GAFA, Geom. funct. anal.
Vol. 7 (1997) 646 – 711
1016-443X/97/040646-66 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

A CLASSIFICATION RESULT FOR APPROXIMATELY HOMOGENEOUS C*-ALGEBRAS OF REAL RANK ZERO

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Dedicated to Prof. Edward Effros on the occasion of his 60th birthday

Abstract

We prove that the total K-theory group $\underline{K}(-) = \bigoplus_{n=0}^{\infty} K_*(-;\mathbb{Z}/n)$ equipped with a natural order structure and acted upon by the Bockstein operations is a complete invariant for a class of approximately subhomogeneous C*-algebras of real rank zero which include the inductive limits of systems of the form

$$P_1 M_{n(1)}(C(X_1))P_1 \longrightarrow P_2 M_{n(2)}(C(X_2))P_2 \longrightarrow \cdots$$

where P_i are selfadjoint projections in $M_{n(i)}(C(X_i))$ and X_i are finite (possibly disconnected) CW complexes whose dimensions satisfy a certain growth condition. The problem of finding suitable invariants for the study of C*-algebras of this type was proposed by Effros. Our result represents a substantial generalization of the classification theorem of approximately finite dimensional (AF) C*-algebras, due to Elliott.

Contents

The first author is partially supported by NSF grant DMS-9303361 and the second is partially supported by NSF grant DMS-9401515.

Introduction

A sweeping conjecture of Elliott asserts that the simple nuclear C*-algebras are classifiable by suitable K-theoretical invariants. Although we may still be far away from a definitive answer, it is quite remarkable that the conjecture was already verified for a number of important classes of C*-algebras. The results available so far suggest that the nuclear C*-algebras of real rank zero manifest certain rigidity properties even in the absence of simplicity. It particular it may turn out that two shape equivalent (asymptotically homotopic) such algebras are necessarily isomorphic cf. [ElGo1], [D2-3] and [Go2-3]. Since the shape theory of C*-algebras is related to E-theory and hence to K-theory [D1], it is then natural to hope that many classes of (not necessarily simple) real rank zero C*-algebras are classifiable by K-theoretical invariants. Our main result gives such a classification in the realm of approximately (sub)homogeneous C*-algebras (see Theorem 9.1). The real rank zero condition is essential and it is quite natural if one is interested in classification results based on purely algebraic invariants. This point can be effectively illustrated if we restrict our considerations to inductive limits with strong slow dimension growth. By changing the connecting morphisms to morphisms with small spectral variation within the same homotopy class it easily seen that any such algebra is shape equivalent to some algebra of real rank zero. The following picture emerges as a consequence of Theorem 9.1. The classes of shape equivalent algebras with strong slow dimension growth are completely determined by $\underline{K}(-)$. Each class contains a real rank zero algebra and this algebra is unique up to an isomorphism.

The AF algebras are inductive limits of sequences of finite dimensional C*-algebras [Br]. It was shown by Elliott [El1], that the ordered scaled K_0 group codifies in an invariant way the combinatorics of Bratteli diagrams, hence being a complete invariant of AF algebras.

In 1982, Effros proposed the problem of finding suitable invariants for studying inductive limits of sequences of matrix algebras of continuous functions. In a slightly generalized version they correspond to systems of the form

$$P_1 M_{n(1)}(C(X_1))P_1 \xrightarrow{\nu_{21}} P_2 M_{n(2)}(C(X_2))P_2 \xrightarrow{\nu_{32}} \cdots$$

where X_i are finite CW complexes (not necessarily connected) and P_i are selfadjoint projections in $M_{n(i)}(C(X_i))$. For the sake of simplicity we call these inductive limits AH algebras, cf. [B4]. As explained in [B4] there is no gain of generality by allowing X_i to be compact separable spaces.

The first results in the early stage of this problem were obtain by [BuDe], [BKu], and [P1-2]. One motivation for this problem comes from the assumption (confirmed recently by a number of spectacular results [Pu], [BrEK1-2], [EvKis], [BrKRS], [ElL1-2], [Bo]) that the AH algebras include important classes of crossed product C*-algebras and hence the perspective of further connections with the field of dynamics, cf. [Pu], [HePuSk], [GPuSk].

As opposed to the case of AF algebras, the connecting map ν_{ji} cannot be classified up to unitary equivalence by K-theory. In fact the description of the stable homotopy classes of such maps, is based on connective K-theory, [Se], [DNe],

On the other hand it was gradually realized that the study of AH algebras should focus on "extremal" objects such as simple or real rank zero C*-algebras. A series of unexpected results [PiVo], [Lo1], [B3], followed by [Ku] and [EvKis], revealed higher dimensional features of AF algebras. They also showed that the simplicity or the real rank zero condition translate into dynamical properties of the connecting maps ν_{ji}, which lead to certain rigidity properties. This phenomenon was exploited in the breakthrough paper of Elliott [El2], where he classifies the real rank zero inductive limits of circle algebras. He also introduces the approximately intertwining argument for proving isomorphism results. This result was extended by Su to more general approximating blocks with one dimensional spectrum [Su]. Further progress in dimensions ≥ 2 required new techniques, which essentially, amounted to reducing the dimensions of local spectra X_i. For contractible spaces this was realized in [ElGLP1], based on an idea of Lin [L1] who solved the two dimensional case. This was extended to certain torsion free spaces with non-zero cohomology in different dimensions in [ElGo1], and [ElGLP2].

Combining results and techniques from [BroDoF], [DNe] [El2], [BBEK], [DNNP], [BDRø], [Ph1], [Su], [GoL], [L1] and [ElGLP1] with new techniques developed in [EG2], Elliott and the second author proved that if two real rank zero AH algebras with slow dimension growth are given by shape equivalent systems, then they are isomorphic. When combined with the homotopy computations of classes of homomorphisms from [DNe] and reduction results from [EG1] and [ElGLP2], this led to the classification of all simple AH algebras of real rank zero with $dim(X_i) \leq 3$ in terms of the ordered, scaled group K_*. The limitation to dimension three came from the fact that the connective KK-theory of [DNe] agrees with KK-theory only for spaces of low dimensions. Soon after, new methods for reducing the dimensions of the local spectra were developed in [D2-3] and [Go3]. The idea was to replace homotopy of $*$-homomorphisms by homotopy of asymptotic $*$-homomorphisms, which were obtained using the E-theory of [CoHi], via a suspension theorem of [DLo1] and to work with approximate intertwinings consisting of completely positive maps. In this way we showed that any real rank zero AH algebra with slow dimension growth is isomorphic to an AH algebra with $dim(X_i) \leq 3$. In conjunction with [ElGo2] this gave a classification of all simple AH algebras of real rank zero and slow dimension growth in terms of K_*. Similar results were obtained for *non-simple* AH algebras under the assumption that K_* is torsion free [ElGo2], [Go1-2], [D2]. However it was shown by the second author, that the ordered, scaled K_*-group is not a complete invariant in the non-simple case when torsion is present. Essentially different examples of this type were constructed in [DLo3]. It was then clear that additional new invariants were needed. They were constructed in several stages. First an order structure was introduced by Loring and the first author on the total K-theory

$$\underline{K}(A) = \bigoplus_{n\geq 0} K_*(A; \mathbb{Z}/n)$$

of a C*-algebra A. And it was proven that if a KK-equivalence $\alpha \in KK(A,B)$ induces an isomorphism $\underline{K}(A) \to \underline{K}(B)$ of the ordered, scaled groups of two real rank zero AD algebras, then $A \cong B$. The AD algebras are inductive limits of certain subhomogeneous C*-algebras with one dimensional spectrum and torsion K_1 groups [El2]. This result led to the question of which morphisms $\underline{K}(A) \to \underline{K}(B)$ are liftable to a KK element. The question was answered in certain special cases by Eilers [Ei] who obtained a complete classification of real rank zero AD algebras with bounded torsion in K_1. Inspired in part by [Ei], Loring and the first author [DLo4] and independently the second author realized that one should add to the order structure on $\underline{K}(A)$, the action of a set of operations Λ consisting of natural coefficient transformations and Bockstein operations. The new invariant was used in [DLo4] to complete the classification of real rank zero AD algebras. There is a universal multi-coefficient theorem which shows that $KK(A,B)$ maps surjectively onto $Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ [DLo4].

Our main result is a classification result for a class of C*-algebras (called ASH and defined in section 1) that includes strictly the AD algebras and the AH algebras of real rank zero whose local spectra satisfy a certain growth condition.

Theorem 9.1. *Let A, B be two ASH algebras of real rank zero and with* *slow dimension growth. Suppose that there is an isomorphism of ordered scaled groups*

$$\alpha : (\underline{K}(A), \underline{K}(A)^+, \Sigma(A)) \to (\underline{K}(B), \underline{K}(B)^+, \Sigma(B))$$

which preserves the action of the Bockstein operations. Then there is a $*$-isomorphism $\varphi : A \to B$ with $\varphi_* = \alpha$.

As a corollary of Theorem 9.1 we show that the ordered scaled group $K_*(-)$ is a complete invariant for the simple ASH algebras of real rank zero with slow dimension growth (see Theorem 9.4).

Thanks are due to S. Eilers and K. Thomsen for a number of useful remarks which helped us to improve the exposition.

The authors are pleased to acknowledge their collaboration over the past few years with George Elliott, Huaxin Lin, Andras Nemethi, Terry Loring and Cornel Pasnicu which resulted in many of the ideas and the techniques on which the present paper is based.

1 Notation and Terminology

We begin by describing the building blocks of the inductive limits we study. For $n \geq 2$, let W_n denote the Moore space obtained by attaching a disk to a circle by a degree-n map. Then $K_0(C(W_n)) = \mathbb{Z} \oplus \mathbb{Z}/n$ and $K_1(C(W_n)) = \{0\}$. Let $\mathbb{I}_n$ denote the nonunital dimension-drop algebra

$$\mathbb{I}_n = \{a \in C([0,1], M_n) : a(0) = 0, a(1) \in \mathbb{C}1_n\} .$$

The unitalization of $\mathbb{I}_n$ is denoted by $\tilde{\mathbb{I}}_n$. It is well known that $K_0(\tilde{\mathbb{I}}_n) = \mathbb{Z}$ and $K_1(\tilde{\mathbb{I}}_n) = \mathbb{Z}/n$. We deal with various C*-algebras of the form

$$\oplus_{j=1}^{s} P_j M_{k(j)}(C(Y_j)) P_j \oplus \oplus_{i=1}^{r} M_{r(i)}(\tilde{\mathbb{I}}_{n(i)}) \tag{1.1}$$

where Y_j are finite connected CW complexes and P_j are selfadjoint projections in $M_{k(j)}(C(Y_j))$. It is not necessary to include here C*-algebras of the form $PM_r(\tilde{\mathbb{I}}_n)P$ as any projection P is unitary equivalent to a multiple of the identity of $\tilde{\mathbb{I}}_n$. The class of C*-algebras isomorphic to C*-algebras of the form (1.1) is denoted by SH. The subclass of SH consisting of C*-algebras of the form $\oplus_{j=1}^{s} P_j M_{k(j)}(C(Y_j)) P_j$ is denoted by H. The subclass of SH consisting of C*-algebras of the form $\oplus_{i=1}^{r} M_{r(i)}(\tilde{\mathbb{I}}_{n(i)})$ is denoted by D. Let $SH(2)$ denote the subclass of SH defined by the condition that the spaces Y_j must be of the form

$$\{pt\},\ S^1,\ \text{or } W_n\ , \qquad n \geq 2\ . \tag{1.2}$$

There are two key features of the C*-algebras in the class $SH(2)$: a good correspondence between the homotopy classes $[A,B]$ and the positive elements in $KK(A,B)$ and the fact that the semigroup $K_0(A)^+$ is finitely generated. The crucial role of these feature will become apparent in section 5.

The following class will be needed for technical reasons. Let $SH(2)^\#$ be the extension of the class $SH(2)$, obtained by adding the two-sphere S^2 to the list (1.2).

Let ASH denote the class of all C*-algebras which are isomorphic to countable inductive limits of C*-algebras in SH. The classes AH, AD, $ASH(2)$ and $ASH(2)^\#$ are defined similarly. Later on we will prove containments of the type $ASH \subset ASH(2)$ for real rank zero C*-algebras with slow dimension growth.

Let us emphasize that the C*-algebras $\tilde{\mathbb{I}}_n$ play an important role in the paper. In our approach their use would be necessary even if we decide to classify just the real rank zero AH-algebras.

If A and B are two C*-algebras, we let $Map(A,B)$ denote the space of all linear, completely positive contractions from A to B. If both A and B are unital, then $Map(A,B)_1$ will denote the subspace of $Map(A,B)$ consisting of unital maps. Let $F \subset A$ be a finite set and let $\delta > 0$. We say that $\varphi \in Map(A,B)$ is δ-multiplicative on F if $\|\varphi(ab) - \varphi(a)\varphi(b)\| < \delta$ for all $a, b \in F$.

2 Large Morphisms and Slow Dimension Growth

In this section we introduce certain technical conditions which make possible the passage from K-theory computations to homotopy computations. Suppose that $A \in SH$ has connected spectrum. Then there is a natural rank map, $rank : K_0(A) \to \mathbb{Z}$. If $A = PM_r(C(X))P$, then this is the map induced on K_0 by any irreducible representation of A. If $A = M_r(\tilde{\mathbb{I}}_n)$, then $rank$ is the map induced on K_0 by the evaluation map at zero, $\delta_0 : M_r(\tilde{\mathbb{I}}_n) \to M_r(\mathbb{C})$. For example $rank\ [1_A] = r$. Let $K_0(A)'$ denote the kernel of the rank map. Then $K_0(A) = \mathbb{Z} \oplus K_0(A)'$. If A, B are C*-algebras in SH with connected spectrum and $\sigma : K_0(A) \to K_0(B)$ is a positive morphism, then arguing as in the proof of Proposition 2.1.3 of [DNe], we see that σ maps $K_0(A)'$ into $K_0(B)'$. Moreover if $x \mapsto kx$ is the map induced by σ on $\mathbb{Z}$, then $k \geq 0$ and $rank\, \sigma(y) = k\, rank\,(y)$ for all $y \in K_0(A)^+$. In particular $\sigma = 0$ if $k = 0$. If $A \in SH$ has connected spectrum and p is a projection in A, then $rank[p]$ will sometimes be denoted by $rank(p)$. Let A, B be C*-algebras in SH with connected spectrum and let $\alpha \in KK(A,B)$. Then α induces a map α_* on the K_0 groups. Let d denote the dimension of the spectrum of B. The KK element α is called *m-large* if either $rank\,\alpha_*[1_A] \geq md\,rank[1_A]$ or $rank\,\alpha_*[1_A] = 0$. The KK element α is called *strictly m-large* if either $rank\,\alpha_*[1_A] \geq m(d+1)\,rank[1_A]$ or $rank\,\alpha_*[1_A] = 0$.

We now extend this definition to elements $\alpha \in KK(A,B)$ where $A, B \in SH$. Let $A = \oplus_i A_i$ and $B = \oplus_j B_j$ where A_i and B_j have connected spectrum. Let $\alpha_{j,i} \in KK(A_i, B_j)$ be the corresponding components of α. The KK-element α is called (strictly) m-large if all $\alpha_{j,i}$ are (strictly) m-large. A $*$-homomorphism $\varphi : A \to B$ is called is called (strictly) m-large if its KK class is (strictly) m-large. Note that if B is a finite dimensional C*-algebra then any $*$-homomorphism from A to B is m-large for any $m \in \mathbb{N}$. The notion of slow dimension growth for a simple AH algebra was introduced in [BDRø]. Subsequently it was extended to the nonsimple case by several authors [Goo1], [MP]. In this paper we work with the following version introduced in [Go3]. Let $(A_n, \nu_{r,n})$ be an inductive system of SH algebras. Write $A_n = \oplus_i A_{n,i}$ where $A_{n,i}$ has connected spectrum denoted by $X_{n,i}$. The partial morphisms of $\nu_{r,n} : A_n \to A_r$ are denoted by $\nu_{r,n}^{j,i}$.

Definition 2.1. *The system $(A_n, \nu_{r,n})$ is said to have slow dimension growth if for any n, there is $d_n \geq 0$ such that*

$$\lim_{r\to\infty} \min_{dim(X_{r,j})>d_n,\, \nu_{r,n}^{j,i}\neq 0} \frac{rank\ \nu_{r,n}^{j,i}(1_{A_{n,i}})}{dim(X_{r,j})} = +\infty\ .$$

(Here we use the convention that the minimum of the empty set is ∞.) An ASH algebra is said to have slow dimension growth if it is isomorphic to the inductive limit of a system with slow dimension growth.

Remarks 2.2. a) If the dimensions of the spectra of A_n form a bounded sequence, then the system $(A_n, \nu_{r,n})$ has slow dimension growth.

b) It follows immediately from 2.1 that if $(A_n, \nu_{r,n})$ has slow dimension growth then for any n, there is an integer $d_n \geq 0$ with the following property. For any integer m there is $r_0 \geq n$ such that for all $r \geq r_0$ each partial homomorphism $\nu_{r,n}^{j,i}$ satisfies either $rank\,\nu_{r,n}^{j,i}(1_{A_{n,i}}) \geq m(dim(X_{r,j})+1)rank(1_{A_{n,i}})$, i.e. $\nu_{r,n}^{j,i}$ is strictly m-large or $rank\,\nu_{r,n}^{j,i}(1_{A_{n,i}}) \leq m(d_n+1)rank(1_{A_{n,i}})$ (cf. [Go3]).

c) Let $(A_n, \nu_{r,n})$ be a system with slow dimension growth. Let $p \in A_m$ be a projection and let $p_n = \nu_{n,m}(p)$, $n \geq m$. Then the system $(p_n A_n p_n, \nu_{r,n})$

has slow dimension growth.

For a number x we let $\langle x\rangle$ denote the smallest integer greater than or equal to x. The following proposition collects a number of well known stability properties of vector bundles, see [Hu, Chapter 8].

PROPOSITION 2.3. *Let $A = PM_N(C(X))P$ where X is a connected CW complex of dimension d. Let e, f be projections in A.*

a) *Suppose that $rank(e) \geq \langle d/2\rangle$. If $[e] = [f]$ in $K_0(A)$, then there is a partial isometry $v \in A$ such that $v^*v = e$ and $vv^* = f$.*
b) *Suppose that $[e] = [f]$ in $K_0(A)$. If $rank(P) - rank(e) \geq \langle d/2\rangle$ and $rank(e) \geq \langle d/2\rangle$, then there is a unitary $u \in A$ such that $ueu^* = f$.*
c) *Suppose that $rank(f) - rank(e) \geq \langle (d-1)/2\rangle$. Then there is a partial isometry $v \in A$ such that $v^*v = e$ and $vv^* \leq f$.*

It follows from Proposition 2.3 that the C*-algebras in $SH(2)^{\#}$ have cancellation of projections.

3 Weak Variation and Spectral Variation

This section is based on results from [El2], [Su] and [ElGo2]. Very often we give only sketchy arguments as they are only minor variations of those given in the quoted papers. The condition of real rank zero for an ASH algebra corresponds to a certain asymptotic behaviour of the spectral variation of the connecting morphisms $\nu_{r,n}$. Eventually this is used to produce a splitting of these morphisms into large morphisms with finite dimensional image and small approximate morphisms carrying the nontrivial part of the topological complexity of the $\nu_{r,n}$.

DEFINITION 3.1 ([D3]). *Let X be a compact connected space and let P be a projection of rank n in $M_N(C(X))$. The weak variation of a finite set $F \subset PM_N(C(X))P$ is defined by*

$$w(F) = \sup_{\pi,\sigma} \inf_{u\in U(n)} \max_{a\in F} \left\| u\pi(a)u^* - \sigma(a)\right\|$$

where π, σ run through the set of irreducible representations of $PM_N(C(X))P$ into M_n.

DEFINITION 3.2. *Let F be a finite subset of $M_r(\tilde{\mathbb{I}}_n)$. The weak variation of F is defined by*

$$w(F) = \sup_{s,t\in(0,1)} \inf_{u\in U(rn)} \max_{a\in F} \left\| ua(s)u^* - a(t)\right\| .$$

Note that if $\kappa : M_r(\tilde{\mathbb{I}}_n) \to M_{rn}(C[0,1])$ is the canonical embedding, then $w(F) = w(\kappa(F))$.

The Definitions 3.1–2 extend to finite subsets F of a C*-algebra $A \in SH$. Thus $w(F)$ is taken to be the maximum of the weak variations in each direct summand. The above definitions are inspired by Definition 1.4.11 in [ElGo2] and [El1].

Lemma 3.3. *Let $\varphi : M_r(\tilde{\mathbb{I}}_n) \to M_m$ be a $*$-homomorphism. There exist integers m_0, m_1 with $0 \le m_i \le n-1$ and a unitary $u \in U(m)$ such that φ is of the form*

$$\varphi(a) = u\ diag(\underbrace{\delta_0(a), \dots, \delta_0(a)}_{m_0-\text{times}}, \underbrace{\delta_1(a), \dots, \delta_1(a)}_{m_1 rm-times}, \hat{\varphi}(a))u^*$$

for all $a \in M_r(\tilde{\mathbb{I}}_n)$, where $\delta_i : M_r(\tilde{\mathbb{I}}_n) \to M_r$ are the irreducible representations corresponding to the endpoints $0, 1$ and $\hat{\varphi}$ is the restriction to $M_r(\tilde{\mathbb{I}}_n)$ of some $*$-homomorphism $\hat{\varphi} : M_r(M_n(C[0,1])) \to M_m$. The integers m_0, m_1 are uniquely determined by the KK class of φ.

Proof. The first part of the lemma is implicit in [El1]. The second part is an obvious consequence of the isomorphism $[\mathbb{I}_n, \mathcal{K}] \cong KK(\mathbb{I}_n, \mathbb{C})$ from [DLo1].

If $\psi : A \to B$ is a $*$-homomorphism with $A, B \in H$ and if F is a finite subset of A, then it is not hard to see that $w(\psi(F)) \le w(F)$. Using Lemma 3.3 one shows that this is also true for $A, B \in SH$.

The spectral variation of a $*$-homomorphism $\varphi : A \to B$ was defined in [ElGo2, 1.4] for C*-algebras in H. The reader is referred to that paper for the precise definition which depends on choosing a metric d on the spectrum X of A. However let us mention that if X is connected, $A = C(X)$, $B = M_k(C(Y))$, then $\varphi : A \to B$ is underlined by a continuous map $\hat{\varphi}$ from Y to the k^{th}-order symmetric product of X. Then $SPV(\varphi)$ coincides with the diameter of the image of $\hat{\varphi}$. One can extend this notion for C*-algebras in SH as follows (cf. [ElGoSu]).

Let $h \in \tilde{\mathbb{I}}_n$ be given by $h(t) = t1_{\tilde{\mathbb{I}}_n}$ for all $t \in [0,1]$. Given a unital $*$-homomorphism $\varphi : \tilde{\mathbb{I}}_n \to PM_N(C(X))P$, $rank\,(P) = p$, for any point $x \in X$, let

$$\lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_p(x)$$

denote the list of all eigenvalues of $\varphi(h)(x) \in M_p$, each eigenvalue being repeated according to its multiplicity. We define the spectral variation of φ by

$$SPV(\varphi) = \sup_{x,y \in X} \max_{1 \le i \le p} \left|\lambda_i(x) - \lambda_i(y)\right| .$$

If $\varphi : \tilde{\mathbb{I}}_n \to M_k(\tilde{\mathbb{I}}_m)$ is a unital $*$-homomorphism we define $SPV(\varphi) = SPV(\kappa\varphi)$, where $\kappa : M_k(\tilde{\mathbb{I}}_m) \to M_{km}(C[0,1])$ denotes the natural embedding. The above definitions extend to nonunital $*$-homomorphisms $\varphi : M_r(\tilde{\mathbb{I}}_n) \to B$, with $B \in SH$ having connected spectrum, by setting $SPV(\varphi) = SPV(\varphi_0)$, where $\varphi_0 : \tilde{\mathbb{I}}_n \to EBE$ is the restriction of φ to $\tilde{\mathbb{I}}_n \otimes e_{11}$ and $E = \varphi(1_{\tilde{\mathbb{I}}_n} \otimes e_{11})$. Finally, given an arbitrary $*$-homomorphism $\varphi : A \to B$, with $A, B \in SH$, we define $SPV(\varphi)$ to be equal to the maximum of the spectral variations of all the partial $*$-homomorphisms of φ.

The next three lemmas are generalizations of results in [ElGo2]. The proofs are similar to those in [ElGo2, 1.4], but one also uses Lemma 3.3 for the maps from $\tilde{\mathbb{I}}_n$.

LEMMA 3.4. *If $A, B, C \in SH$ and $\varphi : A \to B$ and $\psi : B \to C$ are $*$-homomorphisms, then $SPV(\psi\varphi) \leq SPV(\varphi)$.*

LEMMA 3.5. *For any $*$-homomorphism $\varphi : A \to B$, with $A, B \in SH$ and any $\epsilon > 0$, there is $\delta > 0$ such that if $\psi : B \to C$ is any $*$-homomorphism with $C \in SH$ and $SPV(\psi) < \delta$, then $SPV(\psi\varphi) < \epsilon$.*

LEMMA 3.6. *Suppose $A \in SH$ has connected spectrum and let $F \subset A$ be a finite subset. Then for any $*$-homomorphism $\varphi : A \to B$, with $B \in SH$,*

$$w(\varphi(F)) \leq \sup_{d(x,y) \leq SPV(\varphi)} \max_{a \in F} \|a(x) - a(y)\|$$

where $d(x, y)$ is the distance between the points x, y in the spectrum of A.

