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The Bockstein Map is Necessary

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Abstract. We construct two non-isomorphic nuclear, stably finite, real rank zero C^*-algebras E and E' for which there is an isomorphism of ordered groups $\Theta\colon \bigoplus_{n\geq 0} K_\bullet(E;\mathbb{Z}/n) \to \bigoplus_{n\geq 0} K_\bullet(E';\mathbb{Z}/n)$ which is compatible with all the coefficient transformations. The C^*-algebras E and E' are not isomorphic since there is no Θ as above which is also compatible with the Bockstein operations. By tensoring with Cuntz's algebra $\mathcal{O}_\infty$ one obtains a pair of non-isomorphic, real rank zero, purely infinite C^*-algebras with similar properties.

0 Introduction

Elliott has initiated a program for a classification theory of nuclear C^*-algebras. The invariants should be based on K-theory as in the prototypical case of AF algebras. It is also plainly clear that the invariants should encode the ideal structure of the C^*-algebras. For AF algebras A or more generally for C^*-algebras of real rank zero and stable rank one the lattice of ideals is completely described by the order ideals of $K_0(A)$ [16]. This invariant was enlarged by adding an order structure on $K_\bullet(A) = K_0(A) \oplus K_1(A)$ [8], [11]. Examples exhibited by Gong [12] showed that the invariant is not complete even for real rank zero approximate homogeneous C^*-algebras (AH algebras). The effort of several authors [6], [10], [7], [3] led to a refined invariant, the total K-theory group

$$\underline{\mathbf{K}}(A) = \bigoplus_{n\geq 0} K_\bullet(A;\mathbb{Z}/n)$$

endowed with a certain order structure and acted upon by the natural coefficient transformations $\rho_n^i\colon K_i(A) \to K_i(A;\mathbb{Z}/n)$, $\kappa_{m,n}^i\colon K_i(A;\mathbb{Z}/n) \to K_i(A;\mathbb{Z}/m)$ and by the Bockstein maps $\beta_n^i\colon K_i(A;\mathbb{Z}/n) \to K_{i+1}(A)$ studied in [15] and [14]. It turns out that $\underline{\mathbf{K}}(A)$ gives a better description of how the various ideals of A are glued together. That feature was illustrated by a classification result of [3] according to which a large class of C^*-algebras of real rank zero including the AH algebras with slow dimension growth and the AD algebras are classified by $\underline{\mathbf{K}}(A)$. The AD algebras are inductive limits of certain subhomogeneous C^*-algebras with one dimensional spectrum and torsion K_1 groups [11]. Let us emphasize that the effectiveness of $\underline{\mathbf{K}}(-)$ comes from the integration of the coefficient and the Bockstein maps along with the order structure. If one disregards the order, then any KK-equivalent C^*-algebras have isomorphic $\underline{\mathbf{K}}(-)$ groups, irrespective of their ideal structure. On the other hand, the coefficient transformations and the Bockstein maps are a key part of $\underline{\mathbf{K}}(-)$. It is a goal of this note to highlight their role.

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There are a number of interesting cases when $\underline{\mathbf{K}}(A)$ (including its order structure) simplifies dramatically. For real rank zero AD algebras with n tor $K_1(A) = 0$,

$$K_0(A) \xrightarrow{\rho_n^0} K_0(A;\mathbb{Z}/n) \xrightarrow{\beta_n^0} K_1(A)$$

is a complete invariant [10]. Moreover if $nK_1(A) = 0$ then β_n^0 and $K_1(A)$ can be disregarded. For general AD algebras $\underline{\mathbf{K}}(A)$ can be substituted by an invariant based on K-theory with coefficients $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ [9], [1]. Other reduction results were given in [4]. The existence of many reduction results might suggest that due to some interesting interplay between the order and the algebraic structure of the K-groups in question, some of the natural maps in $\underline{\mathbf{K}}(A)$ were redundant. It is relevant in this respect to note that the non-isomorphism of the algebras in the examples of [12] can be proved by just using the ordered groups $K_\bullet(-;\mathbb{Z}\oplus\mathbb{Z}/n)$ of [6]. The fact that ρ_n^i is necessary was proved in [4] for AH algebras and in [9] for AD algebras. However the question whether or not the odd natural transformation β was at all necessary to classify the C^*-algebras studied in the papers mentioned above, turned out to be more difficult and has eluded us for quite a while.

