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Inductive Limits of $C(X)$-Modules and Continuous Fields of AF-Algebras

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Let X be a compact connected space and $(A_i)_{i=1}^{\infty}$ a sequence of finite-dimensional C^*-algebras. Each inductive limit $L = \varinjlim C(X) \otimes A_i$, with $C(X)$-linear connecting $*$-homomorphisms, is $*$-isomorphic as $C(X)$-module to the C^*-algebra defined by a certain continuous field $\mathscr{E}_L$ of AF-algebras. We classify the C^*-algebras L for which $\mathscr{E}_L$ has simple fibres. In the general case the classification is given in the category of the C^*-algebras which are $C(X)$-modules.

INTRODUCTION

In [5] E. G. Effros posed the problem of studying inductive limits of C^*-algebras of the form $C(X) \otimes A$, with A finite-dimensional, as a generalization of the AF-algebras.

Let X be a connected compact space. In this paper we give some classification results concerning inductive limits $\varinjlim C(X) \otimes A_i$, with A_i finite-dimensional, where the bonding homomorphisms are unital, injective, and $C(X)$-linear. The problem here is to measure and to store the possible twistings over X of the embeddings of A_i into A_{i+1}. The $C(X)$-linear $*$-homomorphisms $C(X) \otimes A_i \to C(X) \otimes A_{i+1}$ correspond to homomorphisms $A_i \to C(X) \otimes A_{i+1}$ which are classified, modulo inner equivalence, by matrices of complex vector bundles over X (see Corollary 2.2). Each inductive limit $L = \varinjlim C(X) \otimes A_i$, with $C(X)$-linear connecting $*$-homomorphisms, is isomorphic to the C^*-algebra defined by a continuous field $\mathscr{E}_L$ of AF-algebras canonically associated with L (see Proposition 3.1). This field is not always trivial as it is shown in Proposition 5.1. Moreover, we are able to classify the inductive limits L in the case when the fibres of $\mathscr{E}_L$ are simple, using the semigroup of the homotopy classes of projections in $\bigcup_{n=1}^{\infty} M_n \otimes L$ (see Theorem 4.4). If the canonical map $\mathrm{Vect}(X) \to K^0(X)$ is injective (in particular, this occurs provided that X is a connected finite CW-complex of dimension $\leqslant 3$) this

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result may be given using the pointed ordered group $(K_0(L), K_0(L)_+, [1_L])$ (see Theorem 4.6). Also we classify the C^*-algebras L as $C(X)$-modules (see Theorems 4.3 and 4.5).

1. PRELIMINARIES

If A, B are unital C^*-algebras we shall denote by $\mathrm{Hom}(A, B)$ the space of all unital *-homomorphisms from A to B endowed with the topology of pointwise convergence. Two homomorphisms Φ_1, $\Phi_2 \in \mathrm{Hom}(A, B)$ are said to be inner equivalent if there is a unitary $u \in B$ such that $\Phi_2 = u\Phi_1 u^*$. Let $\mathrm{Hom}(A, B)/\sim$ be the set of classes of inner equivalent homomorphisms from A to B. If A and B are $C(X)$-modules, we shall denote by $\mathrm{Hom}_{C(X)}(A, B)$ the subspace of $\mathrm{Hom}(A, B)$ consisting of all $C(X)$-linear homomorphisms.

We shall use $\mathrm{Vect}(X)$ to denote the set of isomorphism classes of complex vector bundles on X, and $\mathrm{Vect}_k(X)$ to denote the subset of $\mathrm{Vect}(X)$ given by bundles of dimension k. $\mathrm{Vect}(X)$ is a semiring under the operations $\oplus$ and $\otimes$. In $\mathrm{Vect}_k(X)$ we have one naturally distinguished element $[k]$—the class of the trivial bundle of dimension k.

As usual we denote by $G(n, k)$ the Grassmann manifold of all subspaces of $\mathbf{C}^n$ of dimension k and by $U(n)$ the Lie group of all unitaries of M_n. Any continuous map $F: X \to G(n, k)$ defines a vector bundle $E_F = \{(x, F(x)\eta): x \in X,\ \eta \in \mathbf{C}^n\} \subset X \times \mathbf{C}^n$. Let $H^1(X, U(k)_c)$ denote the cohomology set associated with the sheaf of germs of continuous functions $X \to U(k)$. We have a bijection $\mathrm{Vect}_k(X) \to H^1(X, U(k)_c)$ which takes classes of vector bundles to classes of cocycles [8].

We describe below the cocycle of E_F. The fibration

$$U(k) \times U(n-k) \to U(n) \to G(n, k)$$

induces the exact sequence of pointed cohomology sets

$$C(X, U(n)) \longrightarrow C(X, G(n, k)) \xrightarrow{\delta} H^1(X, U(k)_c) \times H^1(X, U(n-k)_c) \longrightarrow H^1(X, U(n)_c)$$

(for details see [2]). Denote $\delta(F) = (\delta_1(F), \delta_2(F))$.

1.1. LEMMA. *The vector bundle E_F is given by the cocycle $\delta_1(F)$.*

Proof. Choose an open covering (U_i) of X and continuous maps $u_i: U_i \to U(n)$ such that

$$F(x) = u_i(x) \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix} u_i(x)^* \qquad \text{on } U_i.$$

Then

$$u_i(x)^* u_j(x) = \begin{bmatrix} u_{ij}(x) & 0 \\ 0 & u'_{ij}(x) \end{bmatrix} \quad \text{on } U_i \cap U_j$$

and $\delta(F) = ((U_i, u_{ij}), (U_i, u'_{ij}))$, by definition. Consider the local trivializations for E_F

$$U_i \times \mathbf{C}^k \xrightarrow{\phi_i} E_F|_{U_i} = \left\{ \left(x, u_i(x) \begin{bmatrix} 1_k & 0 \\ 0 & 0 \end{bmatrix} u_i(x)^* \eta \right) : x \in U_i,\ \eta \in \mathbf{C}^n \right\}$$

given by $\phi_i(x, \xi) = (x, u_i(x)[\begin{smallmatrix}\xi \\ 0\end{smallmatrix}])$, $x \in U_i$, $\xi \in \mathbf{C}^k$.