If $A = QM_N(C(X))Q$, then A is regarded as a subalgebra of $M_N(C(X))$. If $A = M_r(\tilde{\mathbb{I}}_n)$, then A is regarded as a subalgebra of $M_{rn}(C[0, 1])$. In both cases we can therefore make sense of $a(x)$ and $a(y)$.

The following is a slight generalization of Theorem 2.5 of [Su] and Remark 1.4.6 of [ElGo2].

Theorem 3.7. *Let $A = \varinjlim(A_n, \nu_{r,n})$ be an ASH algebra of real rank zero. Then for any n and any $\delta > 0$, there is $m \geq n$ such that $SPV(\nu_{r,n}) < \delta$ for all $r \geq m$.*

Then as in [ElGo2], by combining Lemma 3.6 and Theorem 3.7 we derive the following corollary.

COROLLARY 3.8. *Let $A = \varinjlim(A_n, \nu_{r,n})$ be an ASH algebra of real rank zero. Let $F \subset A_n$ be a finite set. Then $\lim_{r\to\infty} w(\nu_{r,n}(F)) = 0$.*

PROPOSITION 3.9. *Suppose that A is an ASH-algebra of real rank zero which is isomorphic to the inductive limit of a system $(A_n, \nu_{r,n})$ of C*-algebras in SH with slow dimension growth. Let F be a finite subset of*

A_n and let $\epsilon > 0$. For any $m \geq 1$ there is $r \geq n$ such that each partial map $\nu_{r,n}^{j,i}$ satisfies either

(i) *$rank\, \nu_{r,n}^{j,i}(1_{A_{n,i}}) \geq m(dim(X_{r,j}) + 1)\, rank\,(1_{A_{n,i}})$, i.e. $\nu_{r,n}^{j,i}$ is strictly m-large,*

or

(ii) *there is a $*$-homomorphism $\mu : A_{n,i} \to A_{r,j}$ with finite dimensional image such that $\nu_{r,n}^{j,i}(1) = \mu(1)$, $\nu_{r,n}^{j,i}$ is homotopic to μ and $\|\nu_{r,n}^{j,i}(a) - \mu(a)\| < \epsilon$ for all a in the component of F in $A_{n,i}$.*

Proof. Let $\mathcal{C}$ be a class of C*-algebras in SH with the following property denoted by $(*)$. For any $A \in \mathcal{C}$ with connected spectrum, $F \subset A$ a finite set, $M \in \mathbb{N}$ and $\epsilon > 0$ there is $\delta = \delta(A, F, M, \epsilon) > 0$ such that whenever $B \in \mathcal{C}$ has connected spectrum and $\varphi : A \to B$ is a $*$-homomorphism with $SPV(\varphi) < \delta$, at least one of the following holds:

(1) $rank(\varphi(1_A)) > M\, rank(1_A)$

(2) there is a $*$-homomorphism $\mu : A \to B$ with finite dimensional image such that $\varphi(1_A) = \mu(1_A)$ and $\|\varphi(a) - \mu(a)\| < \epsilon$ for all $a \in F$ and φ is homotopic to μ.

Since the spectra of A and B are connected, (1) is equivalent to

(1′) $rank(\varphi(e)) > M\, rank(e)$ for any nonzero projection $e \in A$.

The classes D and H are known to satisfy $(*)$ by [El2, 7.2] and respectively [ElGo2, 2.3, 2.5]. By combining the techniques from these papers one can show that the entire class SH also satisfies $(*)$ and this is easily seen to imply Proposition 3.9. Rather than doing that, we just prove Proposition 3.9 using the fact that the classes D and H satisfy $(*)$.

Fix n and let d_n be given by Remark 2.2 (b). Let m be as in the statement and let r_0 be as in Remark 2.2 (b). For $F \subset A_n$ we let F_i denote the component of F in $A_{n,i}$. Given $F \subset A$ a finite set, $M \in \mathbb{N}$, $M > m(d_n + 1)$ and $\epsilon > 0$, by using Theorem 3.7, we find r, s with $r > s > r_0 > n$, such that

$$SPV(\nu_{s,n}) < \delta_1 = \min_i \delta(A_{n,i}, F_i, M, \epsilon)\ ,$$
$$SPV(\nu_{r,s}) < \delta_2 = \min_\ell \delta(A_{s,\ell}, F'_\ell, M, \epsilon)\ .$$

where $F'_\ell = \cup_i \nu_{s,n}^{\ell,i}(F_i)$. Next we are going to show that each $\nu_{r,n}^{j,i}$ satisfies either (1) or (2). For fixed i, j, any partial morphism $\nu_{r,n}^{j,i}$ is a sum of morphisms θ^ℓ which map $A_{n,i}$ in mutually orthogonal full corners of $A_{r,j}$ and each θ has a factorization of the form

$$A_{n,i} \xrightarrow{\varphi^\ell := \nu_{s,n}^{\ell,i}} e^\ell A_{s,\ell} e^\ell \subset A_{s,\ell} \xrightarrow{\psi^\ell := \nu_{r,s}^{j,\ell}} A_{r,j}\ .$$

Note that at least two out of the three C*-algebras $A_{n,i}$, $A_{s,\ell}$, $A_{r,j}$ must be of the same type, i.e. they are either in the class D or in the class H. Also $SPV(\varphi^\ell) < \delta_1$, $SPV(\theta^\ell) = SPV(\psi^\ell\varphi^\ell) \le SPV(\varphi^\ell) < \delta_1$, $SPV(\psi^\ell) < \delta_2$. Therefore, using our choice of δ_i in connection with property $(*)$ we see that at least one of θ^ℓ, φ^ℓ or ψ^ℓ satisfies the conclusion of property $(*)$ (with $F = F_i$ if this applies to θ^ℓ or φ^ℓ and with $\varphi^\ell(F_i)$ if this applies to ψ^ℓ.) In all situations it is clear that each $\theta^\ell = \psi^\ell\varphi^\ell$ will end up satisfying the conclusion of $(*)$. Note that if at least one of the θ^ℓ satisfies (1), than so does $\nu^{j,i}_{r,n}$. The only other possibility is that each of the θ^ℓ satisfies (2) and hence $\nu^{j,i}_{r,n}$ satisfies (2).

Therefore either

$$rank\, \nu^{j,i}_{r,n}(1_{A_{n,i}}) > M\, rank(1_{A_{n,i}}) > m(d_n + 1)rank(1_{A_{n,i}})$$

in which case $rank\, \nu^{j,i}_{r,n}(1_{A_{n,i}}) \ge m(dim(X_{r,j}) + 1)rank(1_{A_{n,i}})$ by Remark 2.2 (b) since $r \ge r_0$ or else $\nu^{j,i}_{r,n}$ satisfies (2) and hence it can be approximated by a finite dimensional $*$-homomorphism as in the statement of Proposition 3.9. □

An immediate consequence of Proposition 3.9 and Proposition 2.3 is that any ASH algebra of real rank zero and slow dimension growth has cancellation of projections.

The following theorem was proved in [ElGo2] if both A and B are of the form $QM_N(C(X))Q$. If both A and B are matrices over dimension drop algebras, then the corresponding result is implicit in [El2]. For the purposes of this paper we need only those two cases. This is because in applications we are going to deal with morphisms whose partial maps factor through morphisms between blocks of the same type. In any case, a complete proof will appear in [ElGoSu].

Theorem 3.10. *Let $A \in SH$ have connected spectrum. Let $G_1 \subset A$ be finite subset, let $\epsilon_1 > 0$ and let $L \ge 0$ be an integer. There exist $m_1 \ge 1$ and $\delta > 0$ such that if $B \in SH$ and $\nu : A \to B$ is a strictly m_1-large, unital $*$-homomorphism with spectral variation less than δ, then there exist a projection $p \in B$ and a unital $*$-homomorphism $\lambda : A \to (1-p)B(1-p)$ with finite dimensional image such that*

(i) $[\lambda(1)] \ge L[p]$ *in* $K_0(B)$
(ii) $\|\nu(a)p - p\nu(a)\| < \epsilon_1$
(iii) $\|\nu(a) - p\nu(a)p - \lambda(a)\| < \epsilon_1$ *for all* $a \in G_1$.

Notice that if all the elements of G_1 have norm at most one, then the map $\theta(a) = p\nu(a)p$ is ϵ_1-multiplicative on G_1.

The following is a converse to Theorem 3.7.

PROPOSITION 3.11. *Let $A = \varinjlim(A_n, \nu_{r,n})$ be an ASH algebra with the spectrum of A_n of dimension d_n. Suppose that the sequence (d_n) is bounded and that for any n and any $\delta > 0$, there is $r \geq n$ such that $SPV(\nu_{r,n}) < \delta$. Then A has real rank zero.*

Proof. If A is an AH algebra the result is given by [ElGo2, Corollary 2.5]. For AD algebras the result is contained in [El2, Theorem 6.2]. A proof for Proposition 3.11 is obtained by combining in a straightforward manner the arguments given in the proofs of the quoted results.

4 The Invariant

4.1. We begin by defining the total K-theory $\underline{K}(A)$ of a C*-algebra A. The invariant we use to classify the ASH algebras is $\underline{K}(A)$, acted upon by the Bockstein operations [S] and endowed with a natural order structure [DLo3-4]. K-theory with mod-n coefficients for C*-algebras was considered in [Ka], [Cu1], [S], [RosS]. For $n \geq 2$, recall that W_n denotes the Moore space obtained by attaching the disk to the circle by a degree n-map. Each space W_n will be pointed by the image of a fixed interior point of the disk. The C*-algebra $C_0(W_n)$ of continuous functions vanishing at the base point is isomorphic to the mapping cone of the canonical map of degree n from $C_0(S^1)$ to itself. In the setting of [S], the mod-n K-theory groups are defined by

$$K_i(A;\mathbb{Z}/n) = K_i(A \otimes C_0(W_n)) \ .$$

Let $K_*(A;\mathbb{Z}/n) = K_0(A;\mathbb{Z}/n) \oplus K_1(A;\mathbb{Z}/n)$. For $n = 0$ we set $K_*(A;\mathbb{Z}/n) = K_*(A)$ and for $n = 1$, $K_*(A;\mathbb{Z}/n) = 0$.

For a C*-algebra A, one defines the total K-theory of A by

$$\underline{K}(A) = \bigoplus_{n=0}^{\infty} K_*(A;\mathbb{Z}/n) \ .$$

It is a $\mathbb{Z}/2 \times \mathbb{Z}^+$ graded group. It was shown in [S] that the coefficient maps

$$\rho : \mathbb{Z} \to \mathbb{Z}/n \ , \quad \rho(1) = [1] \ ,$$
$$\kappa_{mn,m} : \mathbb{Z}/m \to \mathbb{Z}/mn \ , \quad \kappa_{mn,m}[1] = n[1] \ ,$$
$$\kappa_{n,mn} : \mathbb{Z}/mn \to \mathbb{Z}/n \ , \quad \kappa_{n,mn}[1] = [1] \ ,$$

induce natural transformations

$$\rho_n^i : K_i(A) \to K_i(A;\mathbb{Z}/n) \ ,$$

$$\kappa^i_{mn,m} : K_i(A;\mathbb{Z}/m) \to K_i(A;\mathbb{Z}/mn)\ ,$$
$$\kappa^i_{n,mn} : K_i(A;\mathbb{Z}/mn) \to K_i(A;\mathbb{Z}/n)\ .$$

The Bockstein operation (see [S])

$$\beta^i_n : K_i(A;\mathbb{Z}/n) \to K_{i+1}(A)$$

appears in the six-term exact sequence
(4.1)

$$K_i(A) \xrightarrow{\times n} K_i(A) \xrightarrow{\rho^i_n} K_i(A;\mathbb{Z}/n) \xrightarrow{\beta^i_n} K_{i+1}(A) \xrightarrow{\times n} K_{i+1}(A)$$

induced by the cofibre sequence

$$A \otimes SC_0(S^1) \longrightarrow A \otimes C_0(W_n) \xrightarrow{\beta} A \otimes C_0(S^1) \xrightarrow{n} A \otimes C_0(S^1)\,.$$

There is a second six-term exact sequence involving the Bockstein operations [S]. This is induced by a cofibre sequence

$$A \otimes SC_0(W_n) \longrightarrow A \otimes C_0(W_m) \longrightarrow A \otimes C_0(W_{mn}) \longrightarrow A \otimes C_0(W_n)$$

and takes the form:

$$K_{i+1}(A;\mathbb{Z}/n) \xrightarrow{\beta^{i+1}_{m,n}} K_i(A;\mathbb{Z}/m) \xrightarrow{\kappa^i_{mn,m}} K_i(A;\mathbb{Z}/mn) \xrightarrow{\kappa^i_{n,mn}} K_i(A;\mathbb{Z}/n)$$

where $\beta^i_{m,n} = \rho^{i+1}_m \circ \beta^i_n$.

The collection of all the transformations ρ, β, κ and their compositions is denoted by Λ. Λ can be regarded as the set of morphisms in a category whose objects are the elements of $\mathbb{Z}/2 \times \mathbb{Z}^+$. Abusing the terminology Λ will be called the category of Bockstein operations. Via the Bockstein operations, $\underline{K}(A)$ becomes a Λ-module. It is natural to consider the group $Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ consisting of all $\mathbb{Z}/2 \times \mathbb{Z}^+$ graded group morphisms which are Λ-linear, i.e. preserve the action of the category Λ. Equivalently $Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ consists of sequences (ϕ_n) of $\mathbb{Z}/2$-graded morphisms of groups

$$\phi_n = (\phi^0_n, \phi^1_n) : K_*(A;\mathbb{Z}/n) \to K_*(B;\mathbb{Z}/n)$$

such that the following four diagrams are commutative.

(I)

$$\begin{array}{ccc} K_i(A) & \xrightarrow{\rho^i_n} & K_i(A;\mathbb{Z}/n) \\ {\scriptstyle\phi^i_0}\downarrow & & \downarrow{\scriptstyle\phi^i_n} \\ K_i(B) & \xrightarrow[\rho^i_n]{} & K_i(B;\mathbb{Z}/n) \end{array}$$

$$\text{(II)} \qquad \begin{array}{ccc} K_i(A;\mathbb{Z}/n) & \xrightarrow{\beta_n^i} & K_{i+1}(A) \\ \phi_n^i \big\downarrow & & \big\downarrow \phi_0^{i+1} \\ K_i(B;\mathbb{Z}/n) & \xrightarrow[\beta_n^i]{} & K_{i+1}(B) \end{array}$$

$$\text{(III)} \qquad \begin{array}{ccc} K_i(A;\mathbb{Z}/m) & \xrightarrow{\kappa_{mn,m}^i} & K_{i+1}(A;\mathbb{Z}/mn) \\ \phi_m^i \big\downarrow & & \big\downarrow \phi_{mn}^i \\ K_i(B;\mathbb{Z}/m) & \xrightarrow[\kappa_{mn,m}^i]{} & K_{i+1}(B;\mathbb{Z}/mn) \end{array}$$

$$\text{(IV)} \qquad \begin{array}{ccc} K_i(A;\mathbb{Z}/mn) & \xrightarrow{\kappa_{n,mn}^i} & K_i(A;\mathbb{Z}/n) \\ \phi_{mn}^i \big\downarrow & & \big\downarrow \phi_n^i \\ K_i(B;\mathbb{Z}/mn) & \xrightarrow[\kappa_{n,mn}^i]{} & K_i(B;\mathbb{Z}/n) \end{array}$$

The Kasparov product induces a map

$$\gamma_n^i : KK(A,B) \to Hom\big(K_i(A;\mathbb{Z}/n), K_i(B;\mathbb{Z}/n)\big) \ .$$

Then $\gamma_n = (\gamma_n^0, \gamma_n^1)$ will be a map

$$\gamma_n : KK(A,B) \to Hom\big(K_*(A;\mathbb{Z}/n), K_*(B;\mathbb{Z}/n)\big) \ .$$

Note that if $n = 0$ then $K_*(A,\mathbb{Z}/n) = K_*(A)$ and the map γ_0 is the same as the map γ from the universal coefficient theorem (UCT) of Rosenberg and Schochet [RosS]. We assemble the sequence (γ_n) into a map Γ. Since the Bockstein operations are induced by multiplication with suitable KK elements and since the Kasparov product is associative, we obtain a map

$$\Gamma : KK(A,B) \to Hom_\Lambda(\underline{K}(A), \underline{K}(B)) \ .$$

For the sake of simplicity, if $\alpha \in KK(A,B)$, then $\Gamma(\alpha)$ will be often denoted by α_*. Denote by $\mathcal{N}$ the "bootstrap" category of [RosS]. Using the UCT one can show that $K_*(A;\mathbb{Z}/n) \cong KK(\mathbb{I}_n, A \otimes C(S^1))$ [DLo1]. Accordingly one has a definition of Λ in that setting [DLo4]. The two algebraic constructions are isomorphic.

The following universal coefficient theorem is proved in [DLo4].

Theorem 4.2 ([DLo4]). *Let A, B be C^*-algebras. Suppose that $A \in \mathcal{N}$ and B is σ-unital. Then there is a short exact sequence*

$$0 \longrightarrow Pext(K_*(A), K_*(B)) \xrightarrow{\delta} KK(A,B) \xrightarrow{\Gamma} Hom_\Lambda(\underline{K}(A), \underline{K}(B)) \longrightarrow 0$$

which is natural in each variable.

Here $Pext$ denotes the subgroup of $Ext^1_{\mathbb{Z}}$ consisting of classes of pure extensions. The map δ is the restriction of the map

$$Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK(A,B)$$

from UCT. If $A = B$ then Γ is a (anti)homomorphism of rings.

COROLLARY 4.3. *Let A and B be C*-algebras as in Theorem 4.2. Suppose that $K_*(A)$ is finitely generated. Then $\Gamma{:}KK(A,B) \to Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ is an isomorphism.*

Next we are going to introduce several order structures on $\underline{K}(A)$. While in general they are different, they essentially carry the same type of information for the class of ASH algebras.

Define the group

$$K_*(A, \mathbb{Z} \oplus \mathbb{Z}/n) = K_0(A) \oplus K_1(A) \oplus K_0(A; \mathbb{Z}/n) \oplus K_1(A; \mathbb{Z}/n) \ .$$

There are isomorphisms

$$K_*(A; \mathbb{Z} \oplus \mathbb{Z}/n) \cong K_0\big(A \otimes C(W_n \times S^1)\big) \ ,$$
$$K_*(A; \mathbb{Z} \oplus \mathbb{Z}/n) \cong KK\big(\tilde{\mathbb{I}}_n, A \otimes C(S^1)\big) \ .$$

DEFINITION 4.4 ([DLo3-4]). (a) *Define $K_*(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+$ to be the image of the semigroup $[\mathbb{C}, A \otimes C(W_n \times S^1) \otimes \mathcal{K}]$ into $K_0(A \otimes C(W_n \times S^1)) \cong K_*(A; \mathbb{Z} \oplus \mathbb{Z}/n)$. This means that*

$$K_*(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+ \cong K_0\big(A \otimes C(W_n \times S^1)\big)^+ \ .$$

(b) *Define $K_*(A; \mathbb{Z} \oplus \mathbb{Z}/n)^{++}$ to be the image of the abelian semigroup $[\tilde{\mathbb{I}}_n, A \otimes C(S^1) \otimes \mathcal{K}]$ into $KK(\tilde{\mathbb{I}}_n, A \otimes C(S^1)) \cong K_*(A; \mathbb{Z} \oplus \mathbb{Z}/n)$.*

Recall that

$$\underline{K}(A) = \bigoplus_{n=0}^{\infty} K_*(A; \mathbb{Z}/n) \ .$$

DEFINITION 4.5. (a) *Define $\underline{K}(A)^+$ to be the subsemigroup of $\underline{K}(A)$ generated by the $K_*(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+$, $n \geq 0$.*

(b) *Define $\underline{K}(A)^{++}$ to be the subsemigroup of $\underline{K}(A)$ generated by the $K_*(A; \mathbb{Z} \oplus \mathbb{Z}/n)^{++}$, $n \geq 0$.*

If $\varphi : A \to B$ is a $*$-homomorphism, then $\varphi_*(\underline{K}(A)^+) \subset \underline{K}(B)^+$ and $\varphi_*(\underline{K}(A)^{++}) \subset \underline{K}(B)^{++}$.

In the following, by an ideal of a C*-algebra we mean a closed two-sided ideal. Let A be a C*-algebra with an approximate unit of projections. Recall that any ideal in $A \otimes \mathcal{K}$ is of the form $I \otimes \mathcal{K}$ for some ideal I in A. If

e is a projection in $A\otimes\mathcal{K}$, we denote by $I(e)$ the unique ideal of A such that $I(e)\otimes\mathcal{K}$ is the ideal of $A\otimes\mathcal{K}$ generated by e. For $z\in K_0(A)^+$ we denote by $\mathcal{I}(z)$ the set of all ideals $I(e)$ with the property that $z=[e]\in K_0(A)$ for some projection e in $A\otimes\mathcal{K}$. One may think of $\mathcal{I}(z)$ as being the ideal support of the K-theory class z.

We say that a C^*-algebra A has property $(\mathcal{I})$ if $\mathcal{I}(z)$ is a singleton for any $z\in K_0(A)^+$. Equivalently, A has property $(\mathcal{I})$ if and only if any two projection in $A\otimes K$ with the same K-theory class generate the same ideal. Note that if A has cancellation of projections or if A is an ASH-algebra, then A has property $(\mathcal{I})$.

If $I\otimes\mathcal{K}$ is an ideal of $A\otimes\mathcal{K}$, we let $\underline{K}_I(A)$ denote the image of $\underline{K}(I)$ in $\underline{K}(A)$. An element $x\in\underline{K}(A)$ will be written in the form $x=(x_n^i)$ with $x_n^i\in K_i(A,\mathbb{Z}/n)$, $i=0,1$. In particular the component of x in $K_0(A)$ is denoted by x_0^0.

Definition 4.6. *We define $\underline{K}(A)_+$ to be the subsemigroup of $\underline{K}(A)$ generated by those elements $x=(x_n^i)\in\underline{K}(A)$, with the property that $x_0^0\in K_0(A)^+$ and*

$$x\in\bigcap\left\{\underline{K}_I(A): I\in\mathcal{I}(x_0^0)\right\}.$$

Note that if A is a simple C*-algebra, then

$$\underline{K}(A)_+=\left\{x\in\underline{K}(A): x_0^0\in K_0(A)^+\right\}.$$

Suppose that A and B have property $(\mathcal{I})$ and $\varphi: A\to B$ is a $*$-homomorphism. Then it is not hard to see that $\varphi_*(\underline{K}(A)_+)\subset\underline{K}(B)_+$.

Proposition 4.7. *Let A be a C^*-algebra with an approximate unit consisting of projections. Suppose that A has cancellation of projections (or property $(\mathcal{I})$). Then $\underline{K}(A)^+\subset\underline{K}(A)_+$ and $\underline{K}(A)^{++}\subset\underline{K}(A)_+$.*

Proof. We only prove that $\underline{K}(A)^+\subset\underline{K}(A)_+$. The second part of the proposition can be proven in a similar way. It is enough to show that $K_*(A;\mathbb{Z}\oplus\mathbb{Z}/n)^+\subset\underline{K}(A)_+$. Let $x=[e]$ where e is a projection in $A\otimes C(W_n\times S^1)\otimes\mathcal{K}$. Fix a base point $\omega_0\in W_n\times S^1$. Then $x_0^0=[e(\omega_0)]\in K_0(A)^+$. Let $e_0\in A\otimes\mathcal{K}$ be any projection such that $[e(\omega_0)]=[e_0]$. If A has cancellation of projections, it follows that there is a partial isometry $v\in A\otimes\mathcal{K}$ such that $v^*v=e(\omega_0)$ and $vv^*=e_0$. Therefore $e(\omega_0)\in I\otimes\mathcal{K}$ where $I:=I(e_0)$. We reach the same conclusion if we just assume that A has property $(\mathcal{I})$. Since $W_n\times S^1$ is connected, $e\in I\otimes C(W_n\times S^1)\otimes\mathcal{K}$. We conclude that $x=[e]$ is contained in the image of $\underline{K}(I)$ into $\underline{K}(A)$. □

Proposition 4.8. *Suppose that A has an approximate unit (e_n) consisting of projections. Then*

(i) $\underline{K}(A) = \underline{K}(A)^+ - \underline{K}(A)^+$

(ii) *If A is stably finite, then $\underline{K}(A)^+ \cap (-\underline{K}(A)^+) = \{0\}$, hence $(\underline{K}(A), \underline{K}(A)^+)$ is an ordered group.*

(iii) *For any $x \in \underline{K}(A)$ there are positive integers k, n such that $k[e_n]+x \in \underline{K}(A)^+$.*

Proof. (i) $K_*(A;\mathbb{Z}\oplus\mathbb{Z}/n) \cong K_0(A\otimes C(W_n\otimes S^1)) = K_0(A\otimes C(W_n\otimes S^1))^+ - K_0(A\otimes C(W_n\otimes S^1))^+$ since A has an approximate unit of projections.

