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A STABLY CONTRACTIBLE C*-ALGEBRA WHICH IS NOT CONTRACTIBLE

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ABSTRACT. We exhibit a separable commutative C*-algebra A such that $A\otimes\mathcal{K}$ is homotopy equivalent to zero, without $M_n(A)$ being so for any $n \geq 1$.

1. INTRODUCTION

This note answers a question asked privately by S. Eilers and T. Loring: are there non-contractible C*-algebras which are stably contractible? A stronger version of their question, whether there is a non-contractible C*-algebra A such that $M_2(A)$ is contractible is still open. The motivation for these questions comes from the study of projective and semiprojective C*-algebras. Examples of C*-algebras which are stably homotopy equivalent but not homotopy equivalent have already appeared in author's thesis and in [1]. Ultimately, these results rely on work of G. Segal [5], see also [4]. Hannes Thiel used our example to prove that projectivity does not pass to full hereditary sub-C*-algebras [6].

Let us recall that a space X is called acyclic if it path connected and its singular homology groups $H_n(X;\mathbb{Z}) = 0$ for $n \geq 1$. In [3, Ex. 2.38], a two-dimensional finite CW-complex X is constructed, which is acyclic, but it is not contractible since its fundamental group $\pi_1(X)$ is a nonzero perfect group. Specifically, X is obtained from $S^1 \vee S^1$ by attaching two 2-cells by the words a^5b^{-3} and $b^3(ab)^{-2}$, where a, b are the canonical generators of $\pi_1(S^1 \vee S^1) \cong \mathbb{F}_2$ the free group on two generators. Then $\pi_1(X) \cong \mathbb{F}_2/\{a^5 = b^3 = (ab)^2\}$ surjects onto the alternating group $A(5)$ by

$$a \mapsto (1,2,3,4,5), \quad b \mapsto (1,5,3)$$

and hence it is nonzero. If Y is a closed subspace of X we denote by $C_0(X,Y) \cong C_0(X \setminus Y)$ the C*-algebra of complex valued continuous functions on X which vanish on Y. A nonunital C*-algebra A is called contractible if id_A is homotopic to the zero $*$-homomorphism. Let $\mathcal{K}$ denote the compact operators acting on an infinite dimensional separable complex Hilbert space.

Theorem 1.1. *Let X be a non-contractible acyclic finite CW-complex and let $x_0 \in X$. The C*-algebra $C_0(X,x_0)\otimes\mathcal{K}$ is contractible, while $C_0(X,x_0)\otimes M_n(\mathbb{C})$ is not contractible for any $n \geq 1$.*

We need some preparation for the proof of the theorem. The result from Proposition 1.3 appears already in [2]. We reprove it here by a more direct argument.

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Recall that a map $p : E \to B$ with B path-connected is a quasifibration if the induced map $p_* : \pi_q(E, p^{-1}(b), e) \to \pi_q(B, b)$ is an isomorphism for all $b \in B$, $e \in p^{-1}(b)$ and $q \geq 1$. All the fibers $p^{-1}(b), b \in B$ of a quasifibration are weakly homotopy equivalent [3, p. 479]. Just like in the case of fibration, one derives from the long exact sequence of homotopy groups associated to the pair $(E, F = p^{-1}(b))$ a natural long exact sequence

$$\cdots \to \pi_q(F, e) \to \pi_q(E, e) \to \pi_q(B, b) \to \pi_{q-1}(F, e) \to \cdots \tag{1.1}$$

Let us consider the path fibration ([3, Prop. 4.64])

$$\Omega(B, b) \longrightarrow (PB, b) \xrightarrow{p'} (B, b)$$

where (PB, b) is the path space consisting of all continuous maps $\alpha : [0, 1] \to B$ with $\alpha(0) = b$, $p'(\alpha) = \alpha(1)$ and the fiber $\Omega(B, b)$ is the loop space of B.