The cocycle (U_i, b_{ij}) of E_F can be computed using the local trivializations

$$\begin{aligned} b_{ij}(x)\xi &= (\phi_i^{-1})_x (\phi_j)_x \xi = (\phi_i^{-1})_x u_j(x) \begin{bmatrix} \xi \\ 0 \end{bmatrix} \\ &= (\phi_i^{-1})_x u_i(x) u_i(x)^* u_j(x) \begin{bmatrix} \xi \\ 0 \end{bmatrix} \\ &= (\phi_i^{-1})_x u_i(x) \begin{bmatrix} u_{ij}(x) & 0 \\ 0 & u'_{ij}(x) \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix} \\ &= (\phi_i^{-1})_x u_i(x) \begin{bmatrix} u_{ij}(x)\xi \\ 0 \end{bmatrix} = u_{ij}(x)\xi, \qquad x \in U_i \cap U_j,\ \xi \in \mathbf{C}^k. \end{aligned}$$

1.2. Corollary. *Let $F: X \to G(n, q)$ be a continuous map and define a continuous map $\tilde{F}: X \to G(nk + p, qk)$*

$$\tilde{F}(x) = v_i(x) \begin{bmatrix} F(x) \otimes 1_k & 0 \\ 0 & 0_p \end{bmatrix} v_i(x)^*, \qquad x \in U_i,$$

where (U_i) is an open covering of X and $v_i: U_i \to U(nk + p)$ are continuous maps satisfying

$$v_i(x)^* v_j(x) = \begin{bmatrix} 1_n \otimes a_{ij}(x) & 0 \\ 0 & a'_{ij}(x) \end{bmatrix}, \qquad x \in U_i \cap U_j$$

for some continuous maps $a_{ij}: U_i \cap U_j \to U(k)$ and $a'_{ij}: U_i \cap U_j \to U(p)$. Let H be the vector bundle corresponding to the cocycle (U_i, a_{ij}). Then $E_{\tilde{F}}$ is isomorphic to $E_F \otimes H$.

Proof. We may assume that $F(x) = u_i(x)[\begin{smallmatrix} 1_q & 0 \\ 0 & 0 \end{smallmatrix}] u_i(x)^*$ on U_i, where $u_i: U_i \to U(n)$ are continuous and

$$u_i(x)^* u_j(x) = \begin{bmatrix} u_{ij}(x) & 0 \\ 0 & u'_{ij}(x) \end{bmatrix} \quad \text{on} \quad U_i \cap U_j.$$

We get the following formula for $\tilde{F}$ on U_i:

$$\tilde{F}(x)=v_i(x)\begin{bmatrix} u_i(x)\otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 1_q & 0 \\ 0 & 0 \end{bmatrix}\otimes 1_k & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_i(x)^*\otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix} v_i(x)^*$$

so that we can compute $\delta(\tilde{F})$. Indeed, for $x \in U_i \cap U_j$ we have

$$\begin{aligned}
&\begin{bmatrix} u_i(x)^*\otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix} v_i(x)^*\, v_j(x)\begin{bmatrix} u_j(x)\otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix} \\
&\quad=\begin{bmatrix} u_i(x)^*\otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix}\begin{bmatrix} 1_n\otimes a_{ij}(x) & 0 \\ 0 & a'_{ij}(x) \end{bmatrix}\begin{bmatrix} u_j(x)\otimes 1_k & 0 \\ 0 & 1_p \end{bmatrix} \\
&\quad=\begin{bmatrix} u_{ij}(x)\otimes a_{ij}(x) & 0 & 0 \\ 0 & u'_{ij}(x)\otimes a_{ij}(x) & 0 \\ 0 & 0 & a'_{ij}(x) \end{bmatrix}.
\end{aligned}$$

Hence $E_{\tilde{F}}$ is given by the cocycle $(U_i,\ u_{ij}\otimes a_{ij})$.

2. Homomorphisms of $C(X)$-Modules

In this section we classify the homomorphisms in $\mathrm{Hom}_{C(X)}(C(X)\otimes A,\ C(X)\otimes B)$ within inner equivalence, where $A = M_{n_1}\oplus \cdots \oplus M_{n_r}$, $B = M_{m_1}\oplus \cdots \oplus M_{m_s}$, and X is compact and connected.

Any homomorphism $\Phi \in \mathrm{Hom}_{C(X)}(C(X)\otimes A,\ C(X)\otimes B)$ is uniquely determined by its restriction to A. This allows us to identify $\mathrm{Hom}_{C(X)}(C(X)\otimes A,\ C(X)\otimes B)$ with $\mathrm{Hom}(A, C(X)\otimes B)$ as topological spaces, identification which preserves the inner equivalence classes. By Proposition 1 in [3] it follows that there is a bijection

$$\begin{aligned}
&\delta\colon \mathrm{Hom}_{C(X)}(C(X)\otimes A,\ C(X)\otimes B)/\sim \\
&\qquad \to \{\mathbf{E} = (E_{pq}) \in M_{s\times r}(\mathrm{Vect}(X)) \colon E[\mathbf{n}] = [\mathbf{m}]\}, \qquad (1)
\end{aligned}$$

where $[\mathbf{n}] := ([n_1], ..., [n_r])$, $[\mathbf{m}] := ([m_1], ..., [m_s])$. Explicitly, $\mathbf{E}[\mathbf{n}] = [\mathbf{m}]$ means

$$(E_{p1}\otimes [n_1])\oplus \cdots \oplus (E_{pr}\otimes [n_r]) = [m_p], \qquad p = 1, 2, ..., s.$$