In this note we exhibit examples of real rank zero AD algebras showing that the map β_n^0 is indeed necessary for the classification theory. The construction involves continuous field C^*-algebras where the fibers are glued together using morphisms with highly non-trivial KK-classes. These examples were devised such that by tensoring with Cuntz's algebra $\mathcal{O}_\infty$ we automatically obtain examples with similar properties for nuclear purely infinite C^*-algebras. Therefore the Bockstein maps have to play a role in the classification theory of (non-simple) purely infinite nuclear C^*-algebras, too.

In the final part of the introduction we discuss some notation. Using the functors $K_i(-;\mathbb{Z}/n)$ from C^*-algebras to abelian groups (*cf.* [15]) one may define ordered abelian groups

$$K_\bullet(-;\mathbb{Z}\oplus\mathbb{Z}/n) = K_0(-)\oplus K_1(-)\oplus K_0(-;\mathbb{Z}/n)\oplus K_1(-;\mathbb{Z}/n)$$

as described in [6]. For each i, n and m, there are also natural maps ρ_n^i, $\kappa_{m,n}^i$ and β_n^i and collecting all of the ordered groups and all of the group homomorphisms, we get an invariant $\underline{\mathbf{K}}(-)$ for C^*-algebras. Extracting again the parts of the invariant associated to a fixed integer n we get

$$\mathbf{K}_\bullet(-;n) = [K_\bullet(-;\mathbb{Z}\oplus\mathbb{Z}/n), \rho_n^i, \beta_n^i].$$

Of special interest to us will be the invariant

$$\mathbf{K}(-;n): \quad K_0(A) \xrightarrow{\rho_n^0} K_0(A;\mathbb{Z}/n) \xrightarrow{\beta_n^0} K_1(A).$$

1 Preliminaries

We write αX for the one point (Alexandroff) compactification of a locally compact space X, and denote the point at infinity by ∞_X. All examples in the paper will based on a straightforward construction of continuous field C^*-algebras over $\alpha\mathbb{N}$. The following notation will be convenient for us: when $\varphi_m\colon A \to B$ is a family of $*$-homomorphisms we denote by $\mathcal{F}[(\varphi_m)]$ or $\mathcal{F}[\varphi_1, \varphi_2, \varphi_3, \varphi_4, \ldots]$ the C^*-algebra

$$\Big\{\big(a, (b_m)\big) \in A \oplus \prod_{m=1}^{\infty} B \mid \|b_m - \varphi_m(a)\| \to 0\Big\}.$$

We collect a list of properties of such continuous fields.

Lemma 1.1

(i) If A and B are separable, or of real rank zero, or of stable rank one, or purely infinite, so is $\mathcal{F}[(\varphi_m)]$.
(ii) If A and B are AD *algebras, so is $\mathcal{F}[(\varphi_m)]$.*
(iii) If A and B are separable, simple, nuclear C^*-algebras, then $\mathcal{F}[(\varphi_m)] \otimes \mathcal{O}_\infty$ is a nuclear, purely infinite C^*-algebra of real rank zero.
(iv) If A and B are simple, then $\operatorname{Prim}\big(\mathcal{F}[(\varphi_m)]\big) = \alpha\mathbb{N}$.
(v) There is a split extension

$$0 \longrightarrow \bigoplus\nolimits_1^\infty B \longrightarrow \mathcal{F}[(\varphi_m)] \longrightarrow A \longrightarrow 0.$$

Proof We note that $\mathcal{F}[(\varphi_m)]$ is an inductive limit of C^*-algebras of the form $A \oplus \bigoplus_1^k B_m$ using bonding maps of the form

$$\chi_k(a, b_1, \ldots, b_k) = \big(a, b_1, \ldots, b_k, \varphi_{k+1}(a)\big), \tag{1}$$

and since the properties of (i) are closed under finite direct sums and countable inductive limits, that proves this fact. Similarly, (ii) follows from the fact that countable inductive limits of AD algebras are again AD as a consequence of the local criterion in [5, 1.2]. Notice that (iii) follows from (i), since $A \otimes \mathcal{O}_\infty$ and $B \otimes \mathcal{O}_\infty$ are simple, nuclear and purely infinite by [13]. They have real rank zero by [17]. The remaining claims follow directly from the continuous field structure of $\mathcal{F}[(\varphi_m)]$. ■