(ii) Since A is stably finite it follows that $A\otimes C(W_n\times S^1)$ is stably finite. Since $K_*(A;\mathbb{Z}\oplus\mathbb{Z}/n)^+ \cong K_0(A\otimes C(W_n\times S^1))^+$ it follows from Proposition 6.2.3 in [B1] that $K_*(A;\mathbb{Z}\oplus\mathbb{Z}/n)^+ \cap (-K_*(A;\mathbb{Z}\oplus\mathbb{Z}/n)^+) = \{0\}$.

(iii) We may assume that $x \in K_*(A;\mathbb{Z}\oplus\mathbb{Z}/m)$ for some m. Thus there is k such that $x = [e]-[f]$ for some projections $e, f \in A\otimes C(W_m\times S^1)\otimes M_k$. Since (e_n) is an approximate unit of projections, it follows that for big enough n, f is equivalent to a subprojection g of $e_n\otimes 1\otimes 1_k$. Therefore $x + k[e_n] = [e] + [e_n\otimes 1\otimes 1_k - g]$ is positive. □

Proposition 4.9. *$\underline{K}(A)^{++}$ satisfies Proposition 4.8. If in addition A has property $(\mathcal{I})$, then $\underline{K}(A)_+$ also satisfies Proposition 4.8.*

Proof. The proof for $\underline{K}(A)^{++}$ is very similar to the proof of Proposition 4.8. For $\underline{K}(A)_+$, (i) and (iii) follow from Proposition 4.7 and the corresponding properties of $\underline{K}(A)^+$. For (ii) we notice that if $x \in \underline{K}(A)_+ \cap (-\underline{K}(A)_+) = \{0\}$, then $x_0^0 = 0$ since A is stably finite. Since A has property $(\mathcal{I})$ this implies that $x = 0$. □

The usual continuity property of $K_0(-)^+$ gives the following.

Proposition 4.10. *$\underline{K}(A)$ and $\underline{K}(A)^+$ are continuous functors.*

The continuity of $\underline{K}(-)^{++}$ is a consequence of the semiprojectivity of the dimension-drop algebra [Lo3].

Proposition 4.11. *Suppose that (A_n) is an inductive system of unital C*-algebras with property $(\mathcal{I})$. Then $\underline{K}(\varinjlim A_n)_+ \cong \varinjlim \underline{K}(A_n)_+$.*

Proof. The proof uses the fact that any ideal in $\varinjlim A_n$ is of the form $\varinjlim J_n$ where J_n is a closed ideal in A_n (see [Br]). □

Let A, B be separable C*-algebras of real rank zero and stable rank one. Then A has cancellation of projections and is stably finite [B1]. The lattice of ideals in A is isomorphic to the lattice of order ideals in $K_0(A)$ (see [Z], [Goo2]). A positive morphism of groups $\varphi : K_0(A) \to K_0(B)$ induces a map $I \to I'$ from the ideals of A to the ideals of B. Here I' is the unique ideal in B such that the image of $K_0(I')$ in $K_0(B)$ is the order ideal generated by

$\varphi(K_0(I))$. The map $K_0(I) \to K_0(A)$ is injective, since A has stable rank one.

Proposition 4.12. *Let A,B be separable C^*-algebras of real rank zero and stable rank one. Let $\phi : \underline{K}(A) \to \underline{K}(B)$ be a morphism of $\mathbb{Z}/2 \times \mathbb{Z}^+$-graded groups. The following are equivalent.*

(i) $\phi(\underline{K}(A)_+) \subset \underline{K}(B)_+$.

(ii) $\phi(K_0(A)^+) \subset K_0(B)^+$ *and* $\phi(\underline{K}_I(A)) \subset \underline{K}_{I'}(B)$ *for all ideals I in A.*

Proof. We may assume that both A and B are stable.

(i)⇒(ii) The first part is immediate since $K_0(-) \cap \underline{K}(-)_+ = K_0(-)^+$. For any ideal I in A we are going to identify $K_0(I)$ with its image in $K_0(A)$. Let $x \in \underline{K}_I(A)$. We must show that $\phi(x) \in \underline{K}_{I'}(B)$. Since $\underline{K}(I)_+ - \underline{K}(I)_+ = \underline{K}(I)$, we may assume that x is in the image of $\underline{K}(I)_+$ into $\underline{K}(A)$. Write $x = (x_0^0, \tilde{x})$ with $x_0^0 \in K_0(I)$ and let $\phi(x) = (y_0^0, \tilde{y})$. Therefore $y_0^0 = \phi_0^0(x_0^0) \in K_0(I')$ since $\phi_0^0(K_0(I)) \subset K_0(I')$. Since $(y_0^0, \tilde{y}) \in \underline{K}(B)_+$, we must have $y \in \underline{K}_{I'}(B)$.

(ii)⇒(i) Let $x \in \underline{K}(A)_+$. Then $x = (x_0^0, \tilde{x})$ with $x_0^0 = [e_0]$ for some projection $e_0 \in A$. If I is the ideal generated by e_0 in A, then $x \in \underline{K}_I(A)$. Let I' be the unique ideal in B such that the image of $K_0(I')$ in $K_0(B)$ is the order ideal generated by $\phi(K_0(I))$. Then $\phi(x) = (y_0^0, \tilde{y}) \in \underline{K}_{I'}(B)$ and $y_0^0 \in K_0(B)^+$. It follows that any projection $f_0 \in B$ such that $[f_0] = y_0^0$ will generate I', hence $\phi(x) \in \underline{K}(B)_+$. □

Proposition 4.13. *Let A be a C^*-algebra in the class $\mathcal{N}$ of [RosS]. If the group $K_*(A)$ is finitely generated, then $\underline{K}(A)$ is finitely generated as a Λ-module. That means there are finitely many elements $x_1, \dots, x_r \in \underline{K}(A)$ such that for any $x \in \underline{K}(A)$ there exist $\lambda_i \in \Lambda$ and $k_i \in \mathbb{Z}$ such that $x = \sum_{i=1}^r k_i \lambda_i(x_i)$.*

Proof. The Λ-module $\underline{K}(A)$ is invariant under KK equivalence and $\underline{K}(A \oplus B) \cong \underline{K}(A) \oplus \underline{K}(B)$. Therefore it suffices to consider the case when $K_i(A) = \mathbb{Z}/p^a$, with $p \geq 2$ prime, $a \geq 0$ and $K_{i+1}(A) = 0$. We are going to show that the group generators of $K_i(A) \oplus K_{i+1}(A; \mathbb{Z}/p^a)$ form a system of generators for the Λ-module $\underline{K}(A)$. Let $N \geq 1$. Since $K_{i+1}(A) = 0$ it follows from the exact sequence

$$K_i(A) \xrightarrow{\rho_N^i} K_i(A; \mathbb{Z}/N) \longrightarrow K_{i+1}(A) \xrightarrow{\times N} K_{i+1}(A)$$

that ρ_N^i is surjective. Write $N = p^b s$ with s relatively prime to p. If $b = 0$, then the exact sequence

$$0 = K_{i+1}(A) \xrightarrow{\rho_N^{i+1}} K_{i+1}(A; \mathbb{Z}/N) \longrightarrow K_i(A) \xrightarrow{\times N} K_i(A)$$

shows that ρ_N^{i+1} is surjective, hence $K_{i+1}(A;\mathbb{Z}/N)=0$. If $b\geq a$, it follows from the exact sequence

$$K_{i+1}(A;\mathbb{Z}/p^a)\xrightarrow{\kappa^{i+1}_{N,p^a}}K_{i+1}(A;\mathbb{Z}/N)\xrightarrow{\kappa^{i+1}_{s,N}}K_{i+1}(A;\mathbb{Z}/s)=0$$

that κ^{i+1}_{N,p^a} is surjective. If $b\leq a$, then the map

$$\kappa^{i+1}_{p^bs,p^as}:K_{i+1}(A;\mathbb{Z}/p^as)\longrightarrow K_{i+1}(A;\mathbb{Z}/p^bs)$$

is surjective by Lemma 2.5 (ii) of [DLo4]. Thus the composition

$$K_{i+1}(A;\mathbb{Z}/p^a)\xrightarrow{\kappa^{i+1}_{p^as,p^a}}K_{i+1}(A;\mathbb{Z}/p^as)\xrightarrow{\kappa^{i+1}_{p^bs,p^as}}K_{i+1}(A;\mathbb{Z}/p^bs)$$

is surjective, since we have shown above that κ^{i+1}_{N,p^a} is surjective for $N=p^as$, i.e. for $b=a$. □

5 Existence Results

In this section we investigate the problem of lifting a given KK-theory element satisfying certain positivity conditions to a $*$-homomorphism. The main result is Theorem 5.7.

Theorem 5.1. *Let X,Y be finite, connected CW complexes of dimension at most two. Then the map $[C_0(X),M_m(C_0(Y))]\to KK(C_0(X),C_0(Y))$ is bijective for all $m\geq 6$.*

Proof. The theorem is a consequence of Corollaries 3.4.6 and 6.4.6 of [DNe]. □

Theorem 5.2. *For any $n\geq 2$ and any σ-unital C*-algebra B, the maps*

$$\varinjlim_m\,[\mathbb{I}_n,B\otimes M_m]\to[\mathbb{I}_n,B\otimes\mathcal{K}]\to KK(\mathbb{I}_n,B)$$

are bijections.

Proof. The bijectivity of the first map follows from the semiprojectivity of the dimension drop algebra [Lo2-3]. If B is separable then the second map is bijective by Corollary 7.1 in [DLo4]. It remains to deal with the non-separable case. Let $\mathcal{S}(B)$ denote the set of all separable C*-subalgebras of B. It is easy to see that for any separable C*-algebra A one has $[A,B]=\varinjlim_{B_1\in\mathcal{S}(B)}[A,B_1]$. Using the isomorphism $KK(A,B)\cong[qA,B\otimes\mathcal{K}]$ due to Cuntz [Cu2], we see that $\varinjlim_{B_1\in\mathcal{S}(B)}KK(A,B_1)=KK(A,B)$. In particular we obtain

$$[\mathbb{I}_n,B\otimes\mathcal{K}]=\varinjlim_{B_1\in\mathcal{S}(B)}[\mathbb{I}_n,B_1\otimes\mathcal{K}]\cong\varinjlim_{B_1\in\mathcal{S}(B)}KK(\mathbb{I}_n,B_1)=KK(\mathbb{I}_n,B)\ .$$ □

PROPOSITION 5.3 . *Let X be a finite connected CW complex of dimension at most two. Suppose that the group $K_0(C_0(X))$ is finite. If $m \geq 3$, then the map*

$$Hom(C_0(X), M_m(\tilde{\mathbb{I}}_n)) \to KK(C_0(X), \tilde{\mathbb{I}}_n)$$

is a surjection for all $n \geq 2$.

Proof. Since $K_0(C_0(X))$ is finite, it follows from the universal coefficient theorem that $KK(C_0(X), \mathbb{C}) \cong Ext(K_1(C_0(X)), \mathbb{Z})$. On the other hand $K_1(C_0(X))$ is torsion free since $dim(X) \leq 2$. Therefore $KK(C_0(X), \mathbb{C}) = 0$. The exact sequence

$$0 \to SM_{mn} \xrightarrow{\iota} M_m(\tilde{\mathbb{I}}_n) \longrightarrow M_m(\mathbb{C} \oplus \mathbb{C}) \to 0$$

induces a commutative diagram

$$\begin{array}{ccccc} Hom(C_0(X), SM_{mn}) & \xrightarrow{\iota_1} & Hom(C_0(X), M_m(\tilde{\mathbb{I}}_n)) & & \\ \chi_1 \downarrow & & \downarrow \chi & & \\ KK(C_0(X), SM_{mn}) & \xrightarrow{\iota_*} & KK(C_0(X), M_m(\tilde{\mathbb{I}}_n)) & \longrightarrow & KK(C_0(X), \mathbb{C} \oplus \mathbb{C}) \end{array}$$

with the bottom row exact in the middle. Since $KK(C_0(X), \mathbb{C} \oplus \mathbb{C}) = 0$, it follows that ι_* is surjective. By Theorem 5.1 χ_1 is surjective for $m \geq 3$. Since $\chi \iota_1 = \iota_* \chi_1$, χ must be surjective. □

PROPOSITION 5.4. *Let X be a finite connected CW complex of dimension d and let $n \geq 2$. Let $Q \in M_N((C(X))$ be a projection with $rank\, Q \geq 2\langle (d-1)/2 \rangle + n - 1$. If $\alpha \in KK(\mathbb{I}_n, C(X))$ induces the zero map $K_1(\mathbb{I}_n) \to K_1(C(X))$, then α lifts to a $*$-homomorphism $\varphi : \mathbb{I}_n \to QM_N(C(X))Q$. The statement remains true if $C(X)$ is replaced by $\mathbb{I}_k$ and $d = 1$.*

Proof. Consider the exact sequence of C*-algebras

$$0 \longrightarrow SM_n \xrightarrow{\iota} \mathbb{I}_n \xrightarrow{\delta_1} \mathbb{C} \longrightarrow 0$$

where δ_1 is the evaluation map at 1. Let $A = C(X)$ or $A = \mathbb{I}_k$. By applying $KK(-, A)$ we obtain an exact sequence

$$K_0(A) \xrightarrow{\times n} K_0(A) \xrightarrow{\delta_1^*} KK(\mathbb{I}_n, A) \xrightarrow{\iota^*} K_1(A) \xrightarrow{\times n} K_1(A)\ .$$

Since α induces the zero map on K_1, $\iota^*(\alpha) = 0$, hence there is $x = [e] - [f] \in K_0(A)$ with $\delta_1^*(x) = \alpha$, where e, f are projections. Since $ker(\delta_1^*) = nK_0(A)$, $\delta_1^*(x) = \delta_1^*([g])$, where $g = e \oplus (n-1)f$. Using Proposition 2.3 (c), we can write $g = q \oplus nh$ where $rank(q) \leq \langle (d-1)/2 \rangle + n - 1$ and h is a trivial projection. We have $\alpha = \delta_1^*([q])$. By definition $\delta_1^*([q])$ is the KK-class of the $*$-homomorphism $\varphi : \mathbb{I}_n \to M_N(A)$, $\varphi(a) = a(1)q$. To finish the proof it is enough to show that q is equivalent to a subprojection of Q. Since $rank(Q) - rank(q) \geq \langle (d-1)/2 \rangle$, this follows from Proposition 2.3 (c). □

PROPOSITION 5.5. *Suppose that $C_0 = C_0(W_k)$ or $C_0 = \mathbb{I}_k$. Then there is $m \geq 1$ such that for any C*-algebra D of the form $D = C(W_n)$, or $D = \tilde{\mathbb{I}}_n$, the natural map $\chi : Hom(C_0, M_m(D)) \to KK(C_0, D)$ is surjective.*

Proof. We need to consider the following four cases:

(1) $C_0{=}C_0(W_k)$, $D{=}C(W_n)$. If $m{\geq}6$, then the map $Hom(C_0, M_m(D_0)){\to} KK(C_0, D_0)$ is surjective by Theorem 5.1. Since $KK(C_0, D) = KK(C_0, D_0)$ we are done.
(2) $C_0 = C_0(W_k)$, $D = \tilde{\mathbb{I}}_n$. By Proposition 5.3, the map χ is surjective for $m \geq 3$,
(3) $C_0 = \mathbb{I}_k$, $D = C(W_n)$. In this case $Hom(K_1(\mathbb{I}_k), K_1(C(W_n))) = 0$. By Proposition 5.4 (applied for $Q = 1_{k+1}$, $d \leq 2$) the map χ is surjective for $m \geq k+1$.
(4) $C_0 = \mathbb{I}_k$, $D = \tilde{\mathbb{I}}_n$. In this case one can take $m \geq 2k$ (see [El2] and [Ei]). □

PROPOSITION 5.6. *Suppose that $A \in SH(2)$ has connected spectrum. There is $m_A \geq 1$ such that if $B \in SH(2)$ has connected spectrum, then any $\alpha \in KK(A, B)$ which is strictly m_A-large and satisfies $rank(\alpha_*[1_A]) > 0$ and $\alpha_*[1_A] \leq [1_B]$ can be lifted to a $*$-homomorphism from A to B.*

Proof. Write $A{=}PM_r(C)P$ with $C{=}C(W_k)$ or $C{=}\tilde{\mathbb{I}}_k$, and $B{=}QM_s(D)Q$ with $D = C(W_n)$ or $D = \tilde{\mathbb{I}}_n$. Since A (resp. B) is Morita equivalent to C (resp. D) we may identify $KK(A, B)$ with $KK(C, D)$. Let m be given by Proposition 5.5 applied for C_0 and let $m_A = m + 1$. The isomorphism

$$KK(C, D) \cong KK(C_0, D) \oplus KK(\mathbb{C}, D)$$

gives a decomposition of α, $\alpha = \alpha_0 + \lambda$. By Proposition 5.5, there is a $*$-homomorphism $\varphi_0 : C_0 \to M_m(D)$ such that $[\varphi_0] = \alpha_0$. Let d denote the dimension of the spectrum of D. Since $rank\, \alpha_*[1_A] > 0$ and α is strictly m_A-large, $rank\, \lambda[1_C] \geq (m{+}1)(d{+}1) \geq m{+}d{+}1 = rank(1_m {\otimes} 1_D){+}d{+}1$. It follows from Proposition 2.3 that there is a projection $e \in D \otimes \mathcal{K}$ such that $\lambda[1_C] = [e]$ and $1_m \otimes 1_D \leq e$. Extend φ_0 to a $*$-homomorphism $\varphi : C \to D \otimes \mathcal{K}$ by setting $\varphi(1_C) = e$. Then $[\varphi] = \alpha$ in $KK(C, D)$. Therefore $\varphi \otimes id_r : M_r(C) \to D \otimes \mathcal{K}$ is such that $[\varphi \otimes id_r(P)] \leq [Q]$. Since the C*-algebras in $SH(2)$ have cancellation of projections, there is a partial isometry $v \in D \otimes \mathcal{K}$ such that $v^*v = (\varphi \otimes id_r)(P)$ and $vv^* \leq Q$. It follows that $\phi = v(\varphi \otimes id_r)v^*$ maps A into B and ϕ is a lifting of α. □

Theorem 5.7. *For any C*-algebra $A \in SH(2)$ there exist a finite subset F of $\underline{K}(A)^+$ and a positive integer m_A such that if $B \in SH(2)$ and $\alpha \in KK(A,B) \cong Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ satisfies the following*

(i) *α is strictly m_A-large,*
(ii) *$\alpha_*(F) \subset \underline{K}(B)^+$,*
(iii) *$\alpha_*[1_A] \le [1_B]$,*

then there exists a $*$-homomorphism $\varphi : A \to B$ implementing α.

Proof. First we deal with the case when both A and B have connected spectrum. Let m_A be given by Proposition 5.6. In that case, based on Proposition 5.6, all we have to prove is that either $rank\, \alpha_*[1_A] > 0$ or $\alpha = 0$. We show that this is a consequence of (i) and (ii) provided that F is chosen properly. By Proposition 4.13, since the group $K_*(A)$ is finitely generated, it follows that the Λ-module $\underline{K}(A)$ is finitely generated. Since $\underline{K}(A)=\underline{K}(A)^+ - \underline{K}(A)^+$, there is a finite set $G \subset \underline{K}(A)^+$ which generates $\underline{K}(A)$ as a Λ- module. By Proposition 4.8, there is an integer $n \ge 0$ such that $n[1_A] - x \in \underline{K}(A)^+$ for all $x \in G$. Set

$$F = \{[1_A]\} \cup G \cup \{n[1_A] - x | x \in G\}\ .$$

Assume now that α satisfies the condition (i), (ii) and (iii) for F and m_A, where m_A is given by Proposition 5.6. If $rank\, \alpha_*[1_A] > 0$, then the conclusion of the theorem follows from Proposition 5.6. It remains to deal with the case when $rank(\alpha_*[1_A]) = 0$. Since $[1_A] \in F$, it follows that $\alpha_*[1_A] \in K_0(A)^+$, hence $\alpha_*[1_A]$ must be equal to zero. Next we show that $\alpha_*(G) = \{0\}$. Since $\alpha_*(F) \subset \underline{K}(A)^+$, it follows that $\alpha_*(x) \ge 0$ and $-\alpha_*(x) = \alpha_*(n[1_A] - x) \ge 0$ for all $x \in G$. By Proposition 4.8 $\underline{K}(A)^+ \cap (-\underline{K}(A)^+) = \{0\}$, thus $\alpha_*(x) = 0$ for all $x \in G$. This certainly implies that $\alpha_* = 0$ hence $\alpha = 0$ since G is a generating set of the Λ-module $\underline{K}(A)$. The zero $*$-homomorphism will be an appropriate lifting.

Let us now deal with the general case, when $A, B \in SH(2)$. Write $A = \oplus A_j$ and $B = \oplus B_i$ where A_j and B_i have connected spectrum. Let m_j and F_j be obtained as in the first part of the proof for A_j. Let F be the union of the sets F_j and let m_A be the maximum of the numbers m_j. Assume now that $\alpha \in KK(A,B)$ satisfies (i), (ii) and (iii). Let $\xi_j : A_j \to A$ (resp. $\eta_i : B \to B_i$) be the canonical inclusion (resp. projection). Since ξ_{j*} and η_{i*} are positive maps it follows that for all i, j, the partial map $(\alpha_{i,j})_* = \eta_{i*}\alpha_*\xi_{j*} : \underline{K}(A_j) \to \underline{K}(B_i)$ satisfies $(\alpha_{i,j})_*(F_j) \subset \underline{K}(B_i)^+$ and of course $(\alpha_{i,j})_*$ is strictly m_j-large. By the first part of the proof there exists a $*$-homomorphism $\varphi_{i,j} : A_j \to B_i \otimes \mathcal{K}$ implementing $\alpha_{i,j}$. By taking the direct sum of these $*$-homomorphisms we obtain a $*$-homomorphism $\varphi : A \to B\otimes\mathcal{K}$ with $[\varphi] = \alpha$. Since $\varphi_*[1_A] \le [1_B]$ there is a partial isometry $v \in B \otimes \mathcal{K}$ such that $v^*v = \varphi(1_A)$ and $vv^* \le 1_B$. Then $v\varphi v^* : A \to B$ is a $*$-homomorphism lifting α.

Remark 5.8. The statement and the proof of Theorem 5.7 remain valid if we replace $\underline{K}(A)^+$ by either $\underline{K}(A)^{++}$ or $\underline{K}(A)_+$, the reason being that all the arguments were based on properties that are satisfied by all three of them (see Propositions 4.8 and 4.9).

The following well-known result will be used later for C*-algebras $B \in SH(2)$.

Proposition 5.9. *Let A be a finite dimensional C*-algebra and let B be a unital C*-algebra with cancellation of projections. Suppose that $\sigma : K_0(A) \to K_0(B)$ is a morphism of ordered groups with $\sigma[1] \le [1]$. Then there is a $*$-homomorphism $\varphi : A \to B$ such that $\varphi_* = \sigma$.*

For C*-algebras A, B we let $[[A, B]]$ denote the homotopy classes of asymptotic morphisms from A to B. The E-theory group is defined as $E(A, B) = [[SA, SB\otimes\mathcal{K}]]$. There is a natural transformation $KK(A, B) \to E(A, B)$ which was shown to be an isomorphism if A is nuclear and B is separable (see [CoHi]). We extend this isomorphism to non-separable C*-algebras B. The result is not needed in this paper, Let $\mathcal{S}(B)$ denote the set of all separable C*-subalgebras of B.

Lemma 5.10. *Let A, B be C*-algebras. If A is separable, then the natural map*

$$\chi : \varinjlim_{B_1\in\mathcal{S}(B)} [[A, B_1]] \to [[A, B]]$$

is a bijection.

Proof. Any asymptotic morphism $(\varphi_t) : A \to B$ is homotopic (in fact equivalent) to an asymptotic morphism given by a continuous map $\varphi : A \to C_b([1, \infty), B)$. Thus we may assume that $\varphi(A)$ is a separable topological subspace hence is contained in $C_b([1, \infty), B_1)$ for some $B_1 \in \mathcal{S}(B)$. This shows that χ is surjective. The injectivity of χ is shown using a similar argument for homotopies of asymptotic morphisms. □

The set of SB_1 with $B_1 \in \mathcal{S}(B)$ is cofinal in $\mathcal{S}(SB)$. Thus if A, B are as in Lemma 5.10 we obtain that

$$E(A, B) = \varinjlim_{B_1\in\mathcal{S}(B)} E(A, B_1)\ .$$

As we have seen in the proof Theorem 5.2 that

$$\lim_{\substack{\longrightarrow \\ B_1 \in \mathcal{S}(B)}} KK(A, B_1) = KK(A, B) \ , \tag{5.1}$$

it follows that the isomorphism $KK(A,B) \cong E(A,B)$ established in [CoHi] for A separable, nuclear and B separable extends to the case when B is σ-unital.

Another straightforward consequence of Lemma 5.10 is that the isomorphism $[[A, B \otimes \mathcal{K}]] \cong [[SA, SB \otimes \mathcal{K}]]$ of [DLo1], where A is separable and $[[id_A]]$ is invertible in $[[A, A \otimes \mathcal{K}]]$, remains valid for non-separable C*-algebras B.

