdots \oplus \pi_{j(n)}$. If $k(n)$ denotes the dimension of $H_{j(n)}$, then $H_{j(n)} \otimes B \cong B^{k(n)}$ and $L(H_{j(n)} \otimes B) \cong M_{k(n)}(B)$. The isometry $V_n \in L(B, H_{j(n)} \otimes B)$, $V_n(b) = \xi_n b$ corresponds to a partial isometry $v_n \in M_{k(n)}(B)$ with initial support $e_n = v_n^* v_n = 1_B \otimes e_{11}$ and final support $f_n = v_n v_n^*$. Identifying B with the $(1, 1)$-corner of $M_{k(n)}(B)$, we have from Theorem 1(i), $\|\varphi(a) - v_n^* \rho_n(a)\, v_n\| \to 0$, for all $a \in A$. It is clear that $\rho(a)\, f_n = \rho_n(a)\, f_n$ and as in [Ar, p. 348], one checks that $\|v_n \varphi(a) - \rho_n(a)\, v_n\| \to 0$ for $a \in A$. This follows from the identity

$$\begin{aligned} &(v_n \varphi(a) - \rho_n(a)\, v_n)^* \, (v_n \varphi(a) - \rho_n(a)\, v_n) \\ &\quad = (v_n^* \rho_n(a^*a)\, v_n - \varphi(a^*a)) \\ &\qquad + (\varphi(a^*) - v_n^* \rho_n(a^*)\, v_n)\, \varphi(a) + \varphi(a^*)(\varphi(a) - v_n^* \rho(a)\, v_n). \end{aligned}$$

Moreover, using

$$[\rho_n(a), v_n v_n^*] = (\rho_n(a)\, v_n - v_n \varphi(a))\, v_n^* + v_n(v_n \varphi(a^*) - \rho_n(a^*)\, v_n)^*$$

we see that $\|[\rho_n(a), f_n]\| \to 0$ for all $a \in A$. Set $r(n) = 2k(n)$ and let u_n be a unitary element of $M_{r(n)}(B)$ defined by

$$u_n = \begin{pmatrix} v_n & 1 - f_n \\ 1 - e_n & v_n^* \end{pmatrix},$$

where 1 denotes the unit of $B^{k(n)}$. Define the maps μ_n, τ_n and φ_n by

$$\mu_n(a) = u_n^* \begin{pmatrix} \rho_n(a) & 0 \\ 0 & \rho_n(a) \end{pmatrix} u_n,$$

$$\tau_n(a) = u_n^* \begin{pmatrix} (1 - f_n)\, \rho_n(a)(1 - f_n) & 0 \\ 0 & \rho_n(a) \end{pmatrix} u_n,$$

$$\varphi_n(a) = u_n^* \begin{pmatrix} f_n \rho_n(a)\, f_n & 0 \\ 0 & 0 \end{pmatrix} u_n = \begin{pmatrix} v_n^* \rho_n(a)\, v_n & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\|\varphi_n(a) + \tau_n(a) - \mu_n(a)\| \leqslant \|(1 - f_n)\, \rho_n(a)\, f_n\| + \|f_n \rho_n(a)(1 - f_n)\| \to 0$$

since $\|[\rho_n(a), f_n]\| \to 0$. Since $\varphi_n(a) \to \varphi(a)$, we obtain that

$$\|\varphi(a) + \tau_n(a) - \mu_n(a)\| \to 0$$

as $n \to \infty$, for all $a \in A$. Note that $\varphi(1_A) = 1_B \otimes e_{11} = e_n$ and $\tau_n(1_A) = \left(\begin{smallmatrix} 1 - e_n & 0 \\ 0 & 1 \end{smallmatrix}\right)$, hence we may identify $\varphi + \tau_n$ with $diag(\varphi, \tau_n)$. Finally note that μ_n has the desired form and that both μ_n and τ_n are unital whenever φ is unital. ∎

Proposition 3. *Let A be a separable C^*-algebra and let $\pi \colon A \to L(H)$ be a faithful essential representation. Then A is nuclearly embeddable (exact) and RFD if and only if there exists a sequence $\rho_n \colon A \to L(H)$ of representations with finite dimensional image such that $\|\pi(a) - \rho_n(a)\| \to 0$ for all $a \in A$.*

Proof. ($\Leftarrow$) π is faithful and nuclear since it is approximable by representations with finite dimensional image. This shows that A is nuclearly embeddable. A is RFD since (ρ_n) is a separating family, and each ρ_n is an (infinite) multiple of some finite dimensional representation.