2 Necessity of ρ

In this section we construct an example showing the necessity of the reduction maps ρ; *i.e.*, that if one deletes the ρ maps from $\underline{\mathbf{K}}(-)$ the resulting invariant $\underline{\mathbf{K}}_{\langle\rho\rangle}(-)$ will not be complete, not even for AD algebras of real rank zero where $\underline{\mathbf{K}}(-)$ is known from [7, 3.6] to be complete. More precisely we shall construct for each $n > 2$ a pair of non-isomorphic AD algebras D, D' with $\operatorname{tor} K_1(D)$ and $\operatorname{tor} K_1(D')$ annihilated by n and order isomorphisms fitting in the vertical lines of

$$\begin{array}{lccccccc}
\mathbf{K}_{\langle\rho\rangle}(D;n)\colon & K_1(D;\mathbb{Z}/n) & \xrightarrow{\beta_n^1} & K_0(D) & K_0(D;\mathbb{Z}/n) & \xrightarrow{\beta_n^0} & K_1(D) \\
& \downarrow & & \downarrow & \downarrow & & \downarrow \\
\mathbf{K}_{\langle\rho\rangle}(D';n)\colon & K_1(D';\mathbb{Z}/n) & \xrightarrow[\beta_n^1]{} & K_0(D') & K_0(D';\mathbb{Z}/n) & \xrightarrow[\beta_n^0]{} & K_1(D').
\end{array}$$

Here, the notation $\mathbf{K}_{\langle\rho\rangle}(-;n)$ should be self-explanatory.

AH examples of this kind, with torsion in K_0, have been constructed earlier in [4]. An AD example appeared in [9]. We include an easier example here for two reasons: first, to demonstrate our main techniques; second, to construct an algebra which plays a role in our next example showing necessity of the Bockstein maps β.

Fix $n \in \mathbb{N}$ with $n > 2$. Let A denote the unique simple AD algebra of real rank zero having $K_0(A) = \mathbb{Z}[\frac{1}{n+1}]$ and $K_1(A) = \mathbb{Z}/n$, with $[1_A] = 1$ and

$$K_\bullet(A)_+ = \{(x, y) \mid x > 0 \text{ or } (x = 0, y = 0)\}$$

(where '$>$' refers to the order induced from $\mathbb{R}$). Let B denote the tensor product of the $(n+1)^\infty$ *UHF* algebra with the algebra of $n \times n$ matrices, such that $K_0(B) = \mathbb{Z}[\frac{1}{n+1}]$ and $K_1(B) = 0$ with $[1_B] = n$ and order induced from $\mathbb{R}$ as above. Choosing suitable generators (*not* the same as in [6] and [10]!), we may identify

$$\mathbf{K}(A;n):\quad \mathbb{Z}[\tfrac{1}{n+1}] \xrightarrow{\rho_A} \mathbb{Z}/n \oplus \mathbb{Z}/n \xrightarrow{\beta_A} \mathbb{Z}/n$$

$$\mathbf{K}(B;n):\quad \mathbb{Z}[\tfrac{1}{n+1}] \xrightarrow[\rho_B]{} \mathbb{Z}/n \xrightarrow[\beta_B]{} 0$$

where

$$\rho_A(x) = (\bar{x}, 0) \quad \beta_A(u, v) = v \quad \rho_B(x) = \bar{x} \quad \beta_B(u) = 0.$$

By a one-sided version of [7, 3.6] we may choose unital $*$-homomorphisms $\varphi, \varphi' \colon A \to B$ with

$$\mathbf{K}(\varphi;n) = (n, \begin{bmatrix}0 & 1\end{bmatrix}, 0) \quad \mathbf{K}(\varphi';n) = (n, \begin{bmatrix}0 & -1\end{bmatrix}, 0).$$

Now let D, D' be the C^*-algebras defined by

$$D = \mathcal{F}[\varphi, \varphi, \varphi, \varphi, \ldots] \quad D' = \mathcal{F}[\varphi, \varphi', \varphi, \varphi', \ldots].$$

Lemma 2.1 *Every homeomorphism of* $\mathrm{Prim}(D) = \alpha\mathbb{N}$ *lifts to an automorphism of D which acts trivially on* $K_1(D)$*.*

Proof We identify the ideal spectrum using Lemma 1.1(iii). Denote the homeomorphism by f. Since $\infty_{\mathbb{N}}$ is the only non-isolated point of $\mathrm{Prim}(D)$, $f(\infty_{\mathbb{N}}) = \infty_{\mathbb{N}}$, and so f restricts to a permutation σ of $\mathbb{N}$. We may hence define an automorphism

$$\Xi\big(a, (b_m)\big) = \big(a, (b_{\sigma(m)})\big)$$

of D which clearly has the desired properties. ■

Proposition 2.2 *D is not isomorphic to* D'*. Moreover,* $D \otimes \mathcal{O}_\infty$ *is not isomorphic to* $D' \otimes \mathcal{O}_\infty$*.*

Proof By Lemma 1.1 also $\mathrm{Prim}(D') = \alpha\mathbb{N}$. If D were isomorphic to D' by a $*$-isomorphism Ξ' it would induce a homeomorphism $\sigma_{\Xi'} \colon \alpha\mathbb{N} \to \alpha\mathbb{N}$. By Proposition 2.1 there would then also be an isomorphism Ξ with $\sigma_\Xi = \mathrm{id}$. It is straightforward to check that in this case we may write

$$\Xi\big(a, (b_m)\big) = \Big(\xi(a), \big(\xi_m(b_m)\big)\Big)$$

for unique automorphisms $\xi_0 \colon A \to A$ and $\xi_m \colon B \to B$.