The description of δ can be obtained using the local structure of homomorphisms $A \to C(X)\otimes B$ given in [10] or by Proposition 1 in [3]. For simplicity, suppose that $B = M_m$. Thus, for a homomorphism

$\Phi \in \mathrm{Hom}(A, C(X) \otimes B)$ there are an open covering (U_i) of X, continuous maps $v_i\colon U_i \to U(m)$, and positive integers $k_{11}, ..., k_{1r}$ such that

$$\Phi(a)(x) = v_i(x)(a_1 \otimes 1_{k_{11}} \oplus \cdots \oplus a_r \otimes 1_{k_{1r}})\, v_i(x)^*, \tag{2}$$

where $x \in U_i$, $a = a_1 \oplus \cdots \oplus a_r \in A$ and

$$v_i(x)^*\, v_j(x) = \begin{bmatrix} 1_{n_1} \otimes a_{ij}^1(x) & & 0 \\ & \ddots & \\ 0 & & 1_{n_r} \otimes a_{ij}^r(x) \end{bmatrix} \quad \text{on} \quad U_i \cap U_j.$$

If $\delta(\Phi) = (E_{1q})$ then each vector bundle E_{1q} is given by the cocycle (U_i, a_{ij}^q). Note that rank $E_{1q} = k_{1q}$.

If C is a unital C^*-algebra we shall denote by $D(C)$ the set of homotopy classes of selfadjoint projections in $\bigcup_{n=1}^{\infty} M_n \otimes C$. Recall that $D(C)$ is a semigroup under the operation induced by the direct sum of projections and $D(\cdot)$ is a covariant functor.

Let $C = C(X) \otimes A$. It is known that there is an isomorphism of semigroups $D(C(X) \otimes A) \to \mathrm{Vect}(X)^r$ which maps the class of a projection $F \in C(X) \otimes A \otimes M_n$, having the decomposition

$$F = F_1 \oplus \cdots \oplus F_r \in \bigoplus_{k=1}^{r} C(X) \otimes M_{n_k} \otimes M_n$$

to $(E_{F_1}, ..., E_{F_r}) \in \mathrm{Vect}(X)^r$. Any homomorphism $\Phi \in \mathrm{Hom}_{C(X)}(C(X) \otimes A, C(X) \otimes B)$ induces a map $\Phi_*\colon D(C(X) \otimes A) \to D(C(X) \otimes B)$ or equivalently a map $\Phi_*\colon \mathrm{Vect}(X)^r \to \mathrm{Vect}(X)^s$. $\mathrm{Vect}(X)^r$ is a free module over the unital semiring $\mathrm{Vect}(X)$. Let $e_1, ..., e_r$ be its canonical basis, $e_i = (0, ..., [1], ..., 0)$ with $[1]$ on the ith position. We denote by $\mathrm{Hom}_{\mathrm{Vect}(X)}(\mathrm{Vect}(X)^r, \mathrm{Vect}(X)^s)$ the set of all homomorphisms of $\mathrm{Vect}(X)$-modules $\mathrm{Vect}(V)^r \to \mathrm{Vect}(X)^s$. As usual any element of $\mathrm{Hom}_{\mathrm{Vect}(X)}(\mathrm{Vect}(X)^r, \mathrm{Vect}(X)^s)$ is given by a unique matrix in $M_{s \times r}(\mathrm{Vect}(X))$ with respect to the canonical bases.

2.1. Proposition. *The map Φ_* is* $\mathrm{Vect}(X)$*-linear and its matrix is equal to $\delta(\Phi) = (E_{pq})$.*

Proof. We may assume that $B = M_m$. Using (2) and the canonical bijection $\mathrm{Hom}(A, C(X) \otimes B) \to \mathrm{Hom}_{C(X)}(C(X) \otimes A, C(X) \otimes B)$ we get the following description for Φ:

$$\Phi(G)(x) = v_i(x)(G_1(x) \otimes 1_{k_{11}} \oplus \cdots \oplus G_r(x) \otimes 1_{k_{1r}})\, v_i(x)^*,$$

$x \in U_i$, $G = \bigoplus_{i=1}^{r} G_i \in \bigoplus_{i=1}^{r} C(X) \otimes M_{n_i}$, where $k_{11}, ..., k_{1r}$ are positive integers $(n_1 k_{11} + \cdots + n_r k_{1r} = m)$, (U_i) is an open covering of X, and $v_i\colon U_i \to U(m)$ are continuous maps satisfying

$$v_i(x)^*\, v_j(x) = \begin{bmatrix} 1_{n_1} \otimes a^1_{ij}(x) & & 0 \\ & \ddots & \\ 0 & & 1_{n_r} \otimes a^r_{ij}(x) \end{bmatrix}, \qquad \text{on} \quad x \in U_i \cap U_j.$$