Corollary 5.11. *Let X be a compact, connected metrizable space with base point and let B be a σ-unital C*-algebra. Then $[[C_0(X), B \otimes \mathcal{K}]] \cong KK(C_0(X), B)$.*

Proof. This was proved for separable B and X a finite CW complex in [DLo1] and it was extended to metrizable compact spaces in [D4]. The non-separable case follows from Lemma 5.10 and (5.1). □

6 Uniqueness Results

In this section we show that morphisms with the same K-theory invariants are stably approximately unitarily equivalent. Many important special cases of these results were proved in [L1], [ElGLP1-2], [DLo2], [ElGo2], [D2-3] and [Go3].

Theorem 6.1. *Let $A \in SH$, let B be a unital C*-algebra and let $\varphi, \psi : A \to B$ be two $*$-homomorphisms. Suppose that $[\varphi]=[\psi]$ in $KK(A,B)$. Then for any finite set $F \subset A$ and any $\epsilon > 0$ there exist $k \in \mathbb{N}$, a $*$-homomorphism $\eta : A \to M_k(B)$ with finite dimensional image and a partial isometry $v \in M_{k+1}(B)$ such that $v^*v = \varphi(1) \oplus \eta(1)$, $vv^* = \psi(1) \oplus \eta(1)$ and*

$$\|v(\varphi(a) \oplus \eta(a))v^* - \psi(a) \oplus \eta(a)\| < \epsilon$$

for all $a \in F$.

Proof. We divide the proof into several steps.

(a) The case $A = C(X)$. If both φ and ψ are unital, the theorem was proven in [D2] for separable B. For non-separable B, the same proof remains valid in view of Corollary 5.11. If $\varphi(1) = \psi(1)$, then we can reduce the proof to the unital case by replacing φ and ψ by $\varphi + \mu$ and $\psi + \mu$ where $\mu : C(X) \to B$ is given by $\mu(a) = a(x_0)(1_B - \varphi(1))$, $x_0 \in X$. If

$\varphi(1) \neq \psi(1)$, then we replace φ and ψ by $\varphi_1 = \varphi \oplus \mu$ and $\psi_1 = \psi \oplus \mu$ where $\mu : C(X) \to M_k(B)$ is given by $\mu(a) = a(x_0)(1_B \otimes 1_k)$, $x_0 \in X$. Since $K_0(\varphi) = K_0(\psi)$, by taking k large we find a partial isometry $v \in M_{k+1}(B)$ such that $v\varphi_1(1)v^* = \psi_1(1)$. Then φ_1 and ψ_1 have the same KK class, hence we reduced the proof to a situation that was already discussed.

(b) The case $A = \tilde{\mathbb{I}}_n$. If B has real rank zero, the result is implicit in [DLo2]. Here we modify that argument to cover the case of an arbitrary unital C*-algebra B. First we show that $\tilde{\mathbb{I}}_n$ has the property (H) of [D2-3]. This means that for any finite subset F of $\tilde{\mathbb{I}}_n$ and any $\epsilon > 0$, there exist $r \in \mathbb{N}$, a $*$-homomorphism $\tau : \tilde{\mathbb{I}}_n \to M_{r-1}(\tilde{\mathbb{I}}_n)$ and a $*$-homomorphism $\mu : \tilde{\mathbb{I}}_n \to M_r(\tilde{\mathbb{I}}_n)$ with finite dimensional image, such that $\|a \oplus \tau(a) - \mu(a)\| < \epsilon$ for all $a \in F$. Let $\sigma : \tilde{\mathbb{I}}_n \to \tilde{\mathbb{I}}_n$ be defined by $\sigma(a)(t) = a(1-t)$. Let D be a real rank zero C*-algebra constructed as an inductive limit $D = \varinjlim(D_i, \nu_{j,i})$, where $D_i = M_{(n+2)^i}(\tilde{\mathbb{I}}_n)$, $i \geq 0$ and $\nu_{i+1,i}(a) = a \oplus \sigma(a) \oplus \eta_i(a)$ for suitable $*$-homomorphisms η_i with finite dimensional image. Since $K_1(D) = 0$, it follows from Theorem 1.4 in [DLo2] that the inclusion of $D_0 = \tilde{\mathbb{I}}_n$ into D can be approximated arbitrarily well by $*$-homomorphisms with finite dimensional image. By a standard perturbation argument, for any $\epsilon > 0$ there exist i and a $*$-homomorphism $\eta : \tilde{\mathbb{I}}_n \to D_i$ with finite dimensional image such that $\|\nu_{i,0}(a) - \eta(a)\| < \epsilon$ for all $a \in F$. Obviously $\nu_{i,0}$ is of the form $id \oplus \tau$ for some $*$-homomorphism τ. This shows that $\tilde{\mathbb{I}}_n$ has property (H). This can also be proved by reasoning as in Lemma 2.7 in [Go3] and using the fact that D is a (simple) AF algebra [El2].

Let $\varphi, \psi : \tilde{\mathbb{I}}_n \to B$ be as in the statement and fix $F \subset \tilde{\mathbb{I}}_n$ and $\epsilon > 0$. As in part (a) we may assume that $\varphi(1) = \psi(1) = 1$. Let φ_0, ψ_0 denote the restrictions of φ, ψ to $\mathbb{I}_n$. Since $[\varphi_0] = [\psi_0]$ in $KK(\mathbb{I}_n, B)$, it follows from Theorem 5.2 that there exist $k \in \mathbb{N}$ and a homotopy $\Phi : \mathbb{I}_n \to M_k(B)[0,1]$ such that $\Phi^{(0)} = \varphi_0$ and $\Phi^{(1)} = \psi_0$. Let $\varphi_1, \psi_1 : \tilde{\mathbb{I}}_n \to M_k(B)$ be the unital extensions of φ_0 and ψ_0. Then we can extend the homotopy Φ to a homotopy from φ_1 to ψ_1. Since $\tilde{\mathbb{I}}_n$ has the property (H), it follows from Lemma 1.7 in [D2] that there exist $m \in \mathbb{N}$, a $*$-homomorphism $\eta : \tilde{\mathbb{I}}_n \to M_m(M_k(B))$ with finite dimensional image and a unitary $v \in U_{(m+1)k}(B)$ such that

$$\|v(\varphi(a) \oplus \eta(a))v^* - \psi(a) \oplus \eta(a)\| < \epsilon$$

for all $a \in F$.

(c) The case $A = M_n$. This is a well known folklore type result which follows easily from the definition of the K_0 group. One can choose η and v such that $v(\varphi \oplus \eta)v^* = \psi \oplus \eta$.

(d) The case $A = M_n(D)$ where $D = C(X)$ or $D = \tilde{\mathbb{I}}_n$. Using (c) one reduces the proof to the case when $\varphi(a) = \psi(a)$ for all $a \in M_n(\mathbb{C}1_D) \subset M_n(D)$. In this situation we can write $\varphi = \varphi' \otimes id_n$, $\psi = \psi' \otimes id_n$: $M_n(D) \to M_n(E)$ where E is the commutant of the image of M_n in B. Since $[\varphi'] = [\psi'] \in KK(D, E)$, the result follows from (a) and (b).

(e) The case $A = PM_n(C(X))P$. This is a straightforward consequence of (d) since any $*$-homomorphism $\varphi : A \to B$ can be obtained as the restriction of some $*$-homomorphism $\phi : M_n(C(X)) \to M_N(B)$ having the same KK-class (see Lemma 2.13 of [ElGo2]).

(f) The general case. It is easy to see that the class of unital C*-algebras A satisfying the statement of the Theorem 6.1 is closed under direct sums. Hence the general case follows from (d) and (e). □

Theorem 6.2. *Let $A \in SH$, let $F \subset A$ be a finite set and let $\epsilon > 0$. There exist a finite set $E \subset A$ and $\delta > 0$ such that whenever B is a unital C*-algebra and whenever $\Phi \in Map(A, B[0,1])_1$ is δ-multiplicative on E, there exist $k \in \mathbb{N}$, a unital $*$-homomorphism $\eta : A \to M_k(B)$ with finite dimensional image and a unitary $v \in M_{k+1}(B)$ such that*

$$\|v(\Phi_0(a) \oplus \eta(a))v^* - \Phi_1(a) \oplus \eta(a)\| < \epsilon$$

for all $a \in F$.

Proof. We have seen in the proof of Theorem 6.1 that $\tilde{\mathbb{I}}_n$ has property (H). The class of nuclear algebras with this property is closed with respect to direct sums and tensor products [D3]. Together with Lemma 1.6 of [D3] this implies that any algebra $A \in SH$ has property (H). With these remarks the theorem is a straightforward consequence of Lemma 1.4 of [D2]. One applies that lemma for the sequence $\varphi_j = \Phi_{j/m}$, $0 \le j \le m$, for some large enough m so that $\|\varphi_{j+1}(a) - \varphi_j(a)\| < \epsilon$ for all $a \in A$ and $0 \le j < m$. □

LEMMA 6.3. *Let $B \in SH$ have connected spectrum of dimension d. Let $\varphi, \psi : M_r \to B$ be two unital $*$-homomorphisms inducing the same map on K_0. Suppose that rank $\varphi(e_{11}) \ge \langle d/2 \rangle$. Then there is a unitary $u \in B$ such that $u\varphi u^* = \psi$.*

Proof. If $B = QM_N(C(X))Q$, it follows from Proposition 2.3 (a) that there is a partial isometry $v \in B$ with $v^*v = \varphi(e_{11})$ and $vv^* = \psi(e_{11})$. The same holds true if B is a dimension-drop algebra for all its projections are equivalent to multiples of 1. Set $u = \sum_{i=1}^r \psi(e_{i1})v\varphi(e_{1i})$. Then u is a unitary such that $u\varphi u^* = \psi$. □

LEMMA 6.4. *Let $B \in SH$ have connected spectrum of dimension d. Let $\varphi, \psi : M_r \to B$ be two $*$-homomorphisms. Suppose that φ is unital and $rank\, \varphi(e_{11}) - rank\, \psi(e_{11}) \geq \langle d/2 \rangle$. Then there is a $*$-homomorphism $\eta : M_r \to B$ and a unitary $u \in B$ such that $u\varphi u^* = \psi + \eta$.*

Proof. By Proposition 2.3 (c) there is a projection $e \leq \varphi(e_{11})$ such that $[e] = [\varphi(e_{11})] - [\psi(e_{11})]$. Note that $rank(\varphi(1) - \psi(1)) = n\, rank(e) \geq n\langle d/2 \rangle$. Using Proposition 2.3, we construct inductively mutually orthogonal projections $g_{11}, \ldots, g_{rr}$, equivalent to e and such that $g_{11} + \ldots + g_{rr} = \varphi(1) - \psi(1)$. Let $\eta : M_r \to B$ be a $*$-homomorphism with $\eta(e_{ii}) = g_{ii}$, $i = 1, \ldots, r$. Then the image of η is orthogonal to the image of ψ, so that $\psi + \eta$ is a $*$-homomorphism, and $[\varphi(e_{11})] = [\psi(e_{11}) + \eta(e_{11})]$ in $K_0(B)$. We conclude the proof by applying Lemma 6.3. □

LEMMA 6.5. *Let $B \in SH$ and let $\varphi, \psi : M_r(\tilde{\mathbb{I}}_n) \to B$ be two unital $*$-homomorphisms with finite dimensional image. Suppose that $[\varphi] = [\psi]$ in $KK(A, B)$. Suppose that φ is $(n+1)$-large. Then for any $\epsilon > 0$ and for any finite subset $F \subset M_r(\tilde{\mathbb{I}}_n)$ there is a unitary $u \in B$ such that*

$$\|u\varphi(a)u^* - \psi(a)\| < \epsilon + 6w(F)$$

for all $a \in F$.

Proof. Fix $F \subset M_r(\tilde{\mathbb{I}}_n)$ and $\epsilon > 0$. By dealing separately with each direct summand of B, we may assume that the spectrum of B is connected. Let d denote its dimension. For $t \in (0,1)$, let $\delta_t : M_r(\tilde{\mathbb{I}}_n) \to M_{rn}$ denote the evaluation map at t. Let $\delta_0, \delta_1 : M_r(\tilde{\mathbb{I}}_n) \to M_r$ denote the evaluation maps at the end-points of $[0,1]$. Since φ has finite dimensional image, there are $t_1, \ldots, t_k \in (0,1)$ such that φ factors through

$$\delta_{t_1} \oplus \ldots \oplus \delta_{t_k} \oplus \delta_0 \oplus \delta_1 : M_r(\tilde{\mathbb{I}}_n) \to \oplus_{i=1}^{k} M_{rn} \oplus M_r \oplus M_r\ .$$

Using the definition of the weak variation, we find unitaries $u_i \in M_{rn}$ such that

$$\|u_i\delta_{t_i}(a)u_i^* - \delta_0(a) \otimes 1_n\| < \epsilon + w(F)\ .$$

It follows that there is a unital $*$-homomorphism $\Phi : M_r(\tilde{\mathbb{I}}_n) \to B$ with finite dimensional image such that $\|\varphi(a) - \Phi(a)\| < w(F) + \epsilon$ for all $a \in F$ and Φ factors through $\delta_0 \oplus \delta_1 : M_r(\tilde{\mathbb{I}}_n) \to M_r \oplus M_r$. In addition $[\Phi] = [\varphi]$ in $KK(A, B)$ since δ_{t_i} is homotopic to $\delta_0 \otimes 1_n$. Of course we can approximate ψ by a $*$-homomorphism Ψ with similar properties. Thus, up to an adjustment of $2w(F) + 2\epsilon$ we may assume that both φ and ψ factor through $\delta_0 \oplus \delta_1$. The restrictions of φ and ψ to $M_r(\mathbb{C}) \subset M_r(\tilde{\mathbb{I}}_n)$ induce the same map on K_0 and φ is n-large. In particular this means that

$rank\,\varphi(e_{11}) \geq nd$. Therefore, by Lemma 6.3, there is a unitary $u \in B$ such that $u\varphi(a)\ u^* = \psi(a)$ for all $a \in M_r(\mathbb{C})$. It follows that we may assume that φ and ψ coincide on $M_r(\mathbb{C})$.

If D is a unital C*-algebra and $e \in D$ is a projection let $\gamma_e : M_r(\tilde{\mathbb{I}}_n) \to M_r(D)$ denote the $*$-homomorphism

$$\gamma_e(a) = \delta_0(a) \otimes (1_D - e) + \delta_1(a) \otimes e.$$

With this notation it is easily seen that there is an identification of B with $M_r(D)$ for some C*-algebra $D \in SH$ such that $\varphi = \gamma_p$ and $\psi = \gamma_q$ for suitable projections $p, q \in D$. For the sake of brevity, we will write $\gamma_e \sim \gamma_f$ if $\|u\gamma_e(a)u^* - \gamma_f(a)\| < \epsilon + w(F)$ for some unitary $u \in B$ and all $a \in F$.

Next we reduce the proof to the case when the rank of p and q is less than $\langle (d-1)/2\rangle + n$. If $rank(p) \geq \langle (d-1)/2\rangle + n$, using Proposition 2.3, we find a projection $p_0 \leq p$ with $rank(p_0) < \langle (d-1)/2\rangle + n$ such that $p - p_0 = e_1 + \ldots + e_n$, where e_i are mutually orthogonal equivalent trivial projections in D. Using the definition of $w(F)$, it follows that

$$\big\|u\delta_1(a) \otimes (p-p_0)u^* - \delta_0(a) \otimes (p-p_0)\big\| < \epsilon + w(F)$$

for some partial isometry $u \in M_r(D)$ with $u^*u = uu^* = 1_r \otimes (p - p_0)$. Now it is easily seen that $\gamma_p \sim \gamma_{p_0}$. Similarly, if $rank(q) \geq \langle (d-1)/2\rangle + n$, we find a projection $q_0 \leq q$ with $rank(q_0) < \langle (d-1)/2\rangle + n$ and $\gamma_q \sim \gamma_{q_0}$. Also it is clear that $[\gamma_p] = [\gamma_{p_0}]$ and $[\gamma_q] = [\gamma_{q_0}]$. In particular γ_{p_0} and γ_{q_0} must have the same image in $KK(\mathbb{I}_n, D)$ via the restriction map $KK(\tilde{\mathbb{I}}_n, D) \to KK(\mathbb{I}_n, D)$. The exact sequence

$$K_0(D) \xrightarrow{\times n} K_0(D) \longrightarrow KK(\mathbb{I}_n, D) \longrightarrow K_1(D) \xrightarrow{\times n} K_1(D)$$

shows that $[p_0] - [q_0] \in nK_0(D)$. It follows that there are projections $e, f \in D \otimes \mathcal{K}$ such that $[p_0] + n[e] = [q_0] + n[f]$. Using Proposition 2.3, we can arrange that either $rank(e) = \langle d/2\rangle$ or $rank(f) = \langle d/2\rangle$. By symmetry we may assume that we are in the first case. Note that if $d \leq 1$ then, in addition, we may arrange that either $e = 0$ or $f = 0$. Again, only one of these cases will be discussed. Since φ is unital and $(n+1)$-large, it follows that $rank(1_D) \geq (n+1)d$. Hence for $d \geq 2$ we have

$$\begin{aligned} rank(1_D - q_0) - n\,rank(f) = rank(1_D - p_0) - n\,rank(e) \geq \\ \geq (n+1)d - \left\langle \frac{d-1}{2} \right\rangle - n - n\left\langle \frac{d}{2}\right\rangle \geq \left\langle \frac{d}{2}\right\rangle \ . \end{aligned}$$

On the other hand, if $d \leq 1$ and $f = 0$ we have

$$rank(1_D - p_0) - n\,rank(e) = rank(1_D - q_0) \geq \langle d/2\rangle \ .$$

Using again Proposition 2.3, we find mutually orthogonal projections $e_1, \ldots, e_n$ equivalent to e and $f_1, \ldots, f_n$ equivalent to f such that

$$e_0 = e_1 + \ldots + e_n \le 1_D - p_0 \ , \quad f_0 = f_1 + \ldots + f_n \le 1_D - q_0 \ .$$

Using the definition of $w(F)$ as above, this is easily seen to imply that $\gamma_{p_0} \sim \gamma_{p_0+e_0}$ and $\gamma_{q_0} \sim \gamma_{q_0+f_0}$. By construction $[p_0+e_0] = [q_0+f_0]$ and $rank(p_0+e_0) \ge \langle d/2 \rangle$, $rank(1_D - p_0 - e_0) \ge \langle d/2 \rangle$. Therefore we can find a unitary $v \in B$ such that $v\gamma_{p_0+e_0}v^* = \gamma_{q_0+f_0}$. It follows that there is a unitary $u \in B$ such that

$$\|u\varphi(a)u^* - \psi(a)\| \le 6w(F) + 6\epsilon$$

for all $a \in F$. □

LEMMA 6.6. *Let $B \in SH$ and let $\varphi, \psi : M_r(\tilde{\mathbb{I}}_n) \to B$ be two $*$-homomorphisms with finite dimensional image. Suppose that φ is unital and $(n+1)$-large. Suppose that $[\varphi] - [\psi]$ is strictly n-large. Then for any $\epsilon > 0$ and for any finite subset $F \subset M_r(\tilde{\mathbb{I}}_n)$, there is a unitary $u \in B$ and there is a $*$-homomorphism $\eta : M_r(\tilde{\mathbb{I}}_n) \to B$ with finite dimensional image such that $\varphi(1) = \psi(1) + \eta(1)$ and*

$$\|u\varphi(a)u^* - \psi(a) - \eta(a)\| < \epsilon + 6w(F)$$

for all $a \in F$.

Proof. Again we may assume that the spectrum of B is connected and has dimension d. Let φ_0, ψ_0 denote the restrictions of φ, ψ to $M_r(\mathbb{C}) \subset M_r(\tilde{\mathbb{I}}_n)$. Since $[\varphi] - [\psi]$ is strictly n-large, $rank\, \varphi_0(e_{11}) - rank\, \psi_0(e_{11}) \ge n(d+1) \ge \langle d/2 \rangle$. By Lemma 6.4 there is a $*$-homomorphism $\eta_0 : M_r \to B$ whose image is orthogonal to the image of ψ_0 and such that $v\varphi_0 v^* = \psi_0 + \eta_0$ for some unitary $v \in B$. Now $q := \eta_0(e_{11})$ has rank at least $n(d+1) \ge n - 1 + 2\langle (d-1)/2 \rangle$. By Proposition 5.4 there is a $*$-homomorphism $\eta_1 : \mathbb{I}_n \to qBq$

whose KK-class equals $[\varphi] - [\psi]$. Let $\eta : M_r(\tilde{\mathbb{I}}_n) \to B$ be a $*$-homomorphism extending η_0 and such that its restriction to $\mathbb{I}_n \otimes e_{11}$ equals η_1. Then $\psi + \eta$ is a $*$-homomorphism whose KK-class equals the class of φ. We conclude the proof by applying Lemma 6.5. □

THEOREM 6.7. *For any $A \in SH$, there is $m_A \ge 1$ with the following property. Suppose that $B \in SH$ and $\varphi, \psi : A \to B$ are two $*$-homomorphisms with finite dimensional image. Suppose that φ is (m_A+1)-large. Suppose that $[\varphi] - [\psi]$ is strictly m_A-large. Then for any $\epsilon > 0$ and for any finite subset $F \subset A$ there is a partial isometry $u \in B$ and there is a $*$-homomorphism*

$\eta : A \to B$ whose image is finite dimensional and orthogonal to the image of ψ and such that $u^*u = \varphi(1)$, $uu^* = \psi(1) + \eta(1)$ and

$$\|u\varphi(a)u^* - \psi(a) - \eta(a)\| < \epsilon + 6w(F)$$

for all $a \in F$. If $[\varphi] = [\psi]$, then we may take $\eta = 0$ and we only need to assume that φ is $(m_A + 1)$-large.

Proof. We may assume that the spectrum of B is connected. Let d denote its dimension. First we reduce the proof to the case when the spectrum of A is connected and φ is unital. Write A as a direct sum of basic blocks A_i with connected spectrum. Let e_i denote the unit of A_i. Since $m_A \geq 1$, φ is 1-large so that $rank\, \varphi(e_i) - rank\, \psi(e_i) \geq d$. By Proposition 2.3 (c) there is a partial isometry v_i such that $v_i^* v_i \leq \varphi(e_i)$ and $v_i v_i^* = \psi(e_i)$. Let v be the sum of the v_i. By replacing ψ by $v^*\psi v$ we may assume that $\varphi(e_i) \geq \psi(e_i)$. Thus in order to prove the theorem it is sufficient to deal separately with each pair of $*$-homomorphisms $\varphi|_{A_i}, \psi|_{A_i} : A_i \to \varphi(e_i)B\varphi(e_i)$. Indeed, since $\varphi(e_i)$ is a full projection in B these restrictions will have equal KK-classes whenever φ and ψ have equal KK-classes. Hence for the rest of the proof we may assume that the spectrum of A is connected and φ is unital. If A is a dimension-drop algebra then the statement follows from Lemmas 6.5-6. Suppose now that A is of the form $PM_N(C(X))P$ with $rank(Q) = r$. Let $x_0 \in X$ be fixed and let $\lambda : A \to M_r$ be the evaluation at x_0. By Lemma 1.1 of [D3] there are $*$-homomorphisms $\varphi_0, \psi_0 : M_r \to B$ such that $[\varphi] = [\varphi_0\lambda]$ and $[\psi] = [\psi_0\lambda]$ in $KK(A, B)$ and

$$\|\varphi(a) - \varphi_0\lambda(a)\| < \epsilon + w(F)\ , \quad \|\psi(a) - \psi_0\lambda(a)\| < \epsilon + w(F)$$

for all $a \in F$. In particular φ_0 and ψ_0 induce the same map on K_0 if $[\varphi] = [\psi]$. Let u be the unitary given by Lemmas 6.3-4 applied to φ_0 and ψ_0. Then

$$\|u\varphi(a)u^* - \psi(a) - \eta(a)\| < 2\epsilon + 2w(F)$$

for all $a \in F$. □

Theorem 6.8. *Let $A \in SH$, let $F \subset A$ be a finite set and let $\epsilon > 0$. There exist a finite set $E \subset A$ and a number $\delta > 0$ such that whenever $B \in SH$ and whenever $\Phi \in Map(A, B[0,1])$ is δ-multiplicative on E, with endpoints $\Phi_0 = \varphi$ and $\Phi_1 = \psi$, there exists an integer m such that if $\mu : A \to M_k(B)$ is any m-large $*$-homomorphism with finite dimensional image and all partial $*$-homomorphisms nonzero, then there is a partial isometry $v \in M_{k+1}(B)$ such that $v^*v = 1_B \oplus \mu(1)$, $vv^* = 1_B \oplus \mu(1)$ and*

$$\|v(\varphi(a) \oplus \mu(a))v^* - \psi(a) \oplus \mu(a)\| < \epsilon + 12w(F)$$

for all $a \in F$.