Given $\big(a, (b_m)\big) \in D$ we note that

$$\|\varphi(a) - b_m\| \longrightarrow 0 \quad m \longrightarrow \infty. \tag{2}$$

Since $\Xi(a, b_m) \in D'$ we also get

$$\left.\begin{array}{r} \|\varphi\big(\xi(a)\big) - \xi_{2m+1}(b_{2m+1})\| \longrightarrow 0 \\ \|\varphi'\big(\xi(a)\big) - \xi_{2m}(b_{2m})\| \longrightarrow 0 \end{array}\right\} \quad m \longrightarrow \infty. \tag{3}$$

From (3) we get

$$\left.\begin{array}{r} \Big\|\xi_{2m+1}^{-1}\Big(\varphi\big(\xi(a)\big)\Big) - b_{2m+1}\Big\| \longrightarrow 0 \\ \Big\|\xi_{2m}^{-1}\Big(\varphi'\big(\xi(a)\big)\Big) - b_{2m}\Big\| \longrightarrow 0 \end{array}\right\} \quad m \longrightarrow \infty$$

which combines with (2) to give

$$\|\xi_{2m+1} \circ \xi_{2m}^{-1} \circ \varphi'(a) - \varphi(a)\| \longrightarrow 0 \quad m \longrightarrow \infty.$$

This is true for every a by Lemma 1.1(v) and surjectivity of ξ.

Therefore for any $x \in K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)$

$$\lim_{m\to\infty} (\xi_{2m+1})_* \circ (\xi_{2m})_*^{-1} \circ \varphi'_*(x) = \varphi_*(x)$$

and since $(\xi_k)_* = \mathrm{id}_*$ for every k because the only unital positive automorphism of $\mathbb{Z}[\frac{1}{n+1}]$ is the identity, this combines with the fact that $K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)$ is finitely generated to yield that $\varphi_* = \varphi'_*$, a contradiction since $1 \neq -1$ in $\mathbb{Z}/n$ (by assumption $n > 2$).

The proof for the infinite case follows similarly, since Lemma 2.1 obviously extends to $D \otimes \mathcal{O}_\infty$. ∎

Theorem 2.3 *There exist real rank zero* AD *algebras which are distinguished by* $\underline{\mathbf{K}}(-)$ *but not by* $\underline{\mathbf{K}}_{\langle\rho\rangle}(-)$.

Proof By Lemma 1.1, both D and D' are covered by the classification result in [7] and hence $\underline{\mathbf{K}}(D) \ncong \underline{\mathbf{K}}(D')$ as a consequence of Proposition 2.2. In fact, we may apply [10] to get that $\mathbf{K}(D; n) \ncong \mathbf{K}(D'; n)$. On the other hand, since the family F of quadruples of the form

$$\begin{array}{cccccccc} \mathbf{K}_{\langle\rho\rangle}(B;k) & K_1(B;\mathbb{Z}/k) & \xrightarrow{\beta_k^1} & K_0(B) & & K_0(B'\mathbb{Z}/k) & \xrightarrow{\beta_k^0} & 0 \\ & \| & & \| & & {\scriptstyle -1}\downarrow & & \| \\ \mathbf{K}_{\langle\rho\rangle}(B;k) & K_1(B;\mathbb{Z}/k) & \xrightarrow[\beta_k^1]{} & K_0(B) & & K_0(B;\mathbb{Z}/k) & \xrightarrow[\beta_k^0]{} & 0 \end{array}$$

consists of automorphisms of $K_\bullet(B;\mathbb{Z}\oplus\mathbb{Z}/k)$ which conjugate φ_* to φ'_*, are order preserving and respect the β and κ maps, they induce by continuity of $\underline{\mathbf{K}}_{\langle\rho\rangle}(-)$ an isomorphism $\underline{\mathbf{K}}_{\langle\rho\rangle}(D)\cong\underline{\mathbf{K}}_{\langle\rho\rangle}(D')$ since