($\Rightarrow$) We may assume that π is nondegenerate. By Voiculescu's Theorem [V_1], any two faithful, essential, nondegenerate representations of A are approximately unitarily equivalent. Therefore it suffices to prove that there exists some faithful essential representation π' that has the stated approximation property. We consider first the case when A is unital and π is unit-preserving. Since A is nuclearly embeddable, there is a nuclear unital $*$-monomorphism $\iota: A \to B$ to some separable unital C*-algebra B. Let $F \subset A$ be fixed. Following [Ar], for two maps $\varphi_i: A \to L(H_i)$, $i = 1, 2$ and $\delta > 0$ we will write $\varphi_1 \sim_\delta \varphi_2$ if there is a unitary $v: H_2 \to H_1$ such that

$$\|\varphi_1(a) - v\varphi_2(a)\, v^*\| < \delta$$

for all $a \in F$. This is a symmetric relation and $\varphi_1 \sim_\delta \varphi_2$ together with $\varphi_2 \sim_\eta \varphi_3$ implies that $\varphi_1 \sim_{\delta+\eta} \varphi_3$. Suppose now that $F \subset A$ is finite and fix $\varepsilon > 0$. To prove the proposition it is enough to find a unital representation $\rho': A \to L(H')$ with finite dimensional image and a faithful essential representation $\pi': A \to L(H)$ such that $\pi' \sim_{4\varepsilon} \rho'$.

By Proposition 2, applied for $\varphi = \iota$, there is a unital CP-contraction $\tau: A \to M_{n-1}(B)$, $n \in \mathbb{N}$, and there exist a unital representation $\rho: A \to M_n(\mathbb{C})$ and a unitary $u \in U_n(B)$ such that

$$\|diag(\iota(a), \tau(a)) - u(\rho(a) \otimes 1_B)\, u^*\| < \varepsilon \tag{1}$$

for all $a \in F$. Let $\pi: B \to L(H)$ be a unital, faithful, essential representation. If r is a positive integer, let $\pi_r = id_r \otimes \pi : M_r(B) \to M_r(L(H))$. Let I denote the identity operator on H. Using (1) we have

$$\begin{aligned} &\|diag(\pi\iota(a), \pi_{n-1}\tau(a)) - \pi_n(u)(\rho(a) \otimes I)\, \pi_n(u)^*\| \\ &\quad = \|\pi_n(diag(\iota(a), \tau(a)) - u(\rho(a) \otimes 1_B)\, u^*)\| < \varepsilon \end{aligned} \tag{2}$$

for all $a \in F$. Letting $\pi' = \pi\iota$, $\lambda = \pi_{n-1}\tau$, and $\rho' = \rho \otimes I$, we have from (2)

$$\rho' \sim_\varepsilon \pi' \oplus \lambda. \tag{3}$$

Since π' is unital faithful and essential, by Voiculescu's Theorem, we have

$$\pi' \oplus \pi' \sim_\varepsilon \pi', \tag{4}$$

$$\pi' \oplus \rho' \sim_\varepsilon \pi'. \tag{5}$$

Combining (3), (4), and (5)

$$\rho' \sim_\varepsilon \pi' \oplus \lambda \sim_\varepsilon \pi' \oplus \pi' \oplus \lambda \sim_\varepsilon \pi' \oplus \rho' \sim_\varepsilon \pi'.$$

It follows that $\pi' \sim_{4\varepsilon} \rho'$. This concludes the proof in the unital case, since ρ and hence $\rho' = \rho \otimes I$ has finite dimensional image. Now assume that A is non-unital. Let $\tilde{A}$ be the unitization of A and let $\tilde{\pi}: \tilde{A} \to L(H)$ be the unital extension of π. Since π is faithful and essential, so is $\tilde{\pi}$ (as A is nonunital.) Thus the proof of the non-unital case reduces to the unital case. ∎

If A is RFD, then any of its representations can be approximated by finite dimensional representations in a weaker topology, related to Fell's topology (see [ExL]).