Let $\Phi_n\colon C(X) \otimes A \otimes M_n \to C(X) \otimes M_m \otimes M_n$, $\Phi_n := \Phi \otimes id_{M_n}$, $n \geq 1$. Since Φ_* is a homomorphism of semigroups it is enough to describe the homotopy class of $\Phi_n(F)$ for a projection $F \in C(X) \otimes M_{n_1} \otimes M_n \subset C(X) \otimes A \otimes M_n$. One can easily obtain the following formula:

$$\Phi_n(F)(x) = v_i(x) \otimes 1_n \begin{bmatrix} F(x) \otimes 1_{k_{11}} & 0 \\ 0 & 0_p \end{bmatrix} v_i(x)^* \otimes 1_n, \qquad x \in U_i,$$

where $p = mn - k_{11} n_1 n$. Since

$$(v_i(x) \otimes 1_n)^*\, (v_j(x) \otimes 1_n) = \begin{bmatrix} 1_{nn_1} \otimes a^1_{ij}(x) & 0 \\ 0 & a'_{ij}(x) \end{bmatrix}, \qquad x \in U_i \cap U_j,$$

where $a'_{ij}(x) := \bigoplus_{q=2}^r 1_{nn_q} \otimes a^q_{ij}(x)$, it follows by Corollary 1.2 that $\Phi_n(F)$ gives a vector bundle isomorphic to $E_F \otimes E_{11}$, where E_{11} is the vector bundle corresponding to the cocycle (U_i, a^1_{ij}).

2.2. Corollary. *The map $\Phi \to \Phi_*$ induces a bijection*

$$\operatorname{Hom}_{C(X)} (C(X) \otimes A, C(X) \otimes B)/\sim \;\to \{E \in \operatorname{Hom}_{\mathrm{Vect}(X)} (\mathrm{Vect}(X)^r, \mathrm{Vect}(X)^s) \colon E[\mathbf{n}] = [\mathbf{m}]\}.$$

Proof. Use (1) and Proposition 2.1.

Let $K_0(C(X) \otimes A)$ be the Grothendieck group for the abelian semigroup $D(C(X) \otimes A)$. Let $K_0(C(X) \otimes A)_+$ be the image of $D(C(X) \otimes A)$ in $K_0(C(X) \otimes A)$. $(K_0(C(X) \otimes A), K_0(C(X) \otimes A)_+)$ is an ordered group. The isomorphism $D(C(X) \otimes A) \to \mathrm{Vect}(X)^r$ induces an isomorphism of ordered groups $(K_0(C(X) \otimes A), K_0(C(X) \otimes A)_+) \to (K^0(X)^r, K^0(X)^r_+)$, where $K^0(X)_+$ is the image of $\mathrm{Vect}(X)$ in $K^0(X)$. Recall that $K^0(X)$ has a natural structure of ring. In $K_0(C(X) \otimes A)$ we distinguish the class of the unity $[1_{C(X) \otimes A}] = [\mathbf{n}]$. We shall denote by $\operatorname{Hom}_{K^0(X)} ((K^0(X)^r, K^0(X)^r_+, [\mathbf{n}]), (K^0(X)^s, K^0(X)^s_+, [\mathbf{m}]))$ the set of all pointed ordered group homomorphisms which are $K^0(X)$-linear.

2.3. Corollary. *Assume that the canonical map* $\mathrm{Vect}(X) \to K^0(X)$ *is injective. Then the map* $\Phi \to K_0(\Phi)$ *induces a bijection*

$$\mathrm{Hom}_{C(X)}(C(X) \otimes A, C(X) \otimes B)/\sim$$
$$\to \mathrm{Hom}_{K^0(X)}((K^0(X)^r, K^0(X)^r_+, [\mathbf{n}]), (K^0(X)^s, K^0(X)^s_+, [\mathbf{m}])).$$

3. Continuous Fields of AF-Algebras

Let X be a compact space and let $(A_i)_{i=1}^{\infty}$ be a sequence of finite-dimensional C^*-algebras. We consider a system

$$\cdots \to C(X) \otimes A_i \xrightarrow{\Phi_i} C(X) \otimes A_{i+1} \longrightarrow \cdots, \tag{3}$$

where each $*$-homomorphism Φ_i is unital, injective, and $C(X)$-linear. We show that the corresponding C^*-inductive limit $L = \varinjlim (C(X) \otimes A_i, \Phi_i)$ is $*$-isomorphic, by a $C(X)$-module isomorphism, to the C^*-algebra of the sections of some continuous field of AF-algebras (see [4]).

Since we can canonically identify $\mathrm{Hom}(A_i, C(X) \otimes A_{i+1})$ with $C(X, \mathrm{Hom}(A_i, A_{i+1}))$, each Φ_i defines a continuous map $X \ni x \to \Phi_i(x) \in \mathrm{Hom}(A_i, A_{i+1})$. Note that each $\Phi_i(x)$ is injective.

For any $x \in X$ define the AF-algebra $A(x) = \varinjlim (A_i, \Phi_i(x))$. We want to define a continuous field of AF-algebras $\mathscr{E}_L = ((A(x))_{x \in X}, \Gamma)$. Let L_0 be the algebraic inductive limit of the system (3). Then define $\eta: L_0 \to \prod_{x \in X} A(x)$ by $\eta([F])(x) = [F(x)]$, $\in X$, $F \in L_0$. ($[a]$ denotes the image of a in the corresponding inductive limit.)

Define Γ to be the closure of $\eta(L_0) \subset \prod_{x \in X} A(x)$ with respect to the norm $\|a\| = \sup_{x \in X} \|a(x)\|$. It is easily seen that $\mathscr{E}_L$ is a continuous field of AF-algebras. Moreover, η extends to a $C(X)$-linear $*$-isomorphism from L onto Γ. Thus, we have the following:

3.1. Proposition. *The inductive limit L is $*$-isomorphic to Γ by a $C(X)$-module isomorphism.*

3.2. *Remark.* If each A_i is a factor or if the space X is connected, then $A(x) \cong A(y)$, $x, y \in X$. If X is locally contractible, then the field $\mathscr{E}_L$ is locally trivial.