$$\begin{array}{ccc}
\underline{\mathbf{K}}_{\langle\rho\rangle}(A\oplus\bigoplus_1^{2k}B) & \xrightarrow{(\chi_{2k+1}\circ\chi_{2k})_*} & \underline{\mathbf{K}}_{\langle\rho\rangle}(A\oplus\bigoplus_1^{2k+2}B) \\
{\scriptstyle \mathrm{id}\oplus(\mathrm{id}\oplus F)^k}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\oplus(\mathrm{id}\oplus F)^{k+1}} \\
\underline{\mathbf{K}}_{\langle\rho\rangle}(A\oplus\bigoplus_1^{2k}B) & \xrightarrow[(\chi'_{2k+1}\circ\chi'_{2k})_*]{} & \underline{\mathbf{K}}_{\langle\rho\rangle}(A\oplus\bigoplus_1^{2k+2}B)
\end{array}$$

commutes where the χ_k and χ'_k maps are bonding maps in the inductive limit description of D and D' as in (1) of Lemma 1.1. ∎

Remark 2.4 The C^*-algebras $D\otimes\mathcal{O}_\infty$ and $D'\otimes\mathcal{O}_\infty$ are distinguished by $\underline{\mathbf{K}}(-)$ and its filtration induced by ideals but not by $\underline{\mathbf{K}}_{\langle\rho\rangle}(-)$ and its filtration induced by ideals. More precisely, there is no pair (θ,Θ) consisting of a homeomorphism $\theta\colon \operatorname{Prim}(D\otimes\mathcal{O}_\infty)\to\operatorname{Prim}(D'\otimes\mathcal{O}_\infty)$ and an isomorphism $\Theta\colon\underline{\mathbf{K}}(D\otimes\mathcal{O}_\infty)\to\underline{\mathbf{K}}(D'\otimes\mathcal{O}_\infty)$ such that

$$\Theta\big(\underline{\mathbf{K}}(D\otimes\mathcal{O}_\infty\parallel I)\big)\subset\underline{\mathbf{K}}\big(D'\otimes\mathcal{O}_\infty\parallel\theta(I)\big) \tag{4}$$

for all $I\in\operatorname{Prim}(D\otimes\mathcal{O}_\infty)$. Here we denote by $\underline{\mathbf{K}}(D\otimes\mathcal{O}_\infty\parallel I)$ the image of $\underline{\mathbf{K}}(I)$ into $\underline{\mathbf{K}}(D\otimes\mathcal{O}_\infty)$. However there exists a pair (θ,Θ) with $\Theta\colon\underline{\mathbf{K}}_{\langle\rho\rangle}(D\otimes\mathcal{O}_\infty)\to\underline{\mathbf{K}}_{\langle\rho\rangle}(D'\otimes\mathcal{O}_\infty)$ satisfying (4). The proofs of the above remarks are left to the reader as they run parallel with the proofs of Proposition 2.2 and Theorem 2.3. In this context it is perhaps interesting to note that the space $\operatorname{Prim}(D\otimes\mathcal{O}_\infty)$ of primitive ideals of $D\otimes\mathcal{O}_\infty$ and its topology can be recovered from the semigroup $V(D\otimes\mathcal{O}_\infty\otimes\mathcal{K})$ of Murray-von Neumann equivalence classes of projections in $D\otimes\mathcal{O}_\infty\otimes\mathcal{K}$. Moreover any semigroup isomorphism $V(D\otimes\mathcal{O}_\infty\otimes\mathcal{K})\to V(D'\otimes\mathcal{O}_\infty\otimes\mathcal{K})$ lifts to a unique homeomorphism $\theta\colon\operatorname{Prim}(D\otimes\mathcal{O}_\infty)\to\operatorname{Prim}(D'\otimes\mathcal{O}_\infty)$.

We note the following which shall be used in proving necessity of the β maps:

Proposition 2.5 *Every automorphism of D acts as the identity on $K_1(D)$. Moreover, every automorphism of $D\otimes\mathcal{O}_\infty$ acts as the identity on $K_1(D\otimes\mathcal{O}_\infty)$.*

Proof Let an automorphism Ξ' of D be given. Since Lemma 2.1 ensures the existence of another automorphism, trivial on $K_1(D)$, which acts opposite of Ξ' on $\operatorname{Prim}(D)$, it suffices as in the proof of Proposition 2.2 to prove the claim for Ξ which acts trivially on $\operatorname{Prim}(D)$, and we may write

$$\Xi\big(a,(b_m)\big)=\Big(\xi(a),\big(\xi_m(b_m)\big)\Big).$$

Since

$$\left.\begin{array}{r}\|\varphi(a)-b_m\|\longrightarrow 0\\ \|\varphi\big(\xi(a)\big)-\xi_m(b_m)\|\longrightarrow 0\end{array}\right\}\quad m\longrightarrow\infty.$$