Proof. After replacing A by $\widetilde{A} = A \oplus \mathbb{C}$ and Φ by its unital extension $\widetilde{\Phi} : \widetilde{A} \to B[0,1]$, we may assume that Φ is unital. Fix F and ϵ. Given two $*$-homomorphisms $\Psi, \Psi' : A \to D$ and $s > 0$ we will write $\Psi \sim_s \Psi'$ if there is a partial isometry $v \in D$ with $v^*v = \Psi(1)$, $vv^* = \Psi'(1)$ and

$$\left\| v\Psi(a)v^* - \Psi'(a) \right\| < s$$

for all $a \in F$. Let E and δ be provided by Theorem 6.2. Then it follows by Theorem 6.2 that there exists a $*$-homomorphism $\eta : A \to M_r(B)$ with finite dimensional image such that

$$\varphi \oplus \eta \sim_\epsilon \psi \oplus \eta. \tag{6.4}$$

Let m_A be given by Theorem 6.7. We can find a positive integer $m \geq m_A$ such that if μ is as in the statement of the Theorem 6.8, then $[\mu]$ is (m_A+1)-large and $[\mu] - [\eta]$ is strictly m_A-large. It follows by Theorem 6.7, that

$$\eta \oplus \eta_0 \sim_s \mu \quad \text{with} \quad s = \epsilon + 6w(F) \tag{6.5}$$

for some $*$-homomorphism η_0 with finite dimensional image. From (6.4) and (6.5) it is easily seen that

$$\varphi \oplus \mu \sim_t \psi \oplus \mu$$

with $t = 3\epsilon + 12w(F)$. □

7 KK Shape Equivalence Implies Isomorphism

Our first result in this section shows that two $*$-homomorphisms with the same KK-theory class from an algebra in SH to an ASH algebra of real rank zero are approximately unitarily equivalent on sets with small weak variations. This generalizes results from [ElGo2] and [ElGoSu] which offer a similar conclusion under the stronger assumption that the two $*$-homomorphisms are homotopic. Then we prove an isomorphism result (Theorem 7.3) which shows, roughly speaking, that the isomorphism class of a real rank zero ASH algebra with slow dimension growth depends only on the KK classes of the connecting maps. In view of this result it is natural to expect classification results based on K-theory invariants.

Theorem 7.1. *Let $A \in SH$ and let $B = \varinjlim(B_n, \nu_{r,n})$ be an ASH algebra of real rank zero and with slow dimension growth. Suppose that $\varphi, \psi : A \to B_n$ are two unital $*$-homomorphism such that $[\varphi] = [\psi]$ in $KK(A, B_n)$.*

Then for any finite subset F of A and any $\epsilon > 0$, there exist $r \geq n$ and a unitary $u \in B_r$ such that $u\varphi(1)u^* = \psi(1)$ and

$$\|u\ \nu_{r,n}\varphi(a)\ u^* - \nu_{r,n}\psi(a)\| < \epsilon + 18w(F)$$

for all $a \in F$.

Proof. Since A has cancellation of projections and $[\varphi(1)] = [\psi(1)]$ in $K_0(B_n)$, after increasing n we find a unitary $u \in B_n$ such that $u\varphi(1)u^* = \psi(1)$. Therefore we may assume that $\varphi(1) = \psi(1) = p$. By replacing the system (B_r) with $(B'_r) := (\nu_{r,n}(p)B_r\nu_{r,n}(p))$ we may further assume that both φ and ψ are unital. This is possible as the system (B'_r) has slow dimension growth by Remark 2.2 (c) and its inductive limit has real rank zero by [BroPe]. Moreover, $[\varphi] = [\psi]$ in $KK(A, B'_n)$, since any nonzero block $pB_{n,j}p$ is stably isomorphic to $B_{n,i}$.

For the sake of clarity we divide the rest of the proof into several stages (a)–(l).

(a) Let $F \subset A$ and $\epsilon > 0$ be fixed. We may assume that F contains the units of all the blocks of A. Let $\epsilon_2 > 0$. Since φ and ψ have the same KK class it follows from Theorem 6.1 that there exist $k \in \mathbb{N}$, a $*$-homomorphism $\eta : A \to M_k(B_n)$ with finite dimensional image and a unitary $v \in M_{k+1}(B_n)$ such that

$$\|v(\varphi(a) \oplus \eta(a))v^* - \psi(a) \oplus \eta(a)\| < \epsilon_2 \tag{7.1}$$

for all $a \in F$.

(b) Since $C(S^1)$ and the image of η are semiprojective C*-algebras [B2], they are given by stable relations [Br], [Lo2]. It follows that there exist a finite set $G_1 \subset B_n$ with all elements of norm at most one and $\epsilon_1 > 0$ such that if C is a unital C*-algebra and $\theta \in Map(B_n, C)$ is unital and ϵ_1-multiplicative on G_1, then there exist a unitary $w \in M_{k+1}(C)$ and a $*$-homomorphism $\mu : A \to M_k(C)$ with finite dimensional image such that

$$\|(\theta \otimes id_{k+1})(v) - w\| < \epsilon_2(1 + M)^{-1} \tag{7.2}$$

where $M = \sup\{\|a\| : a \in F\}$ and

$$\|(\theta \otimes id_k)\eta(a) - \mu(a)\| < \epsilon_2 \tag{7.3}$$

for all $a \in F$. For $s \in \mathbb{N}$, let θ_s denote the map $\theta \otimes id_s : M_s(B_n) \to M_s(C)$. By enlarging G_1 and by taking a smaller ϵ_1, we may arrange that

$$\|\theta_{k+1}(v(\varphi(a) \oplus \eta(a))v^*) - \theta_{k+1}(v)\theta_{k+1}(\varphi(a) \oplus \eta(a))\theta_{k+1}(v^*)\| < \epsilon_2$$

for all $a \in F$. Since θ_s is contractive, we obtain from (7.1) and from the above estimate

$$\|\theta_{k+1}(v)(\theta\varphi(a) \oplus \theta_k\eta(a))\theta_{k+1}(v^*) - \theta\psi(a) \oplus \theta_k\eta(a)\| < 2\epsilon_2\ .$$

Using (7.2) and (7.3), we obtain

$$\|w(\theta\varphi(a) \oplus \mu(a))w^* - \theta\psi(a) \oplus \mu(a)\| < 6\epsilon_2$$

for all $a \in F$. Using functional calculus, we find a unitary $w_1 \in M_{k+1}(C)$ which commutes with $\theta\varphi(1) \oplus \mu(1) = \theta\psi(1) \oplus \mu(1) = 1_C \oplus \mu(1)$ and $\|w_1 - w\| < f(\epsilon_2)$, where f is a (universal) function which has limit equal to zero at zero. If ϵ_2 is chosen small enough to insure that $2Mf(\epsilon_2) + 6\epsilon_2 < \epsilon$, then we can replace w by w_1 in the above estimate to the effect that

$$\|w_1(\theta\varphi(a) \oplus \mu(a))w_1^* - \theta\psi(a) \oplus \mu(a)\| < 2Mf(\epsilon_2) + 6\epsilon_2 < \epsilon \tag{7.4}$$

for all $a \in F$.

(c) By compressing the inductive system (B_r) by the successive images of a minimal, nonzero central projection in B_n, it suffices to consider the case when the spectrum of B_n is connected. A is a direct sum of algebras A_i with connected spectrum. Let e_i denote the unit of A_i. We may assume that $\varphi(e_i) \neq 0$ for all i. It is clear that in this case we can find an integer $N > 0$ such that $N[\varphi(e_i)] \geq [1_{B_n}]$ in $K_0(B_n)$, for all i. Let G_1 and $\epsilon_1 < \epsilon$ be as in (b) and let $L = 2kN$ with k defined in (a). Let $G = \varphi(F) \cup \psi(F) \cup G_1$. By applying Theorem 3.10 for B_n, G, ϵ_1 and L, we find an integer $m_1 \geq 0$ and $\delta > 0$ such that if $\nu : B_n \to D$ is a unital $*$-homomorphism which is strictly m_1-large and has spectral variation less than δ, then there exist a projection $p \in D$ and a unital $*$-homomorphism $\lambda : B_n \to (1-p)D(1-p)$ with finite dimensional image such that

$$\|\nu(b)p - p\nu(b)\| < \epsilon_1 \tag{7.5}$$

$$\|\nu(b) - p\nu(b)p - \lambda(b)\| < \epsilon_1 \tag{7.6}$$

for all $b \in G$ and

$$[\lambda(1)] \geq L[p] \quad \text{in} \quad K_0(D)\ . \tag{7.7}$$

Let $C = pDp$ and define a unital map $\theta \in Map(B_n, C)$ by $\theta(b) = p\nu(b)p$. It follows from (7.5) that θ is ϵ_1-multiplicative on G_1, since all the elements of G_1 have norm at most one.

(d) Let G and δ be as in (c). Let m be an integer with $m \geq \max\{m_1, 2m_0+2\}$ where $m_0 = m_A$ is provided by Theorem 6.7 applied to A. By Proposition 3.9 and Theorem 3.7, there exists $r \geq n$ such that each partial morphism $\nu^j_{r,n} : B_n \to B_{r,j}$ has spectral variation less than δ and satisfies either

$$rank\, \nu^j_{r,n}(1) \geq m\big(dim(X_{r,j}) + 1\big)\, rank(1) \tag{7.8}$$

or

$$\|\nu^j_{r,n}(b) - \xi(b)\| < \epsilon \tag{7.9}$$

for all $b \in G$, where $\xi : B_n \to B_r$ is a $*$-homomorphism with finite dimensional image and $\xi(1) = \nu^j_{r,n}(1)$.

For r as above, we are going to show that for each j, there is a unitary $u_j \in \nu^j_{r,n}(1)B_{r,j}\nu^j_{r,n}(1)$ such that

$$\|u_j\nu^j_{r,n}\varphi(a)\ u_j^* - \nu^j_{r,n}\psi(a)\| < 5\epsilon + 18\, w(F)\ .$$

In order to simplify the notation, fix j and set $D = \nu^j_{r,n}(1)B_{r,j}\nu^j_{r,n}(1)$ and $\nu = \nu^j_{r,n}$.

(e) Let ν be as above and assume that it satisfies (7.8). The paragraphs (e)–(k) will be devoted to this case. The $*$-homomorphism ν is strictly m_1-large, since $m \geq m_1$ and satisfies $SPV(\nu) < \delta$. Therefore, as in (c), we find a projection $p \in D$ and a unital $*$-homomorphism $\lambda : B_n \to (1-p)D(1-p)$ with finite dimensional image satisfying (7.5), (7.6) and (7.7). Set $C = pDp$ and define $\theta : B_n \to C$ by $\theta(b) = p\nu(b)p$. It follows from (7.5) that θ is ϵ_1-multiplicative on G_1. Therefore we are able to obtain w_1 and $\mu : A \to M_k(C)$ as in (b). In particular (7.4) holds true. On the other hand, since $\varphi(F) \cup \psi(F) \subset G$, it follows from (7.6) that

$$\|\nu\varphi(a) - \theta\varphi(a) - \lambda\varphi(a)\| < \epsilon_1 < \epsilon \tag{7.10}$$

$$\|\nu\psi(a) - \theta\psi(a) - \lambda\psi(a)\| < \epsilon_1 < \epsilon \tag{7.11}$$

for all $a \in F$.

(f) The next step is to show that λ is strictly $(m_0 + 1)$-large, where m_0 is as in (d), given by Theorem 6.7. Let d denote the dimension of the spectrum of D. The unit of B_n is denoted by 1. Using (7.7) and (7.8),

$$\begin{aligned} rank\,\lambda(1) &= \tfrac{1}{2} rank\,\lambda(1) + \tfrac{1}{2} rank\,\lambda(1) \\ &\geq \tfrac{1}{2}\big(rank(1_D) - rank(p)\big) + \tfrac{1}{2} L\, rank(p) \\ &= \tfrac{1}{2} rank\,\nu(1) + \tfrac{1}{2}(L-1) rank(p) \\ &\geq \tfrac{1}{2} rank\,\nu(1) \geq \tfrac{1}{2} m(d+1) rank(1) \\ &\geq (m_0+1)(d+1) rank(1)\ . \end{aligned}$$

(g) The next step is to show that $[\lambda\varphi] - [\mu]$ is strictly m_0-large. That is

$$rank\,\lambda\varphi(e_i) - rank\,\mu(e_i) \geq m_0(d+1) rank(e_i)\ .$$

Since $\mu : A \to M_k(pDp)$, $rank\,\mu(e_i) \leq k\, rank(p)$. On the other hand

$$N[\lambda\varphi(e_i)] \geq [\lambda(1)] \tag{7.12}$$

in $K_0(D)$ since according to (c), $N[\varphi(e_i)] \geq [1]$ in $K_0(B_n)$. Using (7.7) and

(7.12)

$$\begin{aligned} rank\,\lambda\varphi(e_i) - rank\,\mu(e_i) &= \tfrac{1}{2}rank\,\lambda\varphi(e_i) + \tfrac{1}{2}rank\,\lambda\varphi(e_i) - rank\,\mu(e_i) \\ &\geq \tfrac{1}{2}rank\,\lambda\varphi(e_i) + \tfrac{1}{2N}rank\,\lambda(1) - k\,rank(p) \\ &= \tfrac{1}{2}rank\,\lambda\varphi(e_i) + \tfrac{1}{2N}\big(rank\,\lambda(1) - 2kN\,rank(p)\big) \\ &\geq \tfrac{1}{2}rank\,\lambda\varphi(e_i) \geq \tfrac{1}{2}m(d+1)rank\,\varphi(e_i) \\ &\geq m_0(d+1)rank(e_i) \end{aligned}$$

(h) The $*$-homomorphisms $\lambda\varphi$ and $\lambda\psi$ have finite dimensional image and λ (hence $\lambda\varphi$) is (m_0+1)-large. Since $[\lambda\varphi] = [\lambda\psi]$ in $KK(A,D)$, it follows from Theorem 6.7 that there is a unitary $u_1 \in (1-p)D(1-p)$ such that

$$\|u_1\lambda\varphi(a)\,u_1^* - \lambda\psi(a)\| < \epsilon + 6\,w(F)\ . \tag{7.13}$$

Similarly, since $[\lambda\varphi]$ is (m_0+1)-large and since $[\lambda\varphi]-[\mu]$ is m_0-large, it follows from Theorem 6.7 that there exists a $*$-homomorphism $\mu_0 : A \to M_k(D)$ whose image is finite dimensional and orthogonal to the image of $\mu : A \to M_k(pDp)$ and there exists a partial isometry $u_2 \in M_k(D)$ such that

$$\begin{aligned} &u_2^*u_2 = \lambda\varphi(1) = 1_D - p\ , \quad u_2u_2^* = \mu(1) + \mu_0(1) \\ &\|u_2\,\lambda\varphi(a)u_2^* - \mu(a) - \mu_0(a)\| < \epsilon + 6w(F) \end{aligned} \tag{7.14}$$

for all $a \in F$.

(k) Given two $*$-homomorphisms $\Phi, \Psi : A \to E$ and $s > 0$ we will write $\Phi \sim_s \Psi$ if there is a partial isometry $v \in E$ with $v^*v = \Phi(1)$, $vv^* = \Psi(1)$ and

$$\|v\Phi(a)v^* - \Psi(a)\| < s$$

for all $a \in F$. Note that if $\Phi \sim_s \Psi$ and $\Psi \sim_t \chi$, then $\Phi \sim_{s+t} \chi$. Let $s = \epsilon + 6\,w(F)$. Then

$$\begin{aligned} &\theta\varphi + \lambda\varphi \sim_s \theta\varphi + \mu + \mu_0 \quad \text{by (7.14)} \\ &\theta\varphi + \mu + \mu_0 \sim_\epsilon \theta\psi + \mu + \mu_0 \quad \text{by (7.4)} \\ &\theta\psi + \mu + \mu_0 \sim_s \theta\psi + \lambda\varphi \quad \text{by (7.14)} \\ &\theta\psi + \lambda\varphi \sim_s \theta\psi + \lambda\psi \quad \text{by (7.13)}\ . \end{aligned}$$

Summing up, it follows that

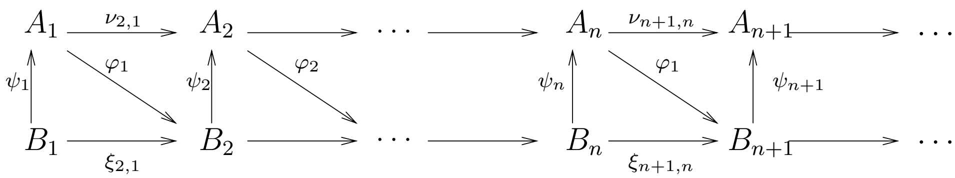
$$\theta\varphi + \lambda\varphi \sim_t \theta\psi + \lambda\psi \tag{7.15}$$

where $t = 3s + \epsilon$. Combining (7.15) with (7.10) and (7.11) we conclude that $\nu\varphi \sim_{t+2\epsilon} \nu\psi$. The proof in the case when ν satisfies (7.8) is now complete.

(ℓ) Let ν be as in (d) and assume that it satisfies (7.9). Since $\varphi(F) \cup \psi(F) \subset G$ it follows that $\nu\varphi \sim_\epsilon \xi\varphi$ and $\nu\psi \sim_\epsilon \xi\psi$. $E = \xi(B_n)$ is a finite dimensional C*-algebra. Since $[\xi\varphi] = [\xi\psi]$ in $KK(A, E)$ and these morphisms are arbitrarily large (since the spectrum of E is zero-dimensional; see the discussion before Definition 2.1) it follows from Theorem 6.7 that $\xi\varphi \sim_s \xi\psi$ where $s = \epsilon + 6\, w(F)$. We conclude that $\nu\varphi \sim_{s+2\epsilon} \nu\psi$. □

The following proposition is a refinement of the approximate intertwining argument of Elliott [El2] (see also [T]).

Proposition 7.2 ([D3]). *Consider a diagram*



where A_n, B_n are finitely generated C-algebras, $\xi_{n+1,n}, \nu_{n+1,n}$ are $*$-homomorphisms, and φ_n, ψ_n are linear, selfadjoint, contractive maps. Let $A = \varinjlim(A_n, \nu_{r,n})$ and $B = \varinjlim(B_n, \xi_{r,n})$ and let $\nu_n : A_n \to A$, $\xi_n : B_n \to B$ denote the canonical maps.*

a) *Let $G_n \subset B_n$ be finite subsets such that $\xi_{n+1,n}(G_n) \subset G_{n+1}$ and $\xi_{n+1,n}(B_n)$ is contained in the C*-subalgebra of B_{n+1} generated by G_{n+1}. Suppose that there is a sequence ϵ_n of positive numbers with $\sum_{n=1}^{\infty} \epsilon_n < \infty$, such that ψ_n is ϵ_n-multiplicative on F_n, and $\|\psi_{n+1}\xi_{n+1,n}(b) - \nu_{n+1,n}\psi_n(b)\| < \epsilon_n$ for $b \in G_n$. Then there is a $*$-homomorphism $\psi : B \to A$ such that*

$$\psi\xi_n(b) = \lim_{r\to\infty} \nu_r\psi_r\xi_{r,n}(b)$$

for all $n \in \mathbb{N}$ and $b \in B_n$.

b) *Let $G_n \subset B_n$ and $F_n \subset A_n$ be finite subsets such that*

$$\nu_{n+1,n}(F_n) \cup \psi_{n+1}(G_{n+1}) \subset F_{n+1} \ , \quad \xi_{n+1,n}(G_n) \cup \varphi_n(F_n) \subset G_{n+1} \ .$$

Suppose that $\xi_{n+1,n}(B_n)$ is contained in the C-subalgebra of B_{n+1} generated by G_{n+1} and $\nu_{n+1,n}(A_n)$ is contained in the C*-subalgebra of A_{n+1} generated by F_{n+1}.*

Suppose that there is a sequence ϵ_n of positive numbers with $\sum_{n=1}^{\infty} \epsilon_n < \infty$, such that φ_n is ϵ_n-multiplicative on F_n, ψ_n is ϵ_n-multiplicative on G_n, and

$$\|\psi_{n+1}\varphi_n(a) - \nu_{n+1,n}(a)\| < \epsilon_n \ , \quad \|\varphi_n\psi_n(b) - \xi_{n+1,n}(b)\| < \epsilon_n$$

for all $a \in F_n$ and $b \in G_n$. Then there is a $*$-isomorphism $\psi : B \to A$ such *that*

$$\psi\xi_n(b) = \lim_{r\to\infty} \nu_r\psi_r\xi_{r,n}(b)$$

for all $n \in \mathbb{N}$ *and* $b \in B_n$.

It was proven in [ElGo2] that if two AH algebras of real rank zero can be written as limits of inductive systems with slow dimension growth which are shape equivalent in the sense of [EK1], then they are isomorphic. We give now a substantial generalization of this result which replaces homotopy by a weaker equivalence relation based on KK theory. Moreover we can keep track of a significant part of the KK class of the induced isomorphism. The result is summarized by the title of the current section.

Theorem 7.3. *Suppose that* $A = \varinjlim(A_n, \nu_{r,n})$ *and* $B = \varinjlim(B_n, \xi_{r,n})$ *are ASH algebras of real rank zero and with slow dimension growth. Let* $\xi_n : B_n \to B$ *and* $\nu_n : A_n \to A$ *be the canonical maps.*

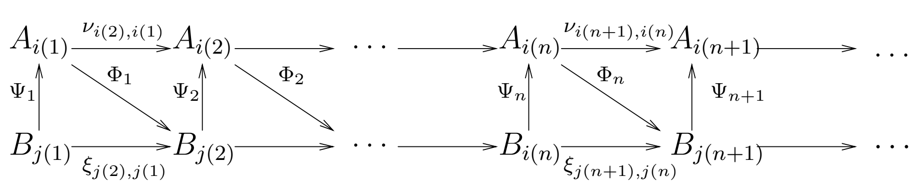
a) *Suppose that there exist* $*$*-homomorphisms* $\psi_n : B_n \to A_n$ *such that* $[\psi_{n+1}\xi_{n+1,n}] = [\nu_{n+1,n}\psi_n]$ *in* $KK(B_n, A_{n+1})$. *Then there is a* $*$*-homomorphism* $\psi : B \to A$ *such that for all* $n \in \mathbb{N}$

$$[\psi\xi_n] = [\nu_n\psi_n] \text{ in } KK(B_n, A) \ .$$

b) *Suppose that there exist* $*$*-homomorphisms* $\varphi_n : A_n \to B_{n+1}$ *and* $\psi_n : B_n \to A_n$ *such that* $[\psi_{n+1}\varphi_n] = [\nu_{n+1,n}]$ *in* $KK(A_n, A_{n+1})$ *and* $[\varphi_n\psi_n] = [\xi_{n+1,n}]$ *in* $KK(B_n, B_{n+1})$ *for all* n. *Then* A *is isomorphic to* B. *Moreover, there is a* $*$*-isomorphism* $\psi : B \to A$ *such that for all* $n \in \mathbb{N}$

$$[\psi\xi_n] = [\nu_n\psi_n] \text{ in } KK(B_n, A) \ .$$

Proof. We prove only part b). The proof uses Theorem 7.1, Proposition 7.2 and Corollary 4.3. We construct by induction two sequences of positive integers $(i(n))$ and $(j(n))$ such that $j(n) < i(n) < j(n+1) < i(n+1)$, and a diagram



As part of the inductive process we construct finite sets $G_n \subset B_{j(n)}$, $F_n \subset A_{i(n)}$ such that

$$w(G_n) < 2^{-n} \ , \quad w(F_n) < 2^{-n} \ ,$$

$$\xi_{j(n+1),j(n)}(G_n) \cup \Phi_n(F_n) \subset G_{n+1} \ , \quad \nu_{i(n+1),i(n)}(F_n) \cup \Psi_{n+1}(G_{n+1}) \subset F_{n+1}$$

$$\xi_{j(n+1),j(n)}(B_{j(n)}) \subset C^*(G_{n+1}) \ , \quad \nu_{i(n+1),i(n)}(A_{i(n)}) \subset C^*(F_{n+1}) \ .$$

Moreover Φ_n and Ψ_n are $*$-homomorphisms of the form

$$\Psi_n = v_n(\nu_{i(n),j(n)}\psi_{j(n)})v_n^* \ , \quad \Phi_n = u_n(\xi_{j(n+1),i(n)+1}\varphi_{i(n)})u_n^* \tag{7.16}$$

where $v_n \in A_{i(n)}$ and $u_n \in B_{j(n+1)}$ are unitaries.

$$\|\Phi_n\Psi_n(b) - \xi_{j(n+1),j(n)}(b)\| < 18 \times 2^{-n} \text{ for all } b \in G_n \ ,$$
$$\|\Psi_{n+1}\Phi_n(a) - \nu_{i(n+1),i(n)}(a)\| < 18 \times 2^{-n} \text{ for all } a \in F_n \ .$$

Suppose that $i(k)$, $j(k)$, Ψ_k, Φ_{k-1}, G_k, and F_k were constructed for $k = 1, \dots, n$. We are now going to construct $i(n+1)$, $j(n+1)$, Ψ_{n+1}, v_{n+1}, Φ_n, u_n, G_{n+1}, and F_{n+1}. Since $[\nu_{n+1,n}\psi_n] = [\psi_{n+1}\varphi_n\psi_n] = [\psi_{n+1}\xi_{n+1,n}]$, it follows that $[\nu_{r,n}\psi_n] = [\psi_r\xi_{r,n}]$ for all $r > n$. Therefore

$$[\varphi_{i(n)}\Psi_n] = [\varphi_{i(n)}][\nu_{i(n),j(n)}\psi_{j(n)}] = [\varphi_{i(n)}\psi_{i(n)}\xi_{i(n),j(n)}] = [\xi_{i(n)+1,j(n)}] \ .$$

It follows from Theorem 7.1 that there exist $r > i(n)$ and a unitary $u \in B_r$ such that

$$\|u\xi_{r,i(n)+1}\varphi_{i(n)}\Psi_n(b)u^* - \xi_{r,j(n)}(b)\| < 18 \times 2^{-n}$$

for all $b \in G_n$.