3.3. Proposition. *Let L, L' be inductive limits of the above type such that the fibres $A(x)$, $A'(x)$ $(x \in X)$ of $\mathscr{E}_L$, $\mathscr{E}_{L'}$ are simple. Then, for any $*$-isomorphism $\Phi: L \to L'$ there is a homeomorphism $\phi: X \to X$ such that*

$$\Phi(f \cdot a) = f \circ \phi \cdot \Phi(a), \qquad f \in C(X),\ a \in L.$$

Proof. Let $\eta: L \to \Gamma$ and $\eta': L' \to \Gamma'$ be the *-isomorphisms constructed in the proof of Proposition 3.1. Let ψ be the *-isomorphism which makes the diagram

$$\begin{array}{ccc} L & \xrightarrow{\eta} & \Gamma \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ L' & \xrightarrow{\eta'} & \Gamma' \end{array}$$

commutative. Since η and η' are $C(X)$-linear, it is enough to prove that $\psi(fa) = f \circ \phi \cdot \psi(a)$, $f \in C(X)$, $a \in \Gamma$.

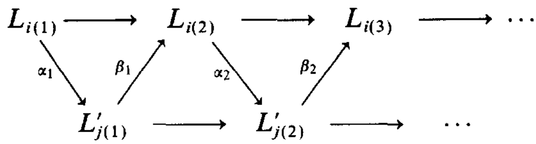
Since each $A(x)$ is simple, the maximal ideals of Γ are of the form $I_x := \{a \in \Gamma: a(x) = 0\}$, $x \in X$. Since ψ is an isomorphism, it induces a homeomorphism $\phi: X \to X$ such that $\psi(I_x) = I'_{\phi^{-1}(x)} := \{a' \in \Gamma': a'(\phi^{-1}(x)) = 0\}$. For $f \in C(X)$ and $a \in \Gamma$ we have $(f - f(x))a \in I_x$ hence $\psi((f - f(x))a)(\phi^{-1}(x)) = 0$, that is, $\psi(fa)(\phi^{-1}(x)) = f(x)\, \psi(a)(\phi^{-1}(x))$. The proof is complete.

3.4. *Remark.* Assume that all the AF-algebras $A(x)$ and $A'(x)$ are simple. Using Proposition 3.1 and a similar argument with that given in the proof of Proposition 3.3 one can see that $L \cong L'$ if and only if the field $\mathscr{E}_L$ is isomorphic to the pullback $\phi^* \mathscr{E}_{L'}$ for some homeomorphism $\phi: X \to X$.

4. Classification Results

Let X be a compact connected space. In this section we shall consider inductive limits $L = \varinjlim (C(X) \otimes A_i, \Phi_i)$, where $(A_i)_{i=1}^{\infty}$ is a sequence of finite-dimensional C^*-algebras and each $\Phi_i \in \mathrm{Hom}_{C(X)}(C(X) \otimes A_i, C(X) \otimes A_{i+1})$ is injective. Note that L inherits a natural structure of $C(X)$-module. Consider $D(L)$, the semigroup of homotopy classes of self-adjoint projections in $\bigcup_{n=1}^{\infty} M_n \otimes L$ (see Section 2). Since $D(L) = \varinjlim D(C(X) \otimes A_i)$, $D(L)$ inherits a natural structure of module over the semiring $\mathrm{Vect}(X)$. Our classification of the inductive limits L will be given in terms of $D(L)$ and $K_0(L)$. Consider two inductive limits $L = \varinjlim (C(X) \otimes A_i, \Phi_i)$ and $L' = \varinjlim (C(X) \otimes A'_i, \Phi'_i)$ of the above type. Set $L_i := C(X) \otimes A_i$ and $L'_i := C(X) \otimes A'_i$.

4.1. Lemma. *Let $\Phi: L \to L'$ be a *-isomorphism such that $\Phi(fa) = f \circ \phi \cdot \Phi(a)$, $f \in C(X)$, $a \in L$, for some homeomorphism $\phi: X \to X$. Then there is a commutative diagram of *-homomorphisms*

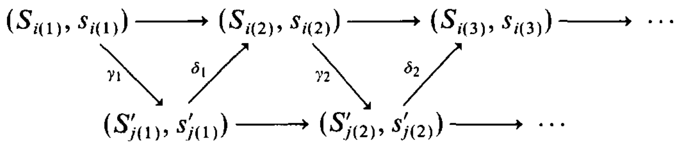


such that $\alpha_i(f) = f \circ \phi$ *and* $\beta_i(f) = f \circ \phi^{-1}$, $f \in C(X)$. *The converse is also true.*

Proof. We prove only the nontrivial implication. Using Glimm's Lemma [7, Lemma 1.8] as in the proof of Lemma 2.6 in [1], we can get suitable unitaries $u_i \in L'$, $v_i \in L$ such that the homomorphisms $\alpha_i = u_i \Phi u_i^*$ and $\beta_i = v_i \Phi^{-1} v_i^*$ have the desired properties.

Let S be a unital semiring. Consider two inductive limits $T = \varinjlim (S^{r_i}, \theta_i)$ and $T' = \varinjlim (S^{r'_i}, \theta'_i)$, where θ_i and θ'_i are homomorphisms (not necessarily injective) of S-modules. Note that T and T' inherit a natural structure of S-modules. Set $S_i = S^{r_i}$ and $S'_i = S^{r'_i}$. We shall distinguish an element s_i (resp. s'_i) in S_i (resp. S'_i) such that $\theta_i(s_i) = s_{i+1}$ (resp. $\theta'_i(s'_i) = s'_{i+1}$). Then T and T' will be pointed in the obvious way, by $t = [s_i]$ and $t' = [s'_i]$. Let $J: S \to S$ be an isomorphism of semirings.