We conclude as above that

$$\lim_{m\to\infty}(\xi_m)_*^{-1}\circ\varphi_*\circ\xi_*(x)=\varphi_*(x)$$

for every $x\in K_\bullet(A;\mathbb{Z}\oplus\mathbb{Z}/n)$, and thus

$$\varphi_*\circ\xi_*=\varphi_*$$

on $K_0(A;\mathbb{Z}/n)$ since every unital automorphism of B must induce the identity on $K_0(B;\mathbb{Z}/n)$. Since the kernel of φ_* does not meet the second summand of $K_0(A;\mathbb{Z}/n)=\mathbb{Z}/n\oplus\mathbb{Z}/n$, ξ_* must act like the identity here, and hence also on its image under β_n^0. This in turn equals $K_1(D)$.

The proof for the infinite case follows similarly. One needs to notice that the class of the unit in $K_0(B\otimes\mathcal{O}_\infty)\cong\mathbb{Z}[\frac{1}{n+1}]$ equals 1. ■

3 Necessity of β

In this section we construct an example showing the necessity of β; *i.e.*, that if one deletes the β maps from $\underline{\mathbf{K}}(-)$ the resulting invariant $\underline{\mathbf{K}}_{\langle\beta\rangle}(-)$ is not complete, not even for AD algebras of real rank zero. More precisely we shall construct for each $n>2$ a pair of non-isomorphic AD algebras E,E' with tor $K_1(E)$ and tor $K_1(E')$ annihilated by n and order isomorphisms giving commutative diagrams

$$\begin{array}{lccccccc}
\mathbf{K}_{\langle\beta\rangle}(E;n)\colon & K_0(E) & \xrightarrow{\rho_n^0} & K_0(E;\mathbb{Z}/n) & K_1(E) & \xrightarrow{\rho_n^1} & K_1(E;\mathbb{Z}/n)\\
& \downarrow & & \downarrow & \downarrow & & \downarrow\\
\mathbf{K}_{\langle\beta\rangle}(E';n)\colon & K_0(E') & \xrightarrow[\rho_n^0]{} & K_0(E';\mathbb{Z}/n) & K_1(E') & \xrightarrow[\rho_n^1]{} & K_1(E';\mathbb{Z}/n).
\end{array}$$

Fix $n>2$. Let C denote the $(n+1)^\infty$ Bunce-Deddens algebra with $K_0(C)=\mathbb{Z}[\frac{1}{n+1}]$ and $K_1(C)=\mathbb{Z}$ with $[1_A]=1$. Choosing suitable generators, we may identify

$$\begin{array}{lccccc}
\mathbf{K}(C;n)\colon & \mathbb{Z}[\frac{1}{n+1}] & \xrightarrow{\rho_C} & \mathbb{Z}/n & \xrightarrow{\beta_C} & \mathbb{Z}\\
\mathbf{K}(D;n)\colon & \{(x,(y_m))\in G_0\mid y_m\to nx\} & \xrightarrow[\rho_D]{} & \{(u,v,(w_m))\in G\mid w_m\to v\} & \xrightarrow[\beta_D]{} & \mathbb{Z}/n
\end{array}$$

where

$$G_0=\mathbb{Z}\Big[\frac{1}{n+1}\Big]\oplus\prod_1^\infty\mathbb{Z}\Big[\frac{1}{n+1}\Big]\qquad G=\mathbb{Z}/n\oplus\mathbb{Z}/n\oplus\prod_1^\infty\mathbb{Z}/n$$

and

$$\rho_C(x)=\bar{x}\quad\beta_C(u)=0\quad\rho_D\big(x,(y_m)\big)=\big(\bar{x},0,(\overline{y_m})\big)\quad\beta_D\big(u,v,(w_m)\big)=v.$$

As in [7, 3.6] we may choose unital $*$-homomorphisms $\eta, \eta' \colon C \to A$ with

$$\mathbf{K}(\eta; n) = \left(1, \begin{bmatrix}1\\0\end{bmatrix}, 1\right) \quad \mathbf{K}(\eta'; n) = \left(1, \begin{bmatrix}1\\0\end{bmatrix}, -1\right).$$

If we define $\psi, \psi' \colon C \to D$ by letting

$$\psi = (\eta, \varphi\eta, \varphi\eta, \dots) \quad \psi' = (\eta, \varphi\eta', \varphi\eta', \dots).$$

then

$$\mathbf{K}(\psi; n) = \left(\begin{bmatrix}1\\n\\n\\\vdots\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\\\vdots\end{bmatrix}, 1 \right) \quad \mathbf{K}(\psi'; n) = \left(\begin{bmatrix}1\\n\\n\\\vdots\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\\\vdots\end{bmatrix}, -1 \right).$$