Recall that a discrete group G is called residually finite if, for any finitely many distinct elements $g_1, g_2, ..., g_n \in G$, there is a morphism $\theta: G \to H$ to a finite group H such that $\theta(g_1), \theta(g_2), ..., \theta(g_n)$ are distinct. A finitely generated group is residually finite if and only if it has a separating family of finite dimensional representations ([Wa_2, p. 25]).

COROLLARY 4. *Let G be a discrete, countable, amenable, residually finite group. If $\lambda: G \to L(\ell^2(G))$ is the left regular representation of G, then there is a sequence of representations $\rho_n: G \to L(\ell^2(G))$ with finite dimensional image such that $\|\lambda(g) - \rho_n(g)\| \to 0$ for all $g \in G$. Conversely, if G is discrete, finitely generated, and λ is approximable by representations with finite dimensional image, then G is residually finite and amenable.*

Proof. We may assume that G is infinite. In that case $C_r^*(G) \cap K(\ell^2(G)) = \{0\}$; (we are indebted to Pierre de la Harpe for showing us a one-line proof of this, based on an argument in [dH]). If G is residually finite, then the natural map $C^*(G) \to C_r^*(G)$ factors through the direct sum of a family of finite dimensional representations of $C^*(G)$. The proof of this fact is similar to the proof of Proposition 3.3 in [Wa_2]. Thus, if in addition G is amenable, then $C^*(G) \cong C_r^*(G)$ is nuclear and RFD. Therefore the first part of the statement follows from Proposition 3.

Suppose now that λ is approximable by (ρ_n) as above. Then $C_r^*(G)$ is quasidiagonal, hence G is amenable by [Ro]. It is also clear that (ρ_n) is a separating family, and each ρ_n is an infinite multiple of some finite dimensional representation. ∎

LEMMA 5. *Let B be a separable C^*-algebra and let A be a subalgebra of B such that $\overline{AB} = B$. Let $\pi: A \to L(H)$ be a faithful, essential representation. Then there exists a sequence $\pi_n: B \to L(H)$ of faithful, essential representations such that $\|\pi(a) - \pi_n(a)\| \to 0$ for all $a \in A$.*

Proof. Without any loss of generality we may assume that π is nondegenerate. Let $\rho: B \to L(K)$ be a nondegenerate, faithful, essential representation. Let ρ_A denote the restriction of ρ to A. Then ρ_A is nondegenerate, as $\overline{AB} = B$.

By Voiculescu's Theorem there is a sequence of unitaries $u_n: K \to H$ such that

$$\|\pi(a) - u_n \rho_A(a) u_n^*\| \to 0$$

for all $a \in A$. Define $\pi_n: B \to L(H)$ by $\pi_n(b) = u_n \rho(b) u_n^*$. Then it is clear that the sequence (π_n) satisfies the conclusion of the lemma. ∎

THEOREM 6. *Let A be a separable exact quasidiagonal C^*-algebra. Let $\pi: A \to L(H)$ be a faithful representation such that $\pi(A) \cap K(H) = \{0\}$. Then there exists a sequence $\varphi_n: A \to L(H)$ of completely positive contractions such that $C^*\{\varphi_n(A)\}$ is finite dimensional for all $n \geqslant 1$ and $\lim_{n \to \infty} \|\pi(a) - \varphi_n(a)\| = 0$ for all $a \in A$.*

Proof. Let $A \subset L(E)$ be a realization of A as a concrete algebra of operators on a separable Hilbert space E, such that $A \cap K(E) = \{0\}$, and $\overline{AE} = E$, hence $\overline{A(A + K(E))} = A + K(E)$. Note that $A + K(E)$ is exact by virtue of being a (semi)split extension of the compacts by the exact algebra A [Ki$_2$]. Indeed if

$$0 \to J \to C \to B \to 0$$

is a semisplit exact sequence of C*-algebras with J and B exact, then it is easy to check that by taking minimal tensor product with any C*-algebra D, one obtains an exact sequence

$$0 \to J \otimes D \to C \otimes D \to B \otimes D \to 0.$$

By the 3×3-Lemma and the definition of exactness, this shows that C is exact.