4.2. LEMMA. *Let* $\Lambda: (T, t) \to (T', t')$ *be an isomorphism of pointed semigroups such that* $\Lambda(sa) = J(s)\,\Lambda(a)$, $s \in S$, $a \in T$. *Then there is a commutative diagram of homomorphisms of pointed semigroups*



such that $\gamma_k(sa) = J(s)\,\gamma_k(a)$, $\delta_k(sb) = J^{-1}(s)\,\delta_k(b)$, $s \in S$, $a \in S_{i(k)}$, $b \in S'_{j(k)}$. *The converse is also true.*

Proof. The proof uses the fact that S_k and S'_k are finitely generated as S-modules.

4.3. THEOREM. *Let* $L = \varinjlim (C(X) \otimes A_i, \phi_i)$, $L' = \varinjlim (C(X) \otimes A'_i, \Phi'_i)$. *Then L and L' are *-isomorphic by a $C(X)$-linear isomorphism if and only if $D(L)$ and $D(L')$ are isomorphic as semigroups, by a* Vect(X)*-linear isomorphism which takes the class of* 1_L *to the class of* $1_{L'}$.

Proof. The proof uses Corollary 2.2, Lemma 4.1 (with $\phi = id_X$), and Lemma 4.2 (with $S = \text{Vect}(X)$ and $J = id_S$).

4.4. THEOREM. *Assume that the fibres of the continuous fields* $\mathscr{E}_L$ *and* $\mathscr{E}_{L'}$

*(see Section 3) are simple. Then L and L′ are *-isomorphic if and only if there is an isomorphism of semigroups* $\Lambda: D(L) \to D(L')$ *which takes the class of* 1_L *to the class of* $1_{L'}$, *and such that*

$$\Lambda(sa) = J(s)\,\Lambda(a), \qquad s \in \mathrm{Vect}(X),\ a \in D(L),$$

where $J: \mathrm{Vect}(X) \to \mathrm{Vect}(X)$ *is an isomorphism of semirings induced by some homeomorphism* $X \to X$.

Proof. The proof uses Corollary 2.2, Proposition 3.3, Lemma 4.1, Lemma 4.2, and the following remarks:

(a) Let A, B be finite dimensional C^*-algebras and let $\Phi \in \mathrm{Hom}(C(X) \otimes A,\ C(X) \otimes B)$ be a *-homomorphism satisfying $\Phi(fa) = f \circ \phi \cdot \Phi(a)$, $f \in C(X)$, $a \in C(X) \otimes A$. Then we have a factorization $\Phi = \Phi_1 \phi^*$

$$C(X) \otimes A \xrightarrow{\phi^*} C(X) \otimes A \xrightarrow{\Phi_1} C(X) \otimes B,$$

where $\phi^*(F) = F \circ \phi$ and Φ_1 is a $C(X)$-linear *-homomorphism.

(b) If $\gamma: \mathrm{Vect}(X)^r \to \mathrm{Vect}(X)^t$ is a semigroup homomorphism satisfying $\gamma(sa) = J(s)\,\gamma(a)$, $s \in \mathrm{Vect}(X)$, $a \in \mathrm{Vect}(X)^r$, then we have the factorization $\gamma = \alpha J^{(r)}$

$$\mathrm{Vect}(X)^r \xrightarrow{J^{(r)}} \mathrm{Vect}(X)^r \xrightarrow{\alpha} \mathrm{Vect}(X)^t,$$

where $J^{(r)}(s_1, ..., s_r) = (J(s_1), ..., J(s_r))$ and α is $\mathrm{Vect}(X)$-linear.

We denote by $K_0(L)_+$ the image of $D(L)$ into $K_0(L)$. Since $K_0(L) = \varinjlim K_0(L_i)$ and $K_0(L)_+ = \varinjlim K_0(L_i)_+$ it follows that $K_0(L)$ inherits a natural structure of $K^0(X)$-module and the triplet $(K_0(L), K_0(L)_+, [1_L])$ is a pointed ordered group. *When the canonical map* $Vect(X) \to K^0(X)$ *is injective the above two Theorems can be formulated in terms of* K_0*-groups in the following way*: (compare with [6])

4.5. THEOREM. *L and L′ are *-isomorphic by a C(X)-linear isomorphism if and only if* $(K_0(L), K_0(L)_+, [1_L])$ *and* $(K_0(L'), K_0(L')_+, [1_{L'}])$ *are isomorphic as pointed ordered groups by a* $K^0(X)$*-linear isomorphism.*

4.6. THEOREM. *Assume that the fibres of the continuous fields* $\mathscr{E}_L$ *and* $\mathscr{E}_{L'}$ *are simple. Then L and L′ are *-isomorphic if and only if there is an isomorphism of pointed ordered groups*

$$\Lambda: (K_0(L), K_0(L)_+, [1_L]) \to (K_0(L'), K_0(L')_+, [1_{L'}])$$

such that $\Lambda(sa) = J(s)\,\Lambda(a)$, $s \in K^0(X)$, $a \in K_0(L)$, *where* $J: K^0(X) \to K^0(X)$ *is a ring isomorphism induced by some homeomorphism* $X \to X$.