Now let E, E' be the C^*-algebras defined by

$$E = \mathcal{F}[\psi, \psi, \psi, \psi, \dots] \quad E' = \mathcal{F}[\psi, \psi', \psi, \psi', \dots].$$

Lemma 3.1 *Every homeomorphism of* $\mathrm{Prim}(E)$ *lifts to an automorphism of E.*

Proof Denote the homeomorphism by f. We have that $\mathrm{Prim}(E) = \alpha(\alpha\mathbb{N} \times \mathbb{N})$, and since the points lying in different parts of the partition of this space into

$$\mathbb{N} \times \mathbb{N} \quad \{\infty_{\mathbb{N}}\} \times \mathbb{N} \quad \{\infty_{\alpha\mathbb{N}\times\mathbb{N}}\}$$

are topologically distinguishable, f must restrict to a permutation σ of $\mathbb{N} \times \mathbb{N}$ and have the property that

$$f(\infty_{\mathbb{N}}, m) = \big(\infty_{\mathbb{N}}, \tau(m)\big)$$

for some permutation τ of $\mathbb{N}$. Note that continuity of f at $(\infty_{\mathbb{N}}, m_0)$ implies that

$$\forall m_0 \in \mathbb{N}\, \exists N \in \mathbb{N}\, \forall n \geq N : \pi_2\big(\sigma(n, m_0)\big) = \tau(m_0), \tag{5}$$

where π_2 denotes projection onto the second coordinate of $\mathbb{N} \times \mathbb{N}$, and that this in turn implies

$$\forall M \in \mathbb{N}\, \exists M' \in \mathbb{N}\, \forall m \geq M'\, \forall n \in \mathbb{N} : \pi_2\big(\sigma(n, m)\big) \geq M. \tag{6}$$

By the definitions, E consists of all tuples

$$\big(c, (a_m), (b_{n,m})\big) \in C \oplus \prod_{\mathbb{N}} A \oplus \prod_{\mathbb{N}\times\mathbb{N}} B$$

which satisfy

$$\|b_{n,m} - \varphi(a_m)\| \longrightarrow 0 \quad n \longrightarrow \infty$$
$$\|a_m - \eta(c)\| \longrightarrow 0 \quad m \longrightarrow \infty$$
$$\sup_{n\in\mathbb{N}}\|b_{n,m} - \varphi\big(\eta(c)\big)\| \longrightarrow 0 \quad m \longrightarrow \infty.$$

It is clear that if we can define an automorphism of E by

$$\Xi\big(c, (a_m), (b_{n,m})\big) = \big(c, (a_{\tau(m)}), (b_{\sigma(n,m)})\big),$$

it will act as f on $\mathrm{Prim}(E)$. We must check

$$\|b_{\sigma(n,m)} - \varphi(a_{\tau(m)})\| \longrightarrow 0 \quad n \longrightarrow \infty \tag{7}$$
$$\|a_{\tau(m)} - \eta(c)\| \longrightarrow 0 \quad m \longrightarrow \infty \tag{8}$$
$$\sup_{n\in\mathbb{N}}\|b_{\sigma(n,m)} - \varphi\big(\eta(c)\big)\| \longrightarrow 0 \quad m \longrightarrow \infty, \tag{9}$$

and here (8) is clear, while (7) and (9) follow from (5) and (6), respectively. ■

Proposition 3.2 *E is not isomorphic to E'. Moreover $E \otimes \mathcal{O}_\infty$ is not isomorphic to $E' \otimes \mathcal{O}_\infty$.*

Proof Arguing as in the proof of Proposition 2.2 we may use Lemma 3.1 to reduce to the case that there is an isomorphism Ξ which acts trivially on the spectrum. Again

$$\Xi\big(c, (d_m)\big) = \Big(\xi(c), \big(\xi_m(d_m)\big)\Big)$$

for automorphisms $\xi\colon C \to C$ and $\xi_m\colon D \to D$, and combining

$$\|\psi(c) - d_m\| \longrightarrow 0 \quad m \longrightarrow \infty$$

with

$$\left.\begin{array}{r}\|\psi\big(\xi(c)\big) - \xi_{2m+1}(d_{2m+1})\| \longrightarrow 0\\ \|\psi'\big(\xi(c)\big) - \xi_{2m}(d_{2m})\| \longrightarrow 0\end{array}\right\} \quad m \longrightarrow \infty$$

we get as in the proof of Proposition 2.2 that

$$\lim_{m\to\infty}(\xi_{2m+1})_* \circ (\xi_{2m})_*^{-1} \circ \psi'_*(x) = \psi_*(x),$$

for $x \in \mathbf{K}(C; n)$. By Proposition 2.5, $(\xi_k)_* = \mathrm{id}$ on $K_1(D)$. But since $\psi_* \neq \psi'_*$ on $K_1(C)$ as $1 \neq -1$ in $\mathbb{Z}/n$, we get the desired contradiction.