Let $G \subset B_r$ be a finite subset such that

$$\xi_{r,j(n)}(G_n) \cup u(\xi_{r,i(n)+1}\varphi_{i(n)}(F_n))u^* \subset G$$

and $\xi_{r,j(n)}(B_{j(n)}) \subset C^*(G)$. Since B has real rank zero, it follows from Corollary 3.8 that, after increasing r we may arrange that $w(G) < 2^{-n-1}$. Finally we define $j(n+1) = r$, $G_{n+1} = G$, $u_n = u$, and

$$\Phi_n = u_n\xi_{j(n),i(n)+1}\varphi_{i(n)}u_n^* \ .$$

The construction of $i(n+1)$, F_{n+1}, v_{n+1} and Ψ_{n+1} are completely similar. By Proposition 7.2 we conclude that A is isomorphic to B by an isomorphism $\psi : B \to A$ such that

$$\psi\xi_{j(n)}(b) = \lim_{r\to\infty} \nu_{i(r)}\Psi_r\xi_{j(r),j(n)}(b)$$

for all $n \in \mathbb{N}$ and $b \in B_{j(n)}$. Therefore passing to $\underline{K}(-)$

$$\psi_*\xi_{j(n)*}(x) = \lim_{r\to\infty} \nu_{i(r)*}\Psi_{r*}\xi_{j(r),j(n)*}(x) \tag{7.17}$$

in the discrete topology of $\underline{K}(A)$, for all $x \in \underline{K}(B_{j(n)})$. Since

$$\nu_{i(r)*}\Psi_{r*} = \nu_{i(r)*}(\nu_{i(r),j(r)}\psi_{j(r)})_* = \nu_{j(r)*}\psi_{j(r)*}$$

we obtain from (7.17)

$$\begin{aligned}\psi_*\xi_{j(n)*}(x) &= \lim_{r\to\infty} \nu_{j(r)*}\psi_{j(r)*}\xi_{j(r),j(n)*}(x) \\ &= \lim_{r\to\infty} \nu_{j(r)*}\nu_{j(r),j(n)*}\psi_{j(n)*}(x) = \nu_{j(n)*}\psi_{j(n)*}(x) \ .\end{aligned}$$

This shows that $\psi_* \xi_{j(n)} = \nu_{j(n)*} \psi_{j(n)*}$. By Corollary 4.3 we conclude that

$$[\psi \xi_{j(n)}] = [\nu_{j(n)} \psi_{j(n)}] \text{ in } KK(B_{j(n)}, A)$$

for all $n \in \mathbb{N}$. □

LEMMA 7.4. *Let A be an ASH algebra of real rank zero and with slow dimension growth. Let m_n be an increasing sequence of positive integers. Then A can be written as an inductive limit $A = \varinjlim(B_n, \xi_{r,n})$ with $B_n \in SH$ and such that each of the partial $*$-homomorphism of $\xi_{n+1,n}$ is either strictly m_n-large or has finite dimensional image.*

Proof. Let $A = \varinjlim(A_n, \nu_{r,n})$. For each n, let F_n be a finite set of generators of A_n such that $\nu_{n+1,n}(F_n) \subset F_{n+1}$. Using Proposition 3.9 we construct inductively a sequence

$$r(1) < r(2) < \ldots < r(n) < \ldots$$

and $*$-homomorphisms $\xi_{n+1,n} : A_{r(n)} \to A_{r(n+1)}$ such that each partial $*$-homomorphism of $\xi_{n+1,n}$ is either m_n-large or has finite dimensional image, and

$$\|\nu_{r(n+1),r(n)}(b) - \xi_{n+1,n}(b)\| < 2^{-n}$$

for all $b \in F_{r(n)}$. Finally we apply Proposition 7.2 to conclude that $A \cong \varinjlim(A_{r(n)}, \xi_{n+1,n})$. □

8 Reduction of Dimension

In this section we prove that any ASH algebra of real rank zero and slow dimension growth is isomorphic to an $ASH(2)$ algebra (see Theorem 8.14). First we make the reduction to $ASH(2)^{\#}$ algebras and then to $ASH(2)$ algebras, by eliminating the two spheres. The main result is Theorem 8.14. A related reduction theorem for AH algebras will appear in [ElGoSu]. The technique we employ is basically that of [D2-3] (cf. [Go2-3]). As in [D2-3] we work with uniformly continuous asymptotic homomorphisms $(\varphi_t) : A \to B \otimes M_\infty \subset B \otimes \mathcal{K}$ such that each individual map φ_t is a completely positive, linear contraction. Moreover we may assume that for each t there is an integer $\alpha(t)$ such that $\varphi_t(A) \subset B \otimes M_{\alpha(t)}$. The map α is said to dominate (φ_t). Any asymptotic morphism is equivalent to one of the above type.

The first part of the section is devoted to proving a basic approximative factorization for morphisms of SH algebras (see Proposition 8.7). We begin by giving homotopy factorization results. Let X be a finite, connected CW complex. Recall from [D3] that a $*$-monomorphism $\gamma : QM_N(C(X))Q \to$

$PM_N(C(X))P$ is called a simple embedding if its kk theory class is equal to the class of $id_{C(X)}$.

LEMMA 8.1. *Let $B_1 \in SH(2)^{\#}$ be of the form $B_1 = D \oplus C$ where $D = \oplus_{i=1}^{r} M_{r(i)}(\tilde{\mathbb{I}}_{n(i)})$ and $C = \oplus_{j=1}^{s} M_{k(j)}(C(Y_j))$. Let $A_1 = M_N(C(X))$ where X is a finite connected CW complexṠuppose that $\psi_1 : B_1 \to A_1$ is a $*$-homomorphism whose restriction to C is homotopic to a direct sum between a strictly 4-large $*$-homomorphism and a $*$-homomorphism with finite dimensional image. Let $G \subset B_1$, $F \subset A_1$ be finite sets. Then for any $\delta > 0$ there exists a diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\ \gamma\ } & A_1' \\ {\scriptstyle\psi_1}\uparrow & \overset{\varphi}{\searrow} & \uparrow{\scriptstyle\psi_2} \\ B_1 & \xrightarrow[\ \xi\ ]{} & B_2 \end{array}$$

where

$A_1' = M_L(C(X))$, $B_2 \in SH(2)^{\#}$

ξ is a $*$-homomorphism, ψ_2 is a unital $*$-homomorphism, and γ is a unital simple embedding;

$\varphi \in Map(A_1, B_2)$ is δ-multiplicative on F;

Moreover there exist homotopies $\Phi \in Map(B_1, B_2[0,1])$ and $\Psi \in Map(A_1, A_1'[0,1])$ such that Φ is δ-multiplicative on G, Ψ is δ-multiplicative on F and $\Phi^{(0)} = \varphi\psi_1$, $\Phi^{(1)} = \xi$, $\Psi^{(0)} = \psi_2\varphi$, $\Psi^{(1)} = \gamma$.

Proof. The proof is divided into several steps:

a) The restriction of ψ_1 to C is denoted by ψ and the restriction of ψ_1 to D is denoted by ψ_0. Let $Q = \psi(1_C)$ and $Q_0 = \psi_0(1_D)$. It is clear that it suffices to prove the statement for any $*$-homomorphism which is homotopic to any $*$-homomorphism unitarily equivalent ψ_1. Thus we may assume that ψ is equal to a direct sum between a strictly 4-large $*$-homomorphism and a $*$-homomorphism with finite dimensional image. As was explained in [D3, pp. 191–193], (in the case of a 4-large $*$-homomorphism) $\psi : C \to QA_1Q$ is homotopic to the sum of two $*$-homomorphisms $\psi' : C \to Q'A_1Q'$ and $\psi_0' : C \to Q_0'A_1Q_0'$, $Q' + Q_0' = Q$ with the following properties. The rank of both Q' and Q_0' is at least $dim(X)$. Q' is equivalent to a trivial projection $1_{N'}$. Modulo the identification of Q' with $1_{N'}$, ψ' is of the form

$$\psi' : \oplus_{j=1}^{s} M_{k(j)}(C(Y_j)) \to \oplus_{j=1}^{s} M_{k(j)m(j)}(C(X)) \subset M_{N'}(C(X))$$

$$\psi' = \psi_1^0 \otimes id_{k(1)} \oplus \ldots \oplus \psi_s^0 \otimes id_{k(s)} \ , \quad N' = \sum_{j=1}^{s} k(j)m(j) \ ,$$

where $\psi_j^0 : C(Y_j) \to M_{m(j)}(C(X))$ are unital $*$-homomorphisms with $\psi_j^0(C_0(Y_j)) \subset M_{m(j)}(C_0(X))$. In addition the unital $*$-homomorphism $\psi_0' : C \to Q_0' M_N(C(X)) Q_0'$ has finite dimensional image. Let $N'' = N - N'$. Since Q' is equivalent to $1_{N'}$, $rank(Q') \geq dim(X)$ and $rank(1_N - Q') \geq rank(Q_0') \geq dim(X)$, using Proposition 2.3 we find a unitary $u \in A_1$ such that $uQ'u^* = 1_{N'}$, hence $u(Q_0' + Q_0)u^* \leq 1_{N''}$. Replacing ψ' by $u\psi' u^*$ and setting $\psi'' = u(\psi_0' + \psi_0)u^* : B_1 \to M_{N''}(C(X))$ we may assume that ψ_1 is a direct sum between a $*$-homomorphism $\psi' : C \to M_{N'}(C(X))$ as described above and a $*$-homomorphism $\psi'' : C \oplus D \to M_{N''}(C(X))$ whose restriction to C has finite dimensional image.

b) Let

$$K_0(C_0(X)) \cong \mathbb{Z}^c \oplus \mathbb{Z}/c(1) \oplus \ldots \oplus \mathbb{Z}/c(r)$$
$$K_1(C_0(X)) \cong \mathbb{Z}^d \oplus \mathbb{Z}/d(1) \oplus \ldots \oplus \mathbb{Z}/d(p)$$

be decompositions of the K-theory groups of $C_0(X)$ into cyclic groups. The C*-algebras in the sequence

$$\overbrace{C_0(S^2), ..., C_0(S^2)}^{c\,\text{times}}, C_0(W_{c(1)}), ..., C_0(W_{c(r)}), \overbrace{C_0(S^1), ..., C_0(S^1)}^{d\,\text{times}}, \mathbb{I}_{d(1)}, ..., \mathbb{I}_{d(p)}$$

are denoted by B_i, $1 \leq i \leq c + r + d + p = R$. If $B_0 := \oplus_{i=1}^R B_i$, then $C_0(X)$ has the same K-theory groups as B_0, hence they are KK-equivalent by [RosS]. By [CoHi] and [DLo3], this KK equivalence is implemented by an asymptotic morphism $\varphi_t : C_0(X) \to B_0 \otimes M_\infty$.

c) For each j, $1 \leq j \leq s$, we can find a $*$-homomorphism ξ_j such that the diagram

$$\begin{array}{ccc} C_0(X) \otimes M_{m(j)} & & \\ \uparrow{\scriptstyle \psi_j^0} & \searrow{\scriptstyle \varphi_t \otimes id_{m(j)}} & \\ C_0(Y_j) & \xrightarrow[\chi]{} & B_0 \otimes M_\infty \otimes M_{m(j)} \end{array}$$

commutes in KK-theory and hence in E-theory. Let us denote by $\xi_{ij} : C_0(Y_j) \to B_i$ the components of ξ_j. If $Y_j \neq S^2$, then the existence of ξ_{ij} is a consequence of of Theorem 5.1 and Proposition 5.3. If $Y_j = S^2$ and B_i is not a dimension drop algebra, then ξ_{ij} is given by Theorem 5.1. Finally if $Y_j = S^2$ and B_i is a dimension drop algebra, then $KK(C_0(Y_j), B_i) = 0$, hence one can take $\xi_{ij} = 0$. Since the above diagram commutes in E-Theory, it follows by Theorem 4.1 in [DLo1] that there exists a homotopy of asymptotic morphisms

$$\Phi_{j,t} : C_0(Y_j) \to M_{m(j)}(B[0,1] \otimes M_\infty)$$

with $\Phi_{j,t}^{(0)} = (\varphi_t \otimes id_{m(j)})\psi_j^0$ and $\Phi_{j,t}^{(1)} = \xi_j$. We may assume that the homotopy is dominated by a function $\alpha : [0,\infty) \to \mathbb{N}$.

d) We have seen that (φ_t) induces a KK-equivalence. Therefore there is $\beta \in KK(B_0, C_0(X)) \cong \oplus_i KK(B_i, C_0(X))$ with $\beta\,[\varphi_t] = [id_{C_0(X)}]$. If $\beta_i \in KK(B_i, C_0(X))$ are the components of β, then by Theorems 5.1 and 5.2 there is $L \geq 1$ and there exist $*$-homomorphisms $\chi_i^0 : B_i \to C_0(X) \otimes M_L$ such that $[\chi_0^i] = \beta_i$. Let $\chi_i : \widetilde{B}_i \to C(X) \otimes M_L$ denote the unital extension of χ_i^0. Set $B := \oplus_i \widetilde{B}_i$ and let $\chi := \oplus \chi_i : B \to C(X) \otimes M_\ell$, $\ell = LR$. Define $\gamma_0 : C_0(X) \to C(X) \otimes M_\ell \otimes M_\infty$, by $\gamma_0(a) = a \otimes e_{11}$ where e_{11} is a rank one scalar projection. Then the diagram

$$\begin{array}{ccc} C_0(X) & \xrightarrow{\gamma_0} & C(X) \otimes M_\ell \otimes M_\infty \\ & \searrow^{\varphi_t} & \uparrow{\scriptstyle \chi \otimes id_\infty} \\ & & B_0 \otimes M_\infty \end{array}$$

commutes in KK theory. It follows from Theorem 4.1 in [DLo1], that there exists a homotopy of asymptotic morphisms

$$\Psi_t : C_0(X) \to C(X)[0,1] \otimes M_\ell \otimes M_\infty$$

with $\Psi_t^{(0)} = (\chi \otimes id_\infty)\varphi_t$ and $\Psi_t^{(1)} = \gamma_0$.

e) We may assume that the maps γ_0, ξ_j, φ_t, $\Phi_{j,t}$ and Ψ_t are dominated by the same function α. For each $t \in [0,\infty)$ we consider the following unital extensions of those maps:

$$\begin{aligned} &\varphi_t^\alpha : C(X) \to B \otimes M_{\alpha(t)} \\ &\xi_j^\alpha : C(Y_j) \to M_{m(j)}(B \otimes M_{\alpha(t)}) \\ &\Phi_{j,t}^\alpha : C(Y_j) \to M_{m(j)}(B[0,1] \otimes M_{\alpha(t)}) \\ &\Psi_t^\alpha : C(X) \to C(X)[0,1] \otimes M_\ell \otimes M_{\alpha(t)} \\ &\gamma_0^\alpha : C(X) \to C(X) \otimes M_\ell \otimes M_{\alpha(t)}\ . \end{aligned}$$

Define the following $*$-homomorphisms:

$$\begin{aligned} &\Phi_t'^{,\alpha} = \oplus_{j=1}^s \Phi_{j,t}^\alpha \otimes id_{k(j)} : \oplus_{j=1}^s M_{k(j)}(C(Y_j)) \to M_{N'}\big(B[0,1] \otimes M_{\alpha(t)}\big) \\ &\xi'^{,\alpha} = \oplus_{j=1}^s \xi_j^\alpha \otimes id_{k(j)} : \oplus_{j=1}^s M_{k(j)}(C(Y_j)) \to M_{N'}(B \otimes M_{\alpha(t)}) \end{aligned}$$

Then $\Phi_t'^{,\alpha}$ is a homotopy from $(\varphi_t^\alpha \otimes id_{N'})\psi'$ to $\xi'^{,\alpha}$.

Similarly, Ψ_t^α is a homotopy from $(\chi \otimes id_{\alpha(t)})\varphi_t^\alpha$ to γ_0^α.

f) Since ψ_0' has finite dimensional image and since D is semiprojective [Lo2], it follows that $\psi'' = u(\psi_0' + \psi_0)u^*$ factors through a semiprojective

C*-algebra. As φ_t is an asymptotic morphism, for each $t \in [0, \infty)$, there is a $*$-homomorphism $\xi_t^{\prime\prime,\alpha} : B_1 \to M_{N''}(B \otimes M_{\alpha(t)})$ such that

$$\lim_{t\to\infty} \left\|(\varphi_t^\alpha \otimes id_{N''})\psi''(b) - \xi_t^{\prime\prime,\alpha}(b)\right\| = 0$$

for all $b \in B_1$. Define $\Phi_t^{\prime\prime(s)} = (1-s)(\varphi_t^\alpha \otimes id_{N''})\psi'' + s\xi_t^{\prime\prime,\alpha}$.

g) If $\Phi_t^\alpha = \Phi_t^{\prime,\alpha} \oplus \Phi_t^{\prime\prime}$, then Φ_t^α is a homotopy from $(\varphi_t^\alpha \otimes id_N)\psi_1$ to $\xi_t^\alpha = \xi^{\prime,\alpha} \oplus \xi_t^{\prime\prime,\alpha}$.

$$\begin{array}{ccc} M_n(C(X)) & \xrightarrow{\gamma_0^\alpha \otimes id_N} & M_N(C(X) \otimes M_{\ell\alpha(t)}) \\ {\scriptstyle\psi_1}\uparrow & \searrow^{\varphi_t^\alpha \otimes id_N} & \uparrow{\scriptstyle \chi\otimes id_{\alpha(t)} \otimes id_N} \\ B_1 & \xrightarrow[\xi_t^\alpha]{} & M_N(B \otimes M_{\alpha(t)}) \end{array}$$

Since the sets G and F are finite, by taking t large enough, we can arrange that $\varphi_t^\alpha \otimes id_N$ and $\Psi_t^\alpha \otimes id_N$ are δ-multiplicative on F and Φ_t^α is δ-multiplicative on G. □

Lemma 8.2. *Let $B_1 \in SH(2)^\#$ be of the form $B_1 = D \oplus C$ where $D = \oplus_{i=1}^r M_{r(i)}(\tilde{\mathbb{I}}_{n(i)})$ and $C = \oplus_{j=1}^s M_{k(j)}(C(Y_j))$. Let $A_1 = M_N(C(X))$ where X is a finite connected CW complex. Suppose that $\psi_1 : B_1 \to A_1$ is a $*$-homomorphism whose restriction to C is homotopic to a direct sum between a strictly 4-large $*$-homomorphism and a $*$-homomorphism with finite dimensional image. Let $G_1 \subset B_1$, $F_1 \subset \widehat{F} \subset A_1$ be finite sets. Then for any $\epsilon > 0$ and any $\delta > 0$ there exists a diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma} & A_1' \\ {\scriptstyle\psi_1}\uparrow & \searrow^{\varphi} & \uparrow{\scriptstyle\psi_2} \\ B_1 & \xrightarrow[\xi]{} & B_2 \end{array}$$

where

$A_1' = M_L(C(X))$, $B_2 \in SH(2)^\#$,

ξ is a $*$-homomorphism, ψ_2 is a unital $*$-homomorphism, and γ is a unital simple embedding;

$\varphi \in Map(A_1, B_2)$ is δ-multiplicative on $\widehat{F}$;

$\|\varphi\psi_1(b) - \xi(b)\| < \epsilon + 12\, w(G_1)$ for all $b \in G_1$.

$\|\psi_2\varphi(a) - \gamma(a)\| < \epsilon + 12\, w(F_1)$ for all $a \in F_1$.

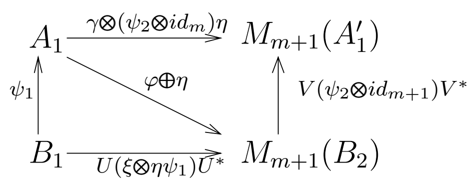
Proof. We may assume that all the partial $*$-homomorphisms of ψ_1 are nonzero. Let (E'', δ'') and (E', δ') be given by Theorem 6.8 for the input data $\epsilon > 0$, $F_1 \subset A_1$ and $\epsilon > 0$, $G_1 \subset B_1$, respectively.

We apply Lemma 8.1 for $\psi_1 : B_1 \to A_1$, with $G = E'$, $F = E'' \cup \widehat{F}$ and $\delta < \min\{\delta', \delta''\}$. Let B_2, A_1', ξ, γ, ψ_2, φ, Φ and Ψ be as in the conclusion of Lemma 8.1. Now the homotopy Φ is δ'-multiplicative on E' and Ψ is δ''-multiplicative on E''. Let m be a multiple of N and let $\eta : A_1 \to M_m(B_2)$ be a unital $*$-homomorphism with finite dimensional image. By taking m large enough, we can apply Theorem 6.8 for the triples $\varphi\psi_1$, ξ, $\mu = \eta\psi_1$ and $\psi_2\varphi$, γ, $\mu = (\psi_2 \otimes id_m)\eta$. Therefore we find unitaries $U \in U_{m+1}(B_2)$ and $V \in U_{m+1}(A_1')$ such that

$$\|U(\xi(b) \oplus \eta\psi_1(b))U^* - \varphi\psi_1(b) \oplus \eta\psi_1(b)\| < \epsilon + 12w(G_1)$$
$$\|V(\psi_2\varphi(a) \oplus (\psi_2 \otimes id_m)\eta(a))V^* - \gamma \oplus (\psi_2 \otimes id_m)\eta(a)\| < \epsilon + 12w(F_1)$$

for all $b \in G_1$ and $a \in F_1$.

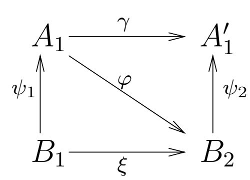
We conclude that the diagram



approximately commutes on G_1, (resp. F_1) to within $\epsilon + 12\,w(G_1)$ (resp. $\epsilon + 12\,w(F_1)$). The $*$-homomorphism $\gamma \oplus (\psi_2 \otimes id_m)\eta$ is a simple embedding since γ is a simple embedding and η has finite dimensional image. □

Next we generalize Lemma 8.2 by allowing B_1 to be an arbitrary C*-algebra in $SH(2)^{\#}$ and A_1 of the form $QM_N(C(X))Q$.

LEMMA 8.3. *There exists a continuous function $f : \mathbb{R} \to \mathbb{R}_+$ with $f(0) = 0$ which has the following property. Let $B_1 \in SH(2)^{\#}$ be of the form $B_1 = D \oplus C$ where $D = \oplus_{i=1}^{r} M_{r(i)}(\tilde{\mathbb{I}}_{n(i)})$ and $C = \oplus_{j=1}^{s} P_j M_{k(j)}(C(Y_j))P_j$. Let $A_1 = QM_N(C(X))Q$ where X is a finite connected CW complex. Suppose that $\psi_1 : B_1 \to A_1$ is a $*$-homomorphism whose restriction to C is homotopic to a direct sum between a strictly 4-large $*$-homomorphism and a $*$-homomorphism with finite dimensional image. Let $G_1 \subset B_1$, $F_1 \subset \widehat{F} \subset A_1$ be finite sets with all elements of norm ≤ 1, and $w(G_1) < 10^{-3}$, $w(F_1) < 10^{-3}$. Then for any $\epsilon > 0$ and any $\delta > 0$ there exists a diagram*



where

$A_1' = Q_1 M_L(C(X))Q_1$, $B_2 \in \mathrm{SH}(2)^{\#}$;

ξ is a $*$-homomorphism, ψ_2 is a unital $*$-homomorphism, and γ is a unital simple embedding;

$\varphi \in Map(A_1, B_2)$ is δ-multiplicative on $\widehat{F}$;

$\|\varphi\psi_1(b) - \xi(b)\| < \epsilon + f(w(G_1))$ for all $b \in G_1$.

$\|\psi_2\varphi(a) - \gamma(a)\| < \epsilon + f(w(F_1))$ for all $a \in F_1$.

Proof. We may assume that $\psi_1(P_j) \neq 0$ for all j. Using Lemma 2.13 in [ElGo2], ψ_1 can be dilated to a $*$-homomorphism $\psi_1' : B' = D \oplus C' \to A'$, where $C' = \oplus_{j=1}^{s} M_{k(j)}(C(Y_j))$ and $A' = M_K(C(X))$ for some K. Hence there is a commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma'} & A' \\ {\scriptstyle\psi_1}\uparrow & & \uparrow{\scriptstyle\psi_1'} \\ B_1 & \xrightarrow[\xi']{} & B' \end{array}$$

where γ' and $\xi'|_C$ are simple embeddings. Moreover the restriction of ψ_1' to C is homotopic to a direct sum between a strictly 4-large $*$-homomorphism and a $*$-homomorphism with finite dimensional image.