Suppose that $p : (E, e) \to (B, b)$ is a continuous map such that the space E deformation retracts to e. This means that there is a continuous map $\Gamma : E \times [0, 1] \to E$ such that $\Gamma(e, t) = e$ for all $t \in [0, 1]$, $\Gamma(x, 0) = e$ and $\Gamma(x, 1) = x$ for all $x \in E$. Define $\gamma : E \to PB$, $x \mapsto \gamma_x$ by $\gamma_x(t) = p(\Gamma(x, t))$ for all $t \in [0, 1]$ and set $F = p^{-1}(b)$. Let us observe that γ maps (F, e) to $\Omega(B, b)$ and that the following diagram is commutative:

$$\begin{array}{ccccc}
(F, e) & \longrightarrow & (E, e) & \xrightarrow{p} & (B, b) \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} & & \| \\
\Omega(B, b) & \longrightarrow & (PB, b) & \xrightarrow{p'} & (B, b)
\end{array}$$

The following Lemma is a variation of [3, Prop. 4.66].

Lemma 1.2. *If $p : E \to B$ is a quasifibration such that the space E is contractible to e, then the map $\gamma : (F, e) \to \Omega(B, b)$ is a weak homotopy equivalence.*

Proof. Since E deformation retracts to e and PB deformation retracts to b, the result follows from the five lemma and the long exact sequences of homotopy groups:

$$\begin{array}{ccccccccc}
\pi_{n+1}(E, e) & \longrightarrow & \pi_{n+1}(B, b) & \longrightarrow & \pi_n(F, e) & \longrightarrow & \pi_n(E, e) & \longrightarrow & \pi_{n-1}(B, b) \\
\downarrow & & \| & & \downarrow{\scriptstyle\gamma_*} & & \downarrow & & \| \\
\pi_{n+1}(PB, b) & \longrightarrow & \pi_{n+1}(B, b) & \longrightarrow & \pi_n(\Omega(B, b)) & \longrightarrow & \pi_n(PB, b) & \longrightarrow & \pi_{n-1}(B, b)
\end{array}$$

□

For a space with base point (X, x_0), let us denote by $(SX, *)$ the reduced suspension of X. Thus SX is the is the one-point compactification of $(0, 1) \times (X \setminus x_0)$ and $*$ denotes the added point. If $X = \{0, 1\}$ then SX is homeomorphic to the unit circle, denoted by S. The suspension of a C*-algebra A is defined by $SA = C_0(S, *) \otimes A$. The suspension map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(SA, SB)$ takes φ to $S\varphi = id_{C_0(S,*)} \otimes \varphi$. Let us note that $C_0(SX, *) \cong C_0(S, *) \otimes C_0(X, x_0)$.

Proposition 1.3. *If (X, x_0) is a finite connected CW complex, then the suspension map*

$$\mathrm{Hom}(C_0(X, x_0), \mathcal{K}) \to \mathrm{Hom}(C_0(SX, *), C_0(S, *) \otimes \mathcal{K}) \cong \Omega(\mathrm{Hom}(C_0(SX, *), \mathcal{K}), 0)$$

is a weak homotopy equivalence.

Proof. Let us introduce the notation

$$F(X) = \mathrm{Hom}(C_0(X, x_0), \mathcal{K}) = \mathrm{Hom}(C_0(X \setminus x_0), \mathcal{K})$$

and choose the null homomorphism as base point. We rely on a result of G. Segal [5] which asserts that if Y is connected CW subcomplex of X, then the inclusion map $(Y, x_0) \hookrightarrow (X, x_0)$ induces a quasifibration $p : F(X) \to F(X/Y)$ with fiber $p^{-1}(0) = F(Y)$. We apply this result for the pair $X = X \times \{1\} \subset CX$ where CX is the reduced cone of (X, x_0). In other words CX is the one-point compactification of $(0, 1] \times (X \setminus x_0)$. Therefore $p : F(CX) \to F(SX)$ is a quasifibration with fiber $p^{-1}(0) = F(X)$. By applying Lemma 1.2 with the specific homotopy $\Gamma : F(CX) \times [0, 1] \to F(CX)$ defined by $\Gamma(\phi, t)(f \otimes g) = \phi(f(t\,\cdot) \otimes g)$, which deformation contracts $F(CX) = \mathrm{Hom}(C_0(0, 1] \otimes C_0(X, x_0), \mathcal{K})$ to the zero homomorphism, we obtain that the induced map $\gamma : (F(X), 0) \to \Omega(F(SX), 0)$ is a weak homotopy equivalence. It remains to identify the map γ with the usual suspension map. If $\phi \in F(CX)$, then $\gamma_\phi(t)(f \otimes g) = \phi(f(t\,\cdot) \otimes g)$. Since the inclusion $j : F(X) \hookrightarrow F(CX)$ is induced by $X \times \{1\} \subset CX$ we have $j(\varphi)(f \otimes g) = f(1)\varphi(g)$. Therefore

