E-THEORY FOR C*-ALGEBRAS OVER TOPOLOGICAL SPACES

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Abstract. We define E-theory for separable C*-algebras over second countable topological spaces and establish its basic properties. This includes an approximation theorem that relates the E-theory over a general space to the E-theories over finite approximations to this space. We obtain effective criteria for determining the invertibility of E-theory elements over possibly infinite-dimensional spaces. Furthermore, we prove a Universal Multicoefficient Theorem for C*-algebras over totally disconnected metrisable compact spaces.

1. Introduction

Eberhard Kirchberg [17] proved a far-reaching classification theorem for non-simple, strongly purely infinite, stable, nuclear, separable C*-algebras. Roughly speaking, two such C*-algebras are isomorphic once they have homeomorphic primitive ideal spaces – call this space X – and are $\mathrm{KK}(X)$-equivalent in a suitable bivariant K-theory for C*-algebras over X. To apply this classification theorem, we need tools to compute this bivariant K-theory. Following Mikael Rørdam [28] and Alexander Bonkat [3], who dealt with the simplest non-trivial case, the non-Hausdorff space with two points, Universal Coefficient Theorems for $\mathrm{KK}(X)$ have now been established over several finite spaces X in [14, 22, 26, 27]. Here we concentrate on the special issues for infinite X.

Recall that Kasparov theory only satisfies excision for C*-algebra extensions with a completely positive section. Similar technical restrictions appear for all variants of Kasparov theory, including Kirchberg's. This is a severe limitation. For instance, excision does not hold in general for extensions of the form $A(U) \rightarrowtail A \twoheadrightarrow A/A(U)$ for an open subset U, where $A(U)$ denotes the restriction of A to U, extended by 0 to a C*-algebra over the original space, even if A is nuclear. In the non-equivariant case, such technical problems are resolved by passing to E-theory, which satisfies excision for all C*-algebra extensions (see [5]). Here we define an analogue of E-theory for separable C*-algebras over a second countable topological

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space X. We establish that our new theory has the expected properties, including a universal property and exactness for all extensions of C*-algebras over X. If X is a locally compact Hausdorff space, then our definitions agree with previous ones by Efton Park and Jody Trout in [24] and by Radu Popescu in [25]. We also formulate sufficient criteria for the natural map $\mathrm{E}_*(X;A,B) \to \mathrm{KK}_*(X;A,B)$ to be invertible. For instance, this works if X is locally compact and Hausdorff and A is a continuous field of nuclear C*-algebras over X.

Our definition of $\mathrm{E}_*(X;A,B)$ is based on asymptotic homomorphisms satisfying an approximate equivariance condition. An asymptotic homomorphism $\varphi_t\colon A \to B$, $t \in [0,\infty)$, is called *approximately X-equivariant* if for each open subset $U \subseteq X$, we have

$$\lim_{t\to\infty} \|\varphi_t(a)\|_{X\setminus U} = 0 \qquad \text{for all } a \in A(U),$$

where $\|\varphi_t(a)\|_{X\setminus U}$ denotes the norm of $\varphi_t(a)$ in the quotient $B(X \setminus U) := B/B(U)$ of B.

Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a countable basis for the topology of X. For each $n \in \mathbb{N}$, the open subsets $U_1,\dots,U_n$ generate a finite topology τ_n on X. Let X_n be the T_0-quotient of (X,τ_n), this is a finite T_0-space. The quotient map $X \twoheadrightarrow X_n$ allows us to view C*-algebras over X as C*-algebras over X_n for all $n \in \mathbb{N}$. Our first main result is a short exact sequence

$$\varprojlim_{n\in\mathbb{N}}{}^1 \mathrm{E}_{*+1}(X_n;A,B) \rightarrowtail \mathrm{E}_*(X;A,B) \twoheadrightarrow \varprojlim_{n\in\mathbb{N}} \mathrm{E}_*(X_n;A,B) \tag{1.1}$$

for all separable C*-algebras A and B over X. This is made plausible by the observation that an asymptotic homomorphism $A \to B$ is approximately X-equivariant if and only if it is approximately X_n-equivariant for all $n \in \mathbb{N}$. Hence the space of approximately X-equivariant asymptotic homomorphisms is the intersection of the spaces of approximately X_n-equivariant asymptotic homomorphisms for $n \in \mathbb{N}$. Since there are, in general, technical problems