5. Applications

Assume that X is a finite connected CW-complex of dimension $\leqslant 3$. Then there is an isomorphism of rings $\chi: K^0(X) \to (\mathbf{Z} \times H^2(X, \mathbf{Z}), +, \cdot)$ given by $\chi[E] = (\operatorname{rank}(E), c_1(E))$, $E \in \operatorname{Vect}(X)$, where $c_1(E)$ is the first Chern class of E. The ring structure on $\mathbf{Z} \times H^2(X, \mathbf{Z})$ is given by

$$(k, \alpha) + (l, \beta) = (k + l, \alpha + \beta)$$

$$(k, \alpha) \cdot (l, \beta) = (kl, l\alpha + k\beta),$$

where $\alpha, \beta \in H^2(X, \mathbf{Z})$, $k, l \in \mathbf{Z}$. Also, in this case the map $\operatorname{Vect}(X) \to K^0(X)$ is injective. These facts follow from the properties of stability of vector bundles (see [9]). When $X = S^2$ we obtain that

$$K^0(S^2) = \{s + tx : s, t \in \mathbf{Z}, x^2 = 0\} = \mathbf{Z}[x]/(x^2)$$

and

$$K^0(S^2)_+ = \{s + tx : (s, t) \in \mathbf{N}^* \times \mathbf{Z} \cup \{(0, 0)\}\}.$$

Let $3 < p_1 < p_2 < \cdots$ be a sequence of prime integers, $a = \left[\begin{smallmatrix} 1 & 4 \\ 1 & 1 \end{smallmatrix}\right]$, $\mathbf{n}_1 = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$ and define

$$\mathbf{n}_i = \begin{bmatrix} n_i' \\ n_i'' \end{bmatrix}$$

by $\mathbf{n}_{i+1} = a_i \mathbf{n}_i$, where $a_i = p_i a$, $i \geqslant 1$. Let $A_i = M_{n_i'} \oplus M_{n_i''}$ and consider a simple AF-algebra A given by the Bratteli system

$$A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \xrightarrow{a_3} \cdots.$$

We shall consider a C^*-algebra $L = \varinjlim (C(S^2) \otimes A_i, \Phi_i)$ whose pointed ordered K_0-group is given by the inductive limit corresponding to the following system of $K^0(S^2)$-linear homomorphisms:

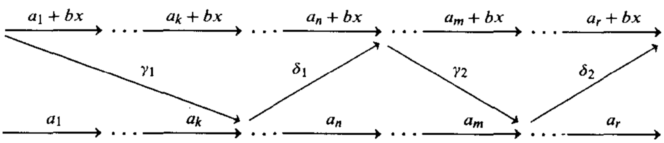
$$K^0(S^2)^2 \xrightarrow{a_1 + bx} K^0(S^2)^2 \xrightarrow{a_2 + bx} K^0(S^2)^2 \xrightarrow{a_3 + bx} \cdots,$$

where $b = \left[\begin{smallmatrix} -2 & 4 \\ 1 & -2 \end{smallmatrix}\right]$. Note that Φ_i is such that $K_0(\Phi_i) = a_i + bx$ and Φ_i is injective.

The following proposition shows that the C^*-algebras studied in this paper do not reduce to the C^*-algebras given by trivial fields of AF-algebras.

5.1. Proposition. *The inductive limit $L = \varinjlim (C(S^2) \otimes A_i, \Phi_i)$ is not *-isomorphic to any C^*-algebra of the form $C(S^2) \otimes B$, with B an AF-algebra.*

Proof. By reasons concerning the primitive spectrum of L, it is enough to show that L is not *-isomorphic to $C(S^2)\otimes A$. To get a contradiction assume that $K_0(L)$ is isomorphic to $K_0(C(S^2)\otimes A)$ as in Theorem 4.6. Since any homeomorphism $\phi\colon S^2\to S^2$ has the degree ± 1, it follows, with the notation of Theorem 4.6, that $J=K^0(\phi)\colon K^0(S^2)\to K^0(S^2)$ is given by $J(s+tx)=s\pm tx$. We shall consider only the case $J(s+tx)=s-tx$. The case $J=id$ is simpler. By Theorem 4.6 and Lemma 4.2 we must have a commutative diagram of the form (we have deleted the spaces $K^0(S^2)^2$)



where $\gamma_1=(c+dx)J^{(2)}$, $\delta_1=(e+fx)J^{(2)}$, $\gamma_2=(c'+d'x)J^{(2)}$, $\delta_2=(e'+f'x)J^{(2)}$, and $J^{(2)}=\left[\begin{smallmatrix} J & 0\\ 0 & J\end{smallmatrix}\right]$.

The following computations use the identities $ab=ba=-b$ and $J^{(2)}(g+hx)J^{(2)}=g-hx$, $g,h\in M_2(\mathbf{Z})$. The commutativity of the above diagram implies

$$ec=a_1\cdots a_k\cdots a_n \tag{4}$$

$$fc-ed=b(a_2a_3\cdots a_n+a_1a_3\cdots a_n+\cdots+a_1a_2\cdots a_{n-1}) \tag{5}$$

$$c'e=a_{k+1}\cdots a_n\cdots a_m \tag{6}$$

$$d'e-c'f=0 \tag{7}$$

$$e'c'ec=a_1\cdots a_k\cdots a_n\cdots a_m\cdots a_r. \tag{8}$$

From (4), (5), (6), and (7) we get

$$\begin{aligned} &d'a_1\cdots a_n-a_{k+1}\cdots a_n\cdots a_m d\\ &\quad=c'(p_2p_3\cdots p_n+\cdots+p_1p_2\cdots p_{n-1})(-1)^{n-1}b \end{aligned}$$

so that we infer that $\left[\begin{smallmatrix} p_n & 0\\ 0 & p_n\end{smallmatrix}\right]$ divides $c'b$ in $M_2(\mathbf{Z})$. It follows that p_n divides $\det(c')$. We obtain from (4) that p_n^2 divides $\det(ec)$ hence p_n^3 divides $\det(e'c'ec)$ which contradicts (8) since $\det(a)=-3$.