$$\Gamma(j(\varphi), t)(f \otimes g) = j(\varphi)(f(t\cdot) \otimes g) = f(t)\varphi(g).$$

Thus γ corresponds to the usual suspension map of $*$-homomorphisms. □

Proof of Theorem 1.1. Setting $A = C_0(X, x_0)$ it suffices to show that $\mathrm{id}_{A \otimes \mathcal{K}}$ is homotopic to zero. Recall that the suspension of a C*-algebra B is defined by $SB = C_0(S, *) \otimes B$. By Proposition 1.3 and by Whitehead's theorem, the map induced by γ

$$[(Y, y_0), \mathrm{Hom}(C_0(X, x_0), \mathcal{K})] \to [(Y, y_0), \mathrm{Hom}(C_0(SX, *), C_0(S, *) \otimes \mathcal{K})]$$

is a bijection for any finite connected CW-complex (Y, y_0). Therefore the suspension map

$$[C_0(Y, y_0), C_0(X, x_0) \otimes \mathcal{K}] \to [C_0(SY, *), C_0(SX, *) \otimes \mathcal{K}]$$

is bijective and in particular so is the map $[A, A \otimes \mathcal{K}] \to [SA, SA \otimes \mathcal{K}]$. On the other hand, $SA \cong C_0(SX, *)$ is homotopic to zero since the CW-complex SX is simply connected (as $\pi_1(SX, *) = 0$ by Freudenthal's suspension theorem) and acyclic (as $H_n(SX, \mathbb{Z}) \cong H_{n-1}(X, \mathbb{Z}) = 0$, for $n \geq 2$) and so SX is contractible by Whitehead's theorem. This shows that $[A, A \otimes \mathcal{K}]$ reduces to a point. Now, for any C*-algebra A, there is a natural bijection $[A, A \otimes \mathcal{K}] \cong [A \otimes \mathcal{K}, A \otimes \mathcal{K}]$, see [7, Lemma 1.4]. Therefore $A \otimes \mathcal{K}$ is contractible.

The C*-algebra $C_0(X, x_0)$ is not homotopy equivalent to the zero C*-algebra since the set of homotopy classes of $*$-homomorphisms $[C_0(X, x_0), C_0(S, *)]$ is in bijection with $\pi_1(X, x_0) \neq 0$. More generally, $C_0(X, x_0) \otimes M_n(\mathbb{C})$ is not homotopy equivalent to the zero C*-algebra for any $n \geq 1$. Seeking a contradiction, let us assume that $\phi_t : M_n(C_0(X, x_0)) \to M_n(C_0(X, x_0))$, $t \in [0, 1]$, is a continuous path of $*$-homomorphisms with $\phi_0 = \mathrm{id}$ and $\phi_1 = 0$. We can view this path as a $*$-homomorphism $\Phi : M_n(C_0(X, x_0)) \to M_n(C_0(X \times [0, 1], X \times \{1\} \cup \{x_0\} \times [0, 1])) \subset M_n(C(X \times [0, 1])$. Let ev_z denote the evaluation map at z. For each $z \in X \times [0, 1]$, $\mathrm{ev}_z \circ \Phi : M_n(C_0(X, x_0)) \to M_n(\mathbb{C})$ is either the zero map or an irreducible representation of $M_n(C_0(X, x_0))$ and hence it is unitarily equivalent to $\mathrm{ev}_{h(z)}$ for some point $h(z) \in X$. Moreover, the map $h : X \times [0, 1] \to X$ must be continuous and $h(X \times \{1\} \cup \{x_0\} \times [0,1]) \subset \{x_0\}$. But the existence of such a map implies contractibility of X, which is a contradiction.

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