Fix $\{a_1, ..., a_m\} \subset A$ and $\varepsilon > 0$. We will find a CP-contraction $\varphi: A \to L(H)$ such that $\|\pi(a_i) - \varphi(a_i)\| < 4\varepsilon$, $1 \leqslant i \leqslant m$, with $C^*\{\varphi(A)\}$ finite dimensional. Since $A \subset L(E)$ is quasidiagonal, reasoning as in the proof of [Ar, Theorem 2], we find operators $x_i \in L(E)$, $1 \leqslant i \leqslant m$, which are simultaneously block-diagonal with respect to a decomposition $E = E_1 \oplus E_2 \oplus \cdots$ with all E_i finite dimensional and such that

$$\|a_i - x_i\| < \varepsilon \qquad \text{and} \qquad a_i - x_i \in K(E), \qquad 1 \leqslant i \leqslant m.$$

If $D = C^*\{x_1, ..., x_m\}$, then $D \subset A + K(E)$ hence D is exact. Moreover, since $D \subset L(E_1) \oplus L(E_2) \oplus \cdots$, we see that D is RFD being a subalgebra of an RFD algebra.

By Lemma 5, there is a faithful essential representation $\sigma: A + K(E) \to L(H)$ such that $\|\pi(a_i) - \sigma(a_i)\| < \varepsilon$, $1 \leqslant i \leqslant m$. Note that the restriction of σ to D is a faithful essential representation. By Proposition 3, there is a representation $\rho: D \to L(H)$ with finite dimensional image and such that $\|\sigma(x_i) - \rho(x_i)\| < \varepsilon$ for $1 \leqslant i \leqslant m$. Let $\varphi: A + K(E) \to \rho(D)$ be a completely positive contractive extension of ρ given by Arveson's Extension Theorem [Pa]. We have

$$\|\pi(a_i)-\varphi(a_i)\| \leqslant \|\pi(a_i)-\sigma(a_i)\| + \|\sigma(a_i)-\sigma(x_i)\| + \|\sigma(x_i)-\rho(x_i)\| + \|\varphi(x_i)-\varphi(a_i)\| < 4\varepsilon$$

for $1 \leqslant i \leqslant m$. We conclude by noting that the C*-algebra generated by $\varphi(A)$ is contained in $\rho(D)$ hence it is finite dimensional. ∎

COROLLARY 7. *Let A be a separable, simple, exact, quasidiagonal C*-algebra. Then any representation of A is a point-norm limit of CP-contractions φ_n such that $C^*\{\varphi_n(A)\}$ is finite dimensional for all $n \geqslant 1$.*

Proof. Let $\pi: A \to L(H)$ be a representation of A. We may assume that $\pi \neq 0$, hence π is faithful as A is simple. Since $\pi(A) \cap K(H)$ is a two-sided closed ideal of $\pi(A)$, we must have either $\pi(A) \cap K(H) = \{0\}$, in which case $\pi(A)$ is quasidiagonal so that the result follows from Theorem 6, or $\pi(A) = \pi(A) \cap K(H) \subset K(H)$, in which case we can take $\varphi_n(a) = p_n \pi(a)\, p_n$, $a \in A$, where (p_n) is a sequence of finite dimensional selfadjoint projections converging strongly to I_H. ∎

COROLLARY 8. *Let $T \in L(H)$ be a quasidiagonal operator such that $C^*\{T\} \cap K(H) = \{0\}$. Then $C^*\{T\}$ is nuclearly embeddable (exact), if and only if for any $\varepsilon > 0$, there is $k \in \mathbb{N}$ and there exists an operator $S \in BD(k)$ (i.e., S is block-diagonal with all blocks of dimension $\leqslant k$) such that $\|T - S\| < \varepsilon$.*

Proof. ($\Rightarrow$) This implication follows from Theorem 6, by elementary facts from the representation theory of finite dimensional C*-algebras.