We may assume that $1_{B_1} \in G_1$, $1_{A_1} \in F_1$ and $\psi_1(G_1) \subset \widehat{F}$. Set $F' = \gamma'(F_1)$, $\widehat{F}' = \gamma'(\widehat{F})$ and $G' = \xi'(G_1)$. Then $w(F') \leq w(F_1)$ and $w(G') \leq w(G_1)$. Applying Lemma 8.2 for ψ_1', F', $\widehat{F}'$, G', $\epsilon < 10^{-3}$, $\delta < 10^{-3}$, we complete the above diagram to an approximately commuting diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{\gamma'} & A' & \xrightarrow{\gamma} & A_1' \\ {\scriptstyle\psi_1}\uparrow & & {\scriptstyle\psi_1'}\uparrow & \overset{\varphi}{\searrow} & \uparrow{\scriptstyle\psi_2} \\ B_1 & \xrightarrow[\xi']{} & B' & \xrightarrow[\xi]{} & B_2 \end{array}$$

where ξ, ψ_2, φ, γ, A_1', B_2 are as in the conclusion of Lemma 8.2. The result is now obtained by a standard approximation argument very similar to that given in the proof of Lemma 2.3 of [D3]. The only difference is that in the various estimates from the proof of that lemma, one replaces $4w(-)$ by $12w(-)$. The function f from the statement can be taken $f(x) = 12x + 16|5x|^{1/2}$. □

Our next goal is to show that in Lemma 8.3 we can arrange that ξ has small spectral variation as well as some other convenient properties. The precise statement is given in Proposition 8.7. We need some preparation.

Lemma 8.4. *Let $G \subset M_r(\tilde{\mathbb{I}}_n)$ be a finite set and let $w > 0$. Let $\psi : M_r(\tilde{\mathbb{I}}_n) \to QM_N(C(X))Q$ be a unital $*$-homomorphism. There is m such* *that if $\gamma : QM_N(C(X))Q \to Q_1M_K(C(X))Q_1$ is a unital $*$-homomorphism of the form $\gamma(a) = a + \mu(a)$, where μ is a strictly m-large unital $*$-homomorphism with finite dimensional image, then there exists a homotopy commutative diagram*

$$\begin{array}{ccc} QM_N(C(X))Q & \xrightarrow{\gamma} & Q_1M_K(C(X))Q_1 \\ \uparrow{\scriptstyle\psi} & & \uparrow{\scriptstyle\sigma} \\ M_r(\tilde{\mathbb{I}}_n) & \xrightarrow[\xi]{} & M_k(M_r(\tilde{\mathbb{I}}_n)) \otimes M_r \otimes M_r \end{array}$$

where σ and ξ are unital $*$-homomorphisms, and $w(\xi(G)) < w$.

Proof. We begin with a remark. Suppose that $\varphi, \psi_1 : M_r(\tilde{\mathbb{I}}_n) \to B$ are $*$-homomorphisms with finite dimensional image as in Lemma 6.6. Keeping in mind that $n\delta_0$ is homotopic to $n\delta_1$, an inspection of the proofs of Lemmas 6.5 and 6.6 shows that we can add to the conclusion of Lemma 6.6 that $u\varphi u^*$ is homotopic to $\psi_1 + \eta$. Let $k = nq + 1$ where $q \geq 1$ is an integer. Define $\xi' : M_r(\tilde{\mathbb{I}}_n) \to M_k(M_r(\tilde{\mathbb{I}}_n))$ by $\xi'(b) = diag(b, b(t_1), \dots, b(t_q))$ where $t_i = i/(q+1)$, $b(t_i) \in M_{rn}(\tilde{\mathbb{I}}_n)$. If q is large enough, then $w(\xi'(G)) < \epsilon$. We fix q and hence k with this property.

By taking $m > k + dim(X)$, since μ is strictly m-large, we see that $Q \otimes 1_{k-1}$ can be identified with a subprojection P of $\mu(Q)$. Consequently we can regard $\sigma' := \psi \otimes 1_k$ as a $*$-homomorphism $\sigma' : M_k(M_r(\tilde{\mathbb{I}}_n)) \to (Q+P)M_K(C(X))(Q+P) \subset Q_1M_K(C(X))Q_1$. Note that $\sigma'\xi'$ is of the form $\sigma'\xi'(a) = \psi(a)+\lambda(a)$, where λ is a $*$-homomorphism with finite dimensional image and $\lambda(1) = P$. If m is taken large enough, then $\mu\psi$ is $(n+1)$-large and $[\mu\psi] - [\lambda]$ is strictly n-large. Therefore by the remark at the beginning of the proof, there is a $*$-homomorphism with finite dimensional image $\eta : M_r(\tilde{\mathbb{I}}_n) \to Q_1M_K(C(X))Q_1$ such that $\mu\psi$ is homotopic to $u^*(\lambda+\eta)u$ for some unitary u. By conjugating σ' by a suitable unitary we may arrange that $\mu\psi$ is homotopic to $\lambda + \eta$. Now since η has finite dimensional image, it is homotopic to $\eta_1 \circ (\delta_0 \oplus \delta_1)$ for some $*$-homomorphism $\eta_1 : M_r \oplus M_r \to Q_1M_K(C(X))Q_1$. If we set $\xi = \xi' \oplus \delta_0 \oplus \delta_1$ and $\sigma = \sigma' \oplus \eta_1$, then $\gamma\psi(a) = \psi(a) + \mu\psi(a)$ and

$$\sigma\xi(a) = \sigma'\xi'(a) \oplus \eta_1 \circ (\delta_0 \oplus \delta_1)(a) = \psi(a) + \lambda(a) + \eta(a).$$

It is then clear that $\sigma\xi$ is homotopy equivalent to $\gamma\psi$. □

LEMMA 8.5. *Lemma 8.4 remains true if $M_r(\tilde{\mathbb{I}}_n)$ is replaced by $PM_n(C(Y))P$ where Y is a finite connected CW complex and $rank\, P = r$.*

Proof. The proof is very similar to the proof of the previous lemma. One uses the homotopy factorization Lemma 1.2 from [D3]. The result is implicitly contained in the proof of Lemma 2.4 in [D3]. □

PROPOSITION 8.6. *Let $B \in SH(2)^{\#}$, let $G \subset B$ be a finite set and let $w > 0$. Let $\psi : B \to A = QM_N(C(X))Q$ be a unital $*$-homomorphism. Then there exists a homotopy commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & A' \\ {\scriptstyle\psi}\uparrow & & \uparrow{\scriptstyle\sigma} \\ B & \xrightarrow[\xi]{} & B' \end{array}$$

with the following properties. $A' = Q_1 M_K(C(X))Q_1$, $B' \in SH(2)^{\#}$, the maps σ and ξ are unital $*$-homomorphisms, γ is a unital simple embedding and $w(\xi(G)) < w$. In addition we may arrange that all the partial morphisms of σ are nonzero.

Proof. This is a straightforward consequence of Lemmas 8.4, 8.5. Let us explain the last part of the proposition. If σ is equal to zero on some block of B' then we can drop that block altogether. Notice that in doing so, we do not increase $w(\xi(G))$. □

PROPOSITION 8.7. *Suppose that we are in the setting of Lemma 8.3 and let $w > 0$. Then in addition to the conclusions of Lemma 8.3, we can arrange that there is a finite set $G_2 \subset B_2$ with all elements of norm ≤ 1 and such that $\xi(G_1) \cup \varphi(F_1) \subset G_2$, $\xi(B_1)$ is contained in the C*-subalgebra of B_2 generated by G_2, and $w(G_2) < w$.*

Proof. Let $\psi_1 : B_1 \to A_1$, G_1, F_1, $\widehat{F}$, ϵ, δ and w be given as in the statement. Let E and δ_E be given by Theorem 6.8 for the input data $F_1 \subset A_1$ and $\epsilon > 0$. We may assume that $\delta < \delta_E$. By applying Lemma 8.3 for G_1, F_1, $\widehat{F}_1 = \widehat{F} \cup E$, ϵ and δ, we obtain an approximately commuting diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma} & A_1' \\ {\scriptstyle\psi_1}\uparrow & \searrow^{\varphi} & \uparrow{\scriptstyle\psi_2} \\ B_1 & \xrightarrow[\xi]{} & B_2 \end{array}$$

with properties as in the conclusion of Lemma 8.3. In particular φ is δ-multiplicative on E and

$$\|\varphi\psi_1(b) - \xi(b)\| < \epsilon + f\big(w(G_1)\big) \tag{8.1}$$

$$\|\psi_2\varphi(a) - \gamma(a)\| < \epsilon + f\big(w(F_1)\big) \tag{8.2}$$

for all $b \in G_1$ and $a \in F_1$. Let $G \subset B_2$ be a finite set with elements of norm ≤ 1 such that $\varphi(F_1) \cup \xi(G_1) \subset G$ and $\xi(B_1) \subset C^*(G)$. By Proposition 8.6 we obtain a homotopy commutative diagram

$$\begin{array}{ccc} A_1' & \xrightarrow{\gamma'} & A_1'' \\ {\scriptstyle \psi_2}\uparrow & & \uparrow{\scriptstyle \sigma} \\ B_2 & \xrightarrow[\xi']{} & B_2' \end{array} \tag{8.3}$$

where γ' is a simple unital embedding, all the partial $*$-homomorphisms of σ are nonzero and $w(\xi'(G)) < w$. We let $\Phi : B_2 \to A_1''[0,1]$ denote the corresponding homotopy. Consider the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma'\gamma} & A_1'' \\ {\scriptstyle \psi_1}\uparrow & \searrow^{\xi'\varphi} & \uparrow{\scriptstyle \sigma} \\ B_1 & \xrightarrow[\xi'\xi]{} & B_2' \end{array}$$

The maps $\gamma'\psi_2\varphi$, $\sigma\xi'\varphi \in Map(A_1, A_1'')$ are connected by $\Phi_t\varphi$. This is a path of maps which are δ_E-multiplicative on E since φ is so and Φ is a $*$-homomorphism. Let m be given by Theorem 6.8. If $\mu : A_1 \to M_k(B_2')$ is any m-large unital $*$-homomorphism with finite dimensional image then $\eta := (\sigma \otimes id_k)\mu : A_1 \to M_k(A_1'')$ is m-large with all partial homomorphisms nonzero. Therefore it follows from Theorem 6.8 that there exists a unitary $v \in M_{k+1}(A_1'')$ such that

$$\|v(\sigma\xi'\varphi(a) \oplus \eta(a))v^* - \gamma'\psi_2\varphi(a) \oplus \eta(a)\| < \epsilon + 12w(F_1)\ . \tag{8.4}$$

On the other hand it follows from (8.2) that

$$\|\gamma'\psi_2\varphi(a) - \gamma'\gamma(a)\| < \epsilon + f\big(w(F_1)\big)\ . \tag{8.5}$$

Define $\gamma_1 = \gamma'\gamma \oplus \eta$, $\varphi_1 = \xi'\varphi \oplus \mu$, $\xi_1 = \xi'\xi \oplus \mu\psi_1$ and $\psi = v(\sigma \otimes id_{k+1})v^*$. Then, it follows from (8.2), (8.4) and (8.5) that the diagram

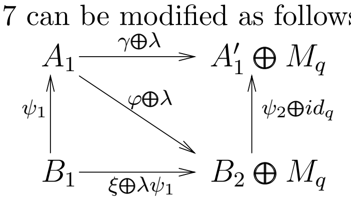
$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma_1} & M_{k+1}(A_1'') \\ {\scriptstyle \psi_1}\uparrow & \searrow^{\varphi_1} & \uparrow{\scriptstyle \psi} \\ B_1 & \xrightarrow[\xi_1]{} & M_{k+1}(B_2') \end{array}$$

is approximately commutative. More precisely

$$\|\varphi_1\psi_1(b) - \xi_1(b)\| < \epsilon + f\big(w(G_1)\big)$$
$$\|\psi\varphi_1(a) - \gamma_1(a)\| < 2\epsilon + 12w(F_1) + f\big(w(F_1)\big)$$

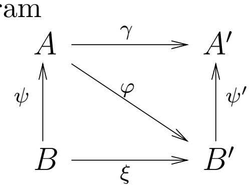
for all $b \in G_1$ and $a \in F_1$. In addition φ_1 is δ-multiplicative on $\widehat{F}$ since $\delta_E < \delta$. Next we set $G_2 = \xi'(G) \oplus G'$, where G' is a finite set of generators of $\mu(A_1)$ such that $\mu(F_1) \cup \mu\psi_1(G_1) \subset G'$. It is clear $\varphi_1(F_1) \cup \xi_1(G_1) \subset G_2$ and $\xi_1(G_1) \subset C^*(G_2)$. Since $w(\xi'(G)) < w$ and μ is a $*$-homomorphism with finite dimensional image, it follows that $w(G_2) \leq w(\xi'(G)) < w$. Since γ and γ' are simple embeddings it follows that γ_1 is a simple embedding. □

Remark 8.8. Consider the set up of Lemma 8.7. Let $q = rank(1_{A_1})$ and let $\lambda : A_1 \to M_q$ be an evaluation map. Then the diagram from the conclusion of Lemma 8.7 can be modified as follows:



Excepting for the form of A_1', the conclusion of Lemma 8.7 remains true for the above diagram.

8.9. We are now ready to prove our first reduction theorem. The following terminology will be useful for handling approximately commutative diagrams. Consider a diagram



where A, A', B and B' are C*-algebras in SH, the maps ψ, ψ', ξ and γ are $*$-homomorphisms with γ a unital simple embedding, and $\varphi \in Map(A, B')$. Suppose that there is given a system of finite subsets $F \subset A$, $G \subset B$, $G' \subset B'$ with the following properties. All their elements have norm at most one,

$\psi(G) \subset F$, $\varphi(F) \cup \xi(G) \subset G'$,

$\psi(B)$ is contained in $C^*(F)$, the C*-subalgebra of A generated by F and

$\xi(B)$ is contained in $C^*(G')$, the C*-subalgebra of B' generated by G'.

Let $\mathbf{c}= (c_1, c_2, c_3, c_4, c_5, c_6)$ where c_i are positive numbers. We say that the above diagram is $\mathbf{c}$-*admisible* if the following conditions are satisfied.

$w(G) < c_1$, $w(F) < c_2$, $w(G') < c_3$,

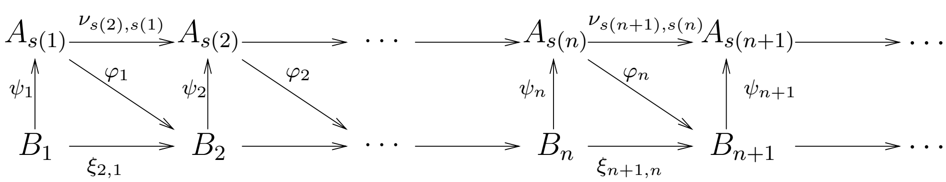
φ is c_4-multiplicative on F,

$\|\varphi\psi(b) - \xi(b)\| < c_5$ for all $b \in G$,

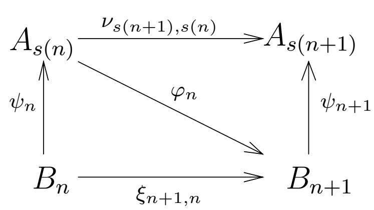
$\|\psi'\varphi(a) - \gamma(a)\| < c_6$ for all $a \in F$.

Theorem 8.10. *Let A be an ASH-algebra of real rank zero and with slow dimension growth. Then A is isomorphic to an $ASH(2)^{\#}$-algebra.*

Proof. Write $A = \varinjlim(A_n, \nu_{r,n})$ with $A \in SH$. We construct inductively a diagram



where $B_n \in SH(2)^{\#}$, ψ_n and $\xi_{n+1,n}$ are $*$-homomorphisms and φ_n are linear selfadjoint contractive maps. The $*$-homomorphism ψ_n is homotopic to a direct sum between a strictly 4-large $*$-homomorphism and a $*$-homomorphism with finite dimensional image. Moreover we construct finite sets $G_n \subset B_n$, $F_n \subset A_{s(n)}$ with elements of norm ≤ 1, with $\nu_{s(n+1),s(n)}(F_n) \subset F_{n+1}$, $\nu_{s(n+1),s(n)}(A_{s(n)}) \subset C^*(F_{n+1})$ and such that the diagram



is $\mathbf{c}_n$-admisible, where

$$\mathbf{c}_n = (w_n, w_{n+1}, w_{n+1}, 2^{-n}, 2^{-n}, 2^{-n}) \ .$$

The numbers w_n form a sequence converging to zero. They are are chosen as follows. After the construction described above is accomplished, the proof is completed by applying Proposition 7.2 (b).

Let f be the real valued function that appears in Lemma 8.7. Then we take (w_n) to be a decreasing sequence of positive numbers less than 10^{-3} such that

$$18\, w_n + f(w_n) < 2^{-n-1} \quad \text{for all } n \ . \tag{8.6}$$

We start the inductive process with $A_{s(1)} = \mathbb{C}$, $B_1 = \mathbb{C}$ and $\psi_1 = id$. Let us assume that $s(k)$, B_k, F_k, G_k, ψ_k, $\xi_{k,k-1}$ and φ_{k-1} have been constructed for all $k \leq n$. We are going to construct $s(n+1)$, B_{n+1}, F_{n+1}, G_{n+1}, ψ_{n+1}, $\xi_{n+1,n}$ and φ_n.

Write $A_{s(n)} = \oplus_{j \in J} A(j)$, where $A(j)$ have connected spectrum. Let J_1 be the set of those j for which $A(j)$ is of the form $Q_j M_N(C(X_j))Q_j$. Let J_2 be the set of those j for which $A(j)$ is a matrix algebra over some dimension-drop algebra. For $j \in J$, let $\psi_n^j : B_n \to A(j)$ be the corresponding partial $*$-homomorphism of ψ_n and let $F_{n,j}$ be the component of F_n in $A(j)$.

Let $j \in J_1$ be fixed. By applying Lemma 8.7 for ψ_n^j, $F_{n,j}$, and G_n with $\epsilon = \delta < 2^{-n-2}$, $w < w_{n+1}$, we produce $B(j) \in SH(2)^{\#}$, a finite subset $G(j)$ of $B(j)$ and a diagram

$$\begin{array}{ccc} A(j) & \xrightarrow{\gamma_0^j} & A(j)' \\ {\scriptstyle \psi_n^j}\uparrow & \overset{\varphi^j}{\searrow} & \uparrow{\scriptstyle \psi^j} \\ B_n & \xrightarrow[\xi^j]{} & B(j) \end{array} \tag{8.7}$$

which is **c**-admisible, with

$$\mathbf{c} = \left(w_n, w_{n+1}, w_{n+1}, 2^{-n-2}, 2^{-n-2} + f(w_n), 2^{-n-2} + f(w_{n+1})\right) .$$

By replacing γ_0^j by $\gamma\gamma_0^j$ and ψ^j by $\gamma_0\psi^j$ where $\gamma : A(j)' \to A'$ is a large unital simple embedding, we can assume that γ_0^j is strictly 4-large. Then we change the diagram (8.7) as in Remark 8.8. Therefore $A(j)'$ has M_q as a direct summand, where q is the rank of the unit of $A(j)$, and γ^j is the direct sum between a simple embedding γ_0^j and an evaluation map λ.

For $r > s(n)$ let $\nu^{\ell,j}$ denote a partial $*$-homomorphism of $\nu_{r,s(n)} : A_{s(n)} \to A_r$. and let $E^{\ell,j} = \nu^{\ell,j}(1)$. Let H be a finite subset of $A(j)$ which contains $F_{n,j}$, has all elements of norm ≤ 1 and generates $A(j)$. Let q' be the rank of the unit of the initial $A(j)'$. It follows from Corollary 3.8 and Proposition 3.9, that there exists $r > s(n)$ such that each partial $*$-homomorphism $\nu^{\ell,j}$ satisfies $w(\nu^{\ell,j}(H)) < w_{n+1}$ and either

(i) $rank\, E^{\ell,j} > 5q'(dim(X_{r,\ell}) + 1)$

or

(ii) there is a $*$-homomorphism $\mu^{\ell,j} : A(j) \to E^{\ell,j} A_{r,\ell} E^{\ell,j}$ with finite dimensional image such that $\|\nu^{\ell,j}(a) - \mu^{\ell,j}(a)\| < 2^{-n-2}$, for all $a \in F_{n,j}$.

Let L_1 (resp. L_2) be the set of those ℓ satisfying (i) (resp. (ii)). Fix $\ell \in L_1$. Suppose first that $A_{r,\ell}$ is of the form $PM_K(C(X_{r,\ell}))P$. Since (i) holds and since $q' = rank\, \gamma_0^j(1) > 4q\, dim(X_j)$ (recall that γ_0^j is strictly 4-large) we can apply the homotopy factorization Lemma 1.2 of [D3], for the $*$-homomorphisms $\nu^{\ell,j} : A(j) \to E^{\ell,j} A_{r,\ell} E^{\ell,j}$ and $\gamma^j : A(j) \to A(j)'$. Therefore there exists a unital $*$-homomorphism $\varphi^{\ell,j} : A(j)' \to E^{\ell,j} A_{r,\ell} E^{\ell,j}$

such that $\nu^{\ell,j}$ is homotopic to $\varphi^{\ell,j}\gamma^j$. By Theorem 7.1, after increasing r and changing notation appropriately, we find a unitary $u \in E^{\ell,j}A_{r,\ell}E^{\ell,j}$ such that

$$\left\|u\varphi^{\ell,j}\gamma^j(a)u^* - \nu^{\ell,j}(a)\right\| < 2^{-n-3} + 18w_{n+1} \tag{8.8}$$

for all $a \in F_{n,j}$. It follows (8.6), (8.7) and (8.8) that the diagram

$$\begin{array}{ccc} A(j) & \xrightarrow{\nu^{\ell,j}} & E^{\ell,j}A_{r,\ell}E^{\ell,j} \\ {\scriptstyle \psi_n^j}\uparrow & \overset{\varphi^j}{\searrow} & \uparrow{\scriptstyle u\varphi^{\ell,j}\psi^j u^*} \\ B_n & \xrightarrow[\xi^j]{} & B(j) \end{array} \tag{8.9}$$

is $\mathbf{c}$-admisible for

$$\mathbf{c} = (w_n, w_{n+1}, w_{n+1}, 2^{-n}, 2^{-n}, 2^{-n}) .$$

Now let us suppose that $A_{r,\ell}$ is of the form $M_K(\tilde{\mathbb{I}}_m)$ and set $G_{\ell,j} := \nu^{\ell,j}(F_{n,j}) \subset E^{\ell,j}A_{r,\ell}E^{\ell,j}$. Then the diagram

$$\begin{array}{ccc} A(j) & \xrightarrow{\nu^{\ell,j}} & E^{\ell,j}A_{r,\ell}E^{\ell,j} \\ {\scriptstyle \psi_n^j}\uparrow & \overset{\nu^{\ell,j}}{\searrow} & \uparrow{\scriptstyle id} \\ B_n & \xrightarrow[\nu^{\ell,j}\psi_n^j]{} & E^{\ell,j}A_{r,\ell}E^{\ell,j} \end{array} \tag{8.10}$$

is $\mathbf{c}$-admisible for

$$\mathbf{c} = (w_n, w_{n+1}, w_{n+1}, 2^{-n}, 2^{-n}, 2^{-n}) .$$

Now let us fix $\ell \in L_2$. Therefore $\nu^{\ell,j}$ satisfies (ii). Then $G_{\ell,j} := \mu^{\ell,j}(F_{n,j}) \subset \mu^{\ell,j}(A(j))$ satisfies $w(G_{\ell,j}) = 0$ being a subset of a finite dimensional C*-algebra. It follows that the diagram

$$\begin{array}{ccc} A(j) & \xrightarrow{\nu^{\ell,j}} & E^{\ell,j}A_{r,\ell}E^{\ell,j} \\ {\scriptstyle \psi_n^j}\uparrow & \overset{\mu^{\ell,j}}{\searrow} & \uparrow \\ B_n & \xrightarrow[\mu^{\ell,j}\psi_n^j]{} & \mu^{\ell,j}(A(j)) \end{array} \tag{8.11}$$

is $\mathbf{c}$-admisible for

$$\mathbf{c} = (w_n, w_{n+1}, w_{n+1}, 2^{-n}, 2^{-n}, 2^{-n}) .$$

After a suitable change of notation, we may assume that for each $j \in J_1$, the numbers r that appear in the diagrams (8.9)–(8.11) are all equal. Now let r be as above and fix $j \in J_2$. For each partial $*$-homomorphism $\nu^{\ell,j}$ of $\nu_{r,n}$, the diagram

$$
\begin{array}{ccc}
A(j) & \xrightarrow{\nu^{\ell,j}} & E^{\ell,j}A_{r,\ell}E^{\ell,j} \\
{\scriptstyle \psi_n^j}\uparrow & \searrow^{id} & \uparrow{\scriptstyle \nu^{\ell,j}} \\
B_n & \xrightarrow[\psi_n^j]{} & A(j)
\end{array}
\tag{8.12}
$$

is commutative. Set $G'_j = F_{n,j}$. Then $\psi_n^j(B_n) \subset C^*(G'_j) = C^*(F_{n,j})$ and $w(G'_j) = w(F_{n,j}) < w_{n+1}$. Thus the diagram 8.12 is **c**-admisible for

$$c = (w_n, w_{n+1}, w_{n+1}, 2^{-n}, 2^{-n}, 2^{-n}) \ .$$

By assembling the diagrams (8.9)–(8.12) we obtain a diagram

$$
\begin{array}{ccc}
A_{s(n)} & \xrightarrow{\nu_{r,s(n)}} & A_r \\
{\scriptstyle \psi_n}\uparrow & \searrow^{\varphi_n} & \uparrow{\scriptstyle \psi_r} \\
B_n & \xrightarrow[\xi_{n+1,n}]{} & B_{n+1}
\end{array}
\tag{8.13}
$$

which is $\mathbf{c}_n$-admisible, where

$$\mathbf{c}_n = (w_n, w_{n+1}, w_{n+1}, 2^{-n}, 2^{-n}, 2^{-n}) \ .$$

The set $G_{n+1} \subset B_{n+1}$ is the union of the sets $G(j)$, $G_{\ell,j}$ and G'_j. Let $F_r \subset A_r$ be a finite set with elements of norm ≤ 1, which generates A_r and such that $\nu_{r,s(n)}(F_n) \cup \psi_{n+1}(G_{n+1}) \subset F_r$. By increasing r we may arrange that $w(F_r) < w_{n+2}$ (see Corollary 3.8) and ψ_r becomes homotopic to a direct sum between a strictly 4-large $*$-homomorphism and a $*$-homomorphism with finite dimensional image. Finally we set $s(n+1) = r$. We conclude the proof by applying Proposition 7.2 (b). □

In the last part of this section we prove that any $ASH(2)^{\#}$ algebra of real rank zero is isomorphic to an $ASH(2)$ algebra. We need the following two Lemmas.