The proof for the infinite case follows similarly. ■

Theorem 3.3 *There exist real rank zero non-isomorphic* AD *algebras which are distinguished by* $\underline{\mathbf{K}}(-)$ *but not by* $\underline{\mathbf{K}}_{\langle\beta\rangle}(-)$.

Proof By Lemma 1.1, both E and E' are covered by the classification results in [3] and [10], so we get that $\underline{\mathbf{K}}(E) \ncong \underline{\mathbf{K}}(E')$ because $\mathbf{K}(E;n) \ncong \mathbf{K}(E';n)$. On the other hand, since the family of quadruples of the form

$$\begin{array}{cccccccc}
\mathbf{K}_{\langle\beta\rangle}(D;k) & K_0(D) & \xrightarrow{\rho_k^0} & K_0(D;\mathbb{Z}/k) & K_1(D) & \xrightarrow{\rho_k^1} & K_1(D;\mathbb{Z}/k) \\
 & \Big\| & & \Big\| & \Big\downarrow{\scriptstyle -1} & & {\scriptstyle -1}\Big\downarrow \\
\mathbf{K}_{\langle\beta\rangle}(D;k) & K_0(D) & \xrightarrow[\rho_k^0]{} & K_0(D;\mathbb{Z}/k) & K_1(D) & \xrightarrow[\rho_k^1]{} & K_1(D;\mathbb{Z};k)
\end{array}$$

consists of automorphisms of $K_\bullet(D;\mathbb{Z}\oplus\mathbb{Z}/k)$ which conjugate ψ_* to ψ'_*, are order preserving and respect the ρ and κ maps, they induce an isomorphism $\underline{\mathbf{K}}_{\langle\beta\rangle}(E) \cong \underline{\mathbf{K}}_{\langle\beta\rangle}(E')$ by continuity exactly as in the proof of Theorem 2.3. ■

Remark 3.4 The C^*-algebras $E \otimes \mathcal{O}_\infty$ and $E' \otimes \mathcal{O}_\infty$ are distinguished by $\underline{\mathbf{K}}(-)$ and its filtration induced by ideals but not by $\underline{\mathbf{K}}_{\langle\beta\rangle}(-)$ and its filtration induced by ideals. More precisely, there is no pair (θ,Θ) consisting of a homeomorphism $\theta\colon \operatorname{Prim}(E\otimes\mathcal{O}_\infty) \to \operatorname{Prim}(E'\otimes\mathcal{O}_\infty)$ and an isomorphism $\Theta\colon \underline{\mathbf{K}}(E\otimes\mathcal{O}_\infty) \to \underline{\mathbf{K}}(E'\otimes\mathcal{O}_\infty)$ such that

$$\Theta([1_{E\otimes\mathcal{O}_\infty}]) = [1_{E'\otimes\mathcal{O}_\infty}] \quad\text{and}\quad \Theta\big(\underline{\mathbf{K}}(E\otimes\mathcal{O}_\infty \parallel I)\big) \subset \underline{\mathbf{K}}(E'\otimes\mathcal{O}_\infty \parallel \theta(I)) \tag{10}$$

for all $I \in \operatorname{Prim}(E\otimes\mathcal{O}_\infty)$. Here we denote by $\underline{\mathbf{K}}(E\otimes\mathcal{O}_\infty \parallel I)$ the image of $\underline{\mathbf{K}}(I)$ into $\underline{\mathbf{K}}(E\otimes\mathcal{O}_\infty)$. However there exists a pair (θ,Θ) with $\Theta\colon \underline{\mathbf{K}}_{\langle\beta\rangle}(E\otimes\mathcal{O}_\infty) \to \underline{\mathbf{K}}_{\langle\beta\rangle}(E'\otimes\mathcal{O}_\infty)$ satisfying (10). The proofs of the above remarks are left to the reader as they are similar to the proofs of Proposition 3.2 and Theorem 3.3.

Remark 3.5 It is relatively easy to see that if K_1 is either torsion or non-torsion, the map β_n^0 carries no information. We hence need a group with nontrivial free and torsion part such as $K_1(E)$ to produce an example of this kind.

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