($\Leftarrow$) This is proved in [Vo_2, Proposition 1.2].

We end this note with the following open question.

QUESTION. *Let A be a separable exact C*-algebra. Suppose that $\pi: A \to L(H)$ is a representation such that $\pi(A)$ is a quasidiagonal set of operators. Does there exist a sequence $\varphi_n: A \to L(H)$ of completely positive contractions such that $\|\pi(a) - \varphi_n(a)\| \to 0$ for all $a \in A$ and $C^*\{\varphi_n(A)\}$ is finite dimensional for all $n \geqslant 1$?*

Note that, by Theorem 6, the above question has a positive answer if one further assumes that $\pi(A) \cap K(H) = \{0\}$.

ACKNOWLEDGMENT

We thank the referee for calling our attention to [KiPh, Wa_1].

REFERENCES

[Ar] W. B. Arveson, Notes on extensions of C*-algebras, *Duke Math. J.* **44** (1977), 329–355.

[BK_1] B. Blackadar and E. Kirchberg, Generalized inductive limits of finite-dimensional C*-algebras, *Math. Ann.* **307** (1997), 343–380.

[BK_2] B. Blackadar and E. Kirchberg, Inner quasidiagonality and strong NF algebras, preprint, 1995.

[ChE] M.-D. Choi and E. G. Effros, The completely positive lifting problem for C*-algebras, *Ann. of Math.* **104** (1976), 585–609.

[DHS] K. Davidson, D. Herrero, and N. Salinas, Quasidiagonal operators, approximation and C*-algebras, *Indiana Univ. Math. J.* **38** (1989), 973–998.

[dH] P. de la Harpe, Operator algebras, free groups and other groups, *Asterisque* **232** (1995), 121–153.

[ExL] R. Exel and T. A. Loring, Finite-dimensional representation of free product C*-algebras, *Internat. J. Math.* **3** (1992), 469–476.

[H] P. R. Halmos, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.* **76** (1970), 887–933.

[Ka] G. G. Kasparov, Hilbert C*-modules: Theorems of Stinespring and Voiculescu, *J. Operator Theory* **4** (1980), 133–150.

[Ki_1] E. Kirchberg, On subalgebras of the CAR-algebra, *J. Funct. Anal.* **129** (1995), 35–63.

[Ki_2] E. Kirchberg, Exact C*-algebras, tensor products, and classification of purely infinite algebras, *in* "Proc. ICM Zurich 1994," pp. 943–954, Birkhäuser, Basel, 1995.

[KiPh] E. Kirchberg and N. C. Phillips, Embedding of exact C*-algebras and continuous fields in the Cuntz algebra $\mathcal{O}_2$, preprint, December 1997.

[Pa] V. Paulsen, "Completely Bounded Maps and Dilations," Pitman Research Notes in Mathematics, Vol. 146, Longman, Harlow/New York, 1986.

[Ro] J. Rosenberg, Quasidiagonality and nuclearity, appendix to strongly quasidiagonal C*-algebras by D. Hadwin, *J. Operator Theory* **18** (1987), 15–18.

[Sa] N. Salinas, Homotopy invariance of $Ext(A)$, *Duke J. Math.* **44** (1977), 777–794.

[Sz] S. J. Szarek, An exotic quasidiagonal operator, *J. Funct. Anal.* **89** (1990), 274–290.

[Vo_1] D. Voiculescu, A non-commutative Weyl–Von Neumann theorem, *Rev. Roumaine Math. Pures Appl.* **21** (1976), 97–113.

[Vo_2] D. Voiculescu, A note on quasidiagonal operators, *in* "Operator Theory: Advances and Applications," Vol. 32, pp. 265–274, Birkhäuser Verlag, Basel, 1988.

[Vo_3] D. Voiculescu, Property T and approximations of operators, *Bull. London Math. Soc.* **22** (1990), 25–30.

[Wa_1] S. Wassermann, Tensor products of free-group C*-algebras, *Bull. London Math. Soc.* **22** (1990), 375–380.

[Wa_2] S. Wassermann, Exact C*-algebras and related topics, *in* "Lecture Notes Ser.," Vol. 19, Seoul Nat. Univ., Seoul.