LEMMA 8.11. *Let $A \in SH(2)$. There is $m \geq 0$ such that for any C*-algebra E of the form $E = QM_N(C(S^2))Q$ and any m-large $*$-homomorphism $\psi : A \to E$, there is a $*$-homomorphism $\mu : A \to E$ with finite dimensional image such that $[\psi] = [\mu]$ in $KK(A, E)$.*

Proof. By dealing separately with each partial $*$-homomorphism we may assume that A has connected spectrum. Thus $A = M_k(\tilde{\mathbb{I}}_n)$ or $A = PM_k(C(W))P$ where W is a point, a circle or $W = W_n$. Reasoning as in the proof of Proposition 5.6 we may further reduce the proof to the cases $A = \mathbb{I}_n$

and $A = C_0(W)$. If $A = \mathbb{I}_n$ then, since $K_1(C(S^2)) = 0$, any element in $KK(\mathbb{I}_n, C(S^2))$ is the class of some $*$-homomorphism $\mu : \mathbb{I}_n \to M_2(C(S^2))$ with finite dimensional image (see Proposition 5.4).

If $A = C_0(W)$ then, since $KK(C_0(W), C(S^2)) = 0$, we can take $\mu = 0$. □

Lemma 8.12. *Let E be a finite direct sum of C^*-algebras of the form $QM_N(C(S^2))Q$. Let F be a finite subset of E and let $\delta > 0$. There is $m \geq 0$ with the property that if $A \in SH(2)$ and if $\sigma : E \to A$ is any strictly m-large $*$-homomorphism then there exist a $*$-homomorphism $\tau : E \to A$ with $[\tau] = [\sigma] \in KK(E, A)$ and a diagram*

$$\begin{array}{ccc} E & \xrightarrow{\ \tau\ } & A \\ & \searrow^{\varphi} & \uparrow{\scriptstyle\theta} \\ & & C \end{array}$$

where C is a finite dimensional algebra, θ is a $*$-homomorphism, $\varphi \in Map(E, C)$ is δ-multiplicative on F and $\|\theta\varphi(a) - \tau(a)\| < \delta$ for all $a \in F$.

Proof. By dealing separately with each of the partial morphisms of σ, we may assume that $E = QM_N(C(S^2))Q$, σ is unital and either $A = M_k(\tilde{\mathbb{I}}_n)$ or $A = PM_k(C(W))P$. Using the dilation Lemma 2.13 of [ElGo2], it is easy to reduce the proof to the case when $E = M_N(C(S^2))$. We identify $K_0(E)$ with $\mathbb{Z} \oplus \mathbb{Z}$ with the generator $(0, 1)$ corresponding to the Bott element. If $A = M_k(\tilde{\mathbb{I}}_n)$, then $KK(E, A) \cong Hom(K_0(E), K_0(A)) = Hom(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z})$ by [RosS]. The evaluation map at zero, $\delta_0 : M_k(\tilde{\mathbb{I}}_n) \to M_k(\mathbb{C})$ induces an isomorphism on K_0. The map $\delta_0\sigma$ is a $*$-homomorphism, hence $(\delta_0\sigma)_*(0, 1) = 0$ by Proposition 2.1.3 in [DNe]. It follows that $\sigma_*(0, 1) = 0$. Therefore we can take τ to be any $*$-homomorphism with finite dimensional image and such that $\tau(1_E) = \sigma(1_E)$.

Let us deal now with the case $A = PM_k(C(W))P$ and $E = M_N(C(S^2))$. The case when W is a point is trivial, so that we may assume that $W = S^1$ or $W = W_k$, $k \geq 2$. First we prove that for any finite set $F \subset E$ and $\delta > 0$, there is L (depending only on E, F and δ) and there is an approximately commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \sigma_L\ } & M_L(E) \\ & \searrow^{\varphi_L} & \uparrow{\scriptstyle\theta_L} \\ & & C_L \end{array}$$

where C_L is a finite dimensional algebra, θ_L is a $*$-homomorphism, σ_L is a $*$-homomorphism of the form $\sigma_L(a) = diag(a, a(x_1), \ldots, a(x_{L-1}))$ with $x_i \in S^2$, $\varphi_L \in Map(E, C)$ is δ-multiplicative on F and $\|\theta_L\varphi_L(a) - \sigma_L(a)\| < \delta$ for all $a \in F$. The above diagram is obtained as an easy consequence of a result of [ElGLP1], according to which, the inductive limit B of the system $(M_{2^n}(E), \gamma_{n,r})$ is an AF algebra whenever (x_n) is a dense sequence in S^2, and $\gamma_{n+1,n}(a) = diag(a, a(x_n))$ (cf. [Go3]). Indeed, if B is AF, then the successive images of $F \subset E_0 = E$ in $M_{2^n}(E)$ are approximately contained in finite dimensional subalgebras C_n, with errors converging to zero. Take $L = 2^n > 8$, $\sigma_L = \gamma_{n,0} : E \to M_{2^n}(E)$ and let θ_L be the inclusion of C_n in $M_{2^n}(E)$. Let φ_L be composition of the conditional expectation of $M_{2^n}(E)$ onto C_n with $\gamma_{n,0}$. Note that σ_L has the desired form and $\|\theta_L\varphi_L(a) - \sigma_L(a)\| < \delta$ for all $a \in F$ if n is large enough.

We are now able to explain how the integer m from the statement is chosen. With $L = L(E, F, \delta) > 8$ found above we set $m = 6L$. Now if $\sigma : E \to A$ is strictly m-large, then $rank(P) \geq m(dim(W) + 1)N \geq 2mN = 12LN$. Since σ_L is a simple embedding in the sense of [D3] (i.e. its kk-class is equal to the class of the identity map of $C_0(S^2)$) and $L > 8$, $rank(P) > 11LN$, we can invoke Lemma 1.2 in [D3] in order to find a homotopy factorization of σ of the form

$$E \xrightarrow{\sigma_L \oplus \lambda} M_L(E) \oplus M_N \xrightarrow{\ \theta\ } A$$

where $\lambda(a) = a(x)$, $x \in S^2$, is an evaluation map. Set $\tau = \theta(\sigma_L \oplus \lambda)$, $C = \theta_L(C_L) \oplus M_N$ and $\varphi = \theta_L\varphi_L \oplus \lambda$. The restriction of θ to C will be also denoted by θ. Then $\varphi \in Map(E, C)$ is δ-multiplicative on F and

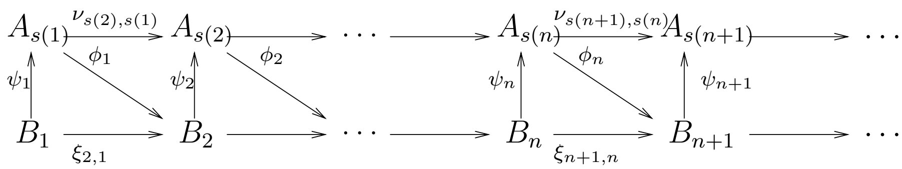
$$\begin{aligned}\|\tau(a) - \theta\varphi(a)\| &= \|\theta(\sigma_L(a) \oplus \lambda(a)) - \theta(\theta_L\varphi_L(a) \oplus \lambda(a))\| \\ &\leq \|\sigma_L(a) - \theta_L\varphi_L(a)\| < \delta\end{aligned}$$

for all $a \in F$. □

Theorem 8.13. *Let $A = \varinjlim(A_n, \nu_{r,n})$ be an $ASH(2)^{\#}$ algebra of real rank zero. Then A is isomorphic to an $ASH(2)$ algebra.*

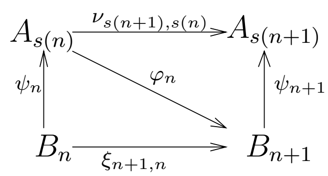
Proof. We may assume that A is as in the conclusion of Lemma 7.4. Moreover we can arrange that any partial $*$-homomorphism of $\nu_{n+1,n}$ acting between any direct summand of A_n in $SH(2)$ (i.e. with spectrum $\neq S^2$) and any direct summand of A_{n+1} with spectrum S^2, has finite dimensional image. Indeed, using Lemma 8.11, we replace the maps between such blocks by $*$-homomorphisms in the same KK-class but with finite dimensional image. As this operation decreases the spectral variation of the connecting maps, the limit of the resulting inductive system will still have real rank zero by Proposition 3.11. By Theorem 7.3 this inductive limit C*-algebra will be isomorphic to A.

We construct inductively a diagram



where B_n are C*-algebras in $SH(2)$, ψ_n are $*$-homomorphisms and φ_n are linear selfadjoint contractive maps.

Moreover we construct finite sets $G_n \subset B_n$, $F_n \subset A_{s(n)}$ with elements of norm at most 1, such that the diagram



is $\mathbf{c}_n$-admisible, where

$$\mathbf{c}_n = (2, 2^{-n}, 2, 2^{-n}, 2^{-n}, 2^{-n+5})$$

and $\nu_{s(n+1),s(n)}(F_n) \subset F_{n+1}$, $\nu_{s(n+1),s(n)}(A_{s(n)}) \subset C^*(F_{n+1})$. After the above construction will be completed we will apply Proposition 7.2 to conclude the isomorphism of A with $B = \varinjlim(B_n, \xi_{r,n})$.

Write $A_{s(n)} = D_n \oplus E_n$ where $D_n \in SH(2)$ and the spectrum of E_n is a disjoint union of copies of S^2. Let $\psi_n' : B_n \to D_n$ and $\psi_n'' : B_n \to E_n$ be the partial homomorphisms of ψ_n. The construction will be done such that ψ_n'' factors through a finite dimensional algebra.

We start the inductive process with $A_{s(1)} = B_1 = \mathbb{C}$ and $\psi_1 = id$. Let us assume that the construction was completed up to the n-th stage. Fix ϵ such that $0 < \epsilon < 2^{-n}$. Since any finite dimensional algebra is semiprojective [B2], and since ψ_n'' factors through such an algebra, there is a positive number $\delta < 2^{-n}$ and there is a finite subset H of $\psi_n''(B_n)$ such that whenever C is a C*-algebra and $\varphi \in Map(E_n, C)$ is δ-multiplicative on H, there exists a $*$-homomorphism $\xi'' : B_n \to C$ such that

$$\|\varphi\psi_n''(b) - \xi''(b)\| < \epsilon \tag{8.14}$$

for all $b \in G_n$. Let m be given by Lemma 8.12 applied for $E = E_n$, $F = H \cup (F_n \cap E_n)$ and δ. We may assume that A is as in the conclusion of Lemma 7.4. Therefore we can find $r > s(n)$ such that any partial $*$-homomorphism of $\nu_{r,s(n)} : A_{s(n)} \to A_r$ is either strictly m-large or has finite dimensional image. Let $\gamma : D_n \to A_r$ and $\sigma : E_n \to A_r$ denote the partial homomorphisms of $\nu_{r,s(n)}$. By Lemma 8.12, there is a $*$-homomorphism $\tau : E_n \to \sigma(1)A_r\sigma(1)$ which has the same KK theory class as σ and such that there is a diagram

$$\begin{array}{ccc} E_n & \xrightarrow{\ \tau\ } & \sigma(1)A_r\sigma(1) \\ & \searrow^{\varphi} & \uparrow{\scriptstyle \theta} \\ & & C \end{array}$$

where C is a finite dimensional algebra, θ is a $*$-homomorphism, $\varphi \in Map(E_n, C)$ is δ-multiplicative on H and

$$\|\theta\varphi(a) - \tau(a)\| < \delta \tag{8.15}$$

for all $a \in F_n$. Since $[\sigma] = [\tau]$, after increasing r, it follows from Theorem 7.1 that there exists a unitary $u \in \sigma(1)A_r\sigma(1)$ such that

$$\|\sigma(a) - u\tau(a)u^*\| < \epsilon + 18w(F_n) < \epsilon + 18 \times 2^{-n} \tag{8.16}$$

for all $a \in F_n$. If $\theta_1 = u\theta u^*$, then it follows from (8.15) and (8.16) that

$$\|\sigma(a) - \theta_1\varphi(a)\| < \delta + \epsilon + 18 \times 2^{-n} < 2^{-n+5}$$

for all $a \in F_n$. On the other hand, since φ is δ-multiplicative on H, there exists $\xi'' : B_n \to C$ satisfying (8.14).

We conclude the proof my making the following choices. We set $s(n+1) = r$, hence $A_{s(n+1)} = A_r$, $B_{n+1} = D_n \oplus C$ and $\xi_{n+1,n} = \psi'_n \oplus \xi''$, $\varphi_n = id_{D_n} \oplus \varphi$, $\psi_{n+1} = \gamma \oplus \theta_1$. Note that ψ''_{n+1} is equal to $\gamma'' \oplus \theta''_1$ where $\gamma'' : D_n \to E_r$ is a partial $*$-homomorphism of γ and $\theta_1 : C \to E_r$ is a partial $*$-homomorphism of θ. It follows from the very first part of the proof and the construction of θ that ψ''_{n+1} factors through a finite dimensional algebra. In the last part of the proof we indicate the construction of G_{n+1} and F_{n+1}. First we take $G_{n+1} \subset B_{n+1}$ to be a finite set with elements of norm ≤ 1 such that G_{n+1} is a generating set for the C*-algebra B_{n+1} and $\xi_{n+1,n}(G_n) \cup \varphi_n(F_n) \subset G_{n+1}$. Then we can find a finite set $F_{n+1} \subset A_{s(n+1)}$ such that $\nu_{s(n+1),s(n)}(A_{s(n)})$ is contained is the C*-algebra generated by F_{n+1} and $\nu_{s(n+1),s(n)}(F_n) \cup \psi_{n+1}(G_{n+1}) \subset F_{n+1}$. Finally, it follows from Lemma 3.7 that by increasing $s(n+1)$ we can arrange that $w(F_{n+1}) < 2^{-n-1}$. □

Our main reduction result is the following.

Theorem 8.14. *Let A be an ASH-algebra of real rank zero and with slow dimension growth. Then A is isomorphic to an $ASH(2)$-algebra.*

Proof. The result follows by combining Theorem 8.10 and Theorem 8.14. □

9 Classification Results

In this section we prove the main result of the paper. By definition any ASH algebra A has an approximate unit consisting of projections. It is also clear that A is stably finite. It follows from Proposition 4.8 that $(\underline{K}(A), \underline{K}(A)^+)$ is an ordered group. Recall that the scale of A is a subset of $K_0(A)$ consisting of all classes of the form $[p]$ where $p \in A$ are projections. Any algebra in $SH(2)$ and hence any $ASH(2)$ algebra has cancellation of projections. It follows from Theorem 8.14 that the ASH algebras of real rank zero with slow dimension growth have cancellation of projections. Therefore if A and B are unital C*-algebras of this type, then

$$\Sigma(A) = \{x \in K_0(A) : 0 \leq x \leq [1_A]\}$$

and a positive morphism $\alpha : K_0(A) \to K_0(B)$ preserves the scale iff $\alpha[1_A] \leq [1_B]$.

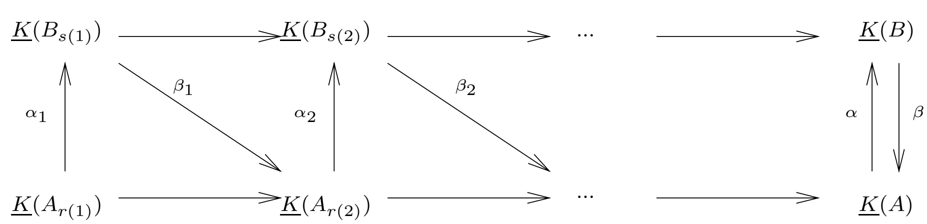
Theorem 9.1. *Let A, B be two ASH algebras of real rank zero, with slow dimension growth. Suppose that there is an isomorphism of ordered scaled groups*

$$\alpha : (\underline{K}(A), \underline{K}(A)^+, \Sigma(A)) \to (\underline{K}(B), \underline{K}(B)^+, \Sigma(B))$$

which preserves the action of the Bockstein operations. Then there is a $*$-isomorphism $\varphi : A \to B$ with $\varphi_* = \alpha$. If A and B are unital, the same conclusion remains valid if one assumes that $\alpha : \underline{K}(A) \to \underline{K}(B)$ is an isomorphism of ordered groups such that $\alpha[1_A] = [1_B]$.

Proof. By Theorem 8.14 we may assume that both A and B are $ASH(2)$ algebras. Write $A = \varinjlim(A_n, \nu_{r,n})$ and $B = \varinjlim(B_n, \gamma_{r,n})$. Let $\nu_n : A_n \to A$ and $\gamma_n : B_n \to B$ be the obvious maps. Let $\beta = \alpha^{-1}$.

We construct a commutative diagram



where α_n and β_n are liftable to $*$-homomorphisms $\varphi_n : A_{r(n)} \to B_{s(n)}$ and $\psi_n : B_{s(n)} \to A_{r(n+1)}$. The conclusion of the theorem will follow from Corollary 4.3 and Theorem 7.3. The construction is done inductively. We may assume that $A_{r(1)} = B_{s(1)} = \{0\}$ hence take $\alpha_1 = \beta_1 = 0$. Assume now that α_i and β_i have been constructed for $i \le n-1$. For the C*-algebra $A_{r(n)}$, let m and $F \subset \underline{K}(A_{r(n)})$ be provided by Theorem 5.7. We may assume that both A and B are given by inductive systems like in the conclusion of Lemma 7.4. Therefore there is $r > r(n)$ such that each partial morphism of the connecting map $\nu_{r,r(n)}$ is either strictly $3m$-large or it has finite dimensional image. Thus we can arrange that $\nu_{r,r(n)}$ is of the form $\nu' + \nu''$ where $\nu' : A_{r(n)} \to A'$ and $\nu'' : A_{r(n)} \to A''$ are $*$-homomorphisms such that ν' is strictly $3m$-large and ν'' has finite dimensional image. Here A' and A'' are orthogonal C*-subalgebras of A_r with $A' = \nu'(1)A_r\nu'(1)$ and A'' finite dimensional. Since the set F is finite and since, by Proposition 4.13, the Λ-module $\underline{K}(A_r)$ is finitely generated, there is $k > s(n-1)$ and there is a $\xi \in Hom_\Lambda(\underline{K}(A_r), \underline{K}(B_k))$ such that $\gamma_{k*}\xi = \alpha\nu_{r*}$, $\xi\nu_{r,r(n)*}\beta_{n-1} = \gamma_{k,s(n-1)*}$, $\xi\nu'_*(F) \subset \underline{K}(B_k)^+$ and $\xi[1_{A_r}] \le [1_{B_k}]$. In addition, since the semigroup $K_0(A_r)^+$ is finitely generated, we can arrange that the restriction of ξ to $K_0(A_r)$ is positive. In particular this implies that $\xi\nu'_*$ is strictly m-large since ν'_* is strictly $3m$-large and the spectrum of B_k has dimension at most 2 (see section 2). Let τ' (resp. τ'') denote the inclusion map of A' (resp. A'') into A_r. We can identify $\xi\nu'_*$ with $\xi\tau'_*\nu'_*$. Then $\xi\tau'_*\nu'_*$ satisfies all the conditions of Theorem 5.7, hence it lifts to a $*$-homomorphism $\varphi' : A_{r(n)} \to B_k$. On the other hand, by Proposition 5.9, $\xi\tau''_*$ lifts to a $*$-homomorphism $\varphi'' : A'' \to B_k$, since A'' is finite dimensional. It follows that $\varphi = \varphi' \oplus \varphi''\nu'' : A_{r(n)} \to M_2(B_k)$ is a lifting of $\xi\nu_{r,r(n)*}$. Since $\varphi_*[1_{A_{r(n)}}] = \xi\nu_{r,r(n)*}[1_{A_{r(n)}}] \le [1_{B_k}]$, after conjugating φ by a suitable partial isometry we can arrange that φ maps into B_k. We conclude the construction of α_n by setting $k = s(n)$ and $\alpha_n = \xi\nu_{r,r(n)*}$. It is then clear that $\alpha_n\beta_{n-1} = \gamma_{s(n),s(n-1)}$, $\alpha\nu_{n*} = \gamma_{s(n)*}\alpha_n$, and $\varphi_n := \varphi$ is a $*$-homomorphism lifting α_n. The construction of β_n is completely similar. We conclude the proof by applying Theorem 7.3. Recall that by Corollary 4.3 we can identify $Hom_\Lambda(\underline{K}(A_i), \underline{K}(B_j))$ with $KK(A_i, B_j)$. □

The following corollary is derived from the proof of Theorem 9.1.

COROLLARY 9.2. *Let A, B, α be as in Theorem 9.1, but assume that α is a morphism (rather than an isomorphism). Then α can be lifted to a $*$-homomorphism $\varphi : A \to B$.*

REMARK 9.3. a) In view of Remark 5.8 it is clear that Theorem 9.1 and

Corollary 9.2 remain true if we replace $\underline{K}(-)^+$ (in the statement and the proof) by either $\underline{K}(-)^{++}$ or $\underline{K}(-)_+$.

In general these order structures are not the same. However it is not hard to prove that they are isomorphic for those ASH algebras which can be written as limits of an inductive system with all connecting maps strictly 2-large.

b) Let A and B be as in Theorem 9.1 and let $\alpha : \underline{K}(A) \to \underline{K}(B)$ be a Λ-linear morphism. It follows from Corollary 9.2 and the first part of Remark 9.3 that α preserves the positive cone $\underline{K}(-)^+$ iff α preserves $\underline{K}(-)^{++}$ iff α preserves $\underline{K}(-)_+$.

Since by definition $AH \subset ASH$, Theorem 9.1 gives a complete invariant for the AH algebras of real rank zero and slow dimension growth. We will show elsewhere that these classes are distinct.

As a corollary of Theorem 9.1 we show that the ordered scaled group $K_*(-)$ is a complete invariant for the simple ASH algebras of real rank zero with slow dimension growth.

Theorem 9.4. *Let A, B be two simple ASH algebras of real rank zero, with slow dimension growth. Suppose that there exist isomorphisms*

$$\beta_0 : \big(K_0(A), K_0(A)^+, \Sigma(A)\big) \to \big(K_0(B), K_0(B)^+, \Sigma(B)\big)$$
$$\beta_1 : K_1(A) \to K_1(B)\ .$$

Then there is a $*$-isomorphism $\varphi : A \to B$ with $K_i(\varphi) = \beta_i$ for $i = 0, 1$.

Proof. Using the universal coefficient theorem of [RosS], we lift (β_0, β_1) to a KK element $\alpha \in KK(A, B)$. If $(\alpha_n^i) : \underline{K}(A) \to \underline{K}(B)$ is the map induced by α, then $\alpha_0^i = \beta_i$. Since the only ideals of A are $\{0\}$ and A, it follows from the very definition of $\underline{K}(A)_+$, that $\underline{K}(A)_+$ consists exactly of the zero element and of those $x = (x_n^i)$ whose $K_0(A)$ component x_0^0 is positive and nonzero. This shows that

$$\alpha : \big(\underline{K}(A), \underline{K}(A)_+, \Sigma(A)\big) \to \big(\underline{K}(B), \underline{K}(B)_+, \Sigma(B)\big)$$

is an isomorphism of ordered, scaled groups. Therefore we can apply Theorem 9.1 (in its version for $\underline{K}(-)_+$, explained in Remark 9.3 (a)) to obtain a $*$-isomorphism $\varphi : A \to B$ implementing α. In particular $K_i(\varphi) = \beta_i$ for $i = 0, 1$. □

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Submitted: June 1996
Revised version: February 1997
Final revision: June 1997