In contrast with the above Proposition we have the following:

5.2. Proposition. *Let X be a connected finite CW-complex of dimension $\leqslant 3$ and let A be a UHF-algebra, $A=\varinjlim A_i$, where each A_i is a finite discrete factor. Then $L=\varinjlim (C(X)\otimes A_i,\Phi_i)$ is *-isomorphic by a $C(X)$-linear isomorphism to $C(X)\otimes A$, for any choice of $\Phi_i\in \mathrm{Hom}_{C(X)}(C(X)\otimes A_i, C(X)\otimes A_{i+1})$.*

Proof. By hypothesis we have $\mathrm{Vect}(X) \simeq \mathbf{N}^* \times H^2(X, \mathbf{Z}) \cup \{0\} = \{s + \eta x : s \in \mathbf{N}^*,\ \eta \in H^2(X, \mathbf{Z}),\ x^2 = 0\} \cup \{0\}$. Hence

$$\begin{aligned}
&\mathrm{Hom}_{C(X)}(C(X) \otimes A_i, C(X) \otimes A_{i+1})/\sim \\
&\quad \simeq \{E \in \mathrm{Vect}(X) : E \otimes [n_i] = [n_{i+1}]\} \\
&\quad \simeq \{s_i + \eta x : \eta \in H^2(X, \mathbf{Z}),\ n_i \eta = 0,\ x^2 = 0\},
\end{aligned}$$

where $A_i = M_{n_i}$ and $s_i = n_{i+1}/n_i$ (see Corollary 2.2). Consider an arbitrary inductive limit $L' = \varinjlim (C(X) \otimes A_i, \Phi'_i)$ of the same type as L. We shall apply Theorem 4.3 to show that $L \cong L'$ as $C(X)$-modules. To prove that $(D(L), [1_L]) \simeq (D(L'), [1_{L'}])$ as pointed $\mathrm{Vect}(X)$-modules we shall use Lemma 4.2; i.e., we shall construct a commutative diagram of the type

$$\begin{array}{ccccccccc}
D_1 & \xrightarrow{s_1 + \eta_1 x} \cdots \xrightarrow{s_m + \eta_m x} & D_{m+1} & \to & \cdots \xrightarrow{s_r + \eta_r x} & D_{r+1} & \to \cdots \\
 & \searrow^{\gamma_1} & \uparrow{\scriptstyle \delta_1} & \searrow^{\gamma_2} & & \uparrow{\scriptstyle \delta_2} & \\
D'_1 & \xrightarrow{s_1 + \eta'_1 x} \cdots \xrightarrow{s_k + \eta'_k x} & D'_{k+1} & \to & \cdots \xrightarrow{s_q + \eta_q x} & D'_{q+1} & \to \cdots,
\end{array}$$

where $D_i = D'_i = (\mathbf{N}^* \times H^2(X, \mathbf{Z}) \cup \{0\}, n_i)$, $(\Phi_i)_* = s_i + \eta_i x$, $(\Phi'_i)_* = s_i + \eta'_i x$, $\gamma_1 = s_1 \cdots s_k + \xi_1 x$, $\delta_1 = s_{k+1} \cdots s_m + \xi_2 x$, $\gamma_2 = s_{m+1} \cdots s_q + \xi_3 x$, etc. Let $T_i := \{\eta \in H^2(X, \mathbf{Z}) : n_i \eta = 0\}$. The torsion part of $H^2(X, \mathbf{Z})$ is finite. Hence the sequence $T_1 \subset T_2 \subset \cdots$ stops. Since $\eta_i \in T_i$ we may assume that $\eta_i \in T_1$, $i \geqslant 1$. After dropping finitely many terms in the sequence $s_1, s_2, s_3, \ldots$ we may also assume that any class $\hat{s}_i \in \mathbf{Z}/n_1\mathbf{Z}$ occurs infinitely many times. With these assumptions, the sequence $(\xi_i)_{i=1}^{\infty}$, $\xi_1 = 0$, is constructed inductively, using the following remark: given $u < v$ and $\xi \in T_1$ there are $w > v$ and $\xi' \in T_1$ such that if $\gamma = s_u \cdots s_v + \xi x$ and $\delta = s_{v+1} \cdots s_w + \xi' x$, the diagram

$$\begin{array}{ccc}
D_u & \xrightarrow{s_u + \eta_u x} \cdots \xrightarrow{s_w + \eta_w x} & D_{w+1} \\
 & {}_{\gamma}\searrow \quad \nearrow_{\delta} & \\
 & D_{v+1} &
\end{array}$$

commutes, i.e.,

$$\prod_{i=u}^{w} (s_i + \eta_i x) = (s_{v+1} \cdots s_w + \xi' x)(s_u \cdots s_v + \xi x).$$

To prove this we choose w large enough such that

$$(s_u \cdots s_v)^\wedge \text{ divides } (s_{v+1} \cdots s_w)^\wedge \quad \text{in} \quad \mathbf{Z}/n_1\mathbf{Z}.$$

Note added in proof. After this paper was circulated as a preprint, INCREST 1986, we made the following remarks:

(a) The conclusion of Theorem 4.5 remains also true if instead of the injectivity of the canonical map $\mathrm{Vect}(X) \to K^0(X)$ we assume that X is a finite CW-complex and that the K_0-groups of the AF-fibres of the continuous fields $\mathscr{E}_L$ and $\mathscr{E}_{L'}$ are with large denominators, in the sense of V. Nistor: On the homotopy group of the automorphisms group of AF-C^*-algebras (to appear in *J. Operator Theory*).

(b) Since the simple AF-algebras have the K_0-groups with large denominators, the conclusion of Theorem 4.6 also holds if instead of the injectivity of the canonical map $\mathrm{Vect}(X) \to K^0(X)$ we assume that X is a finite CW-complex.

In addition to the previous arguments, the proofs of these statements use the stability properties of vector bundles over finite CW-complexes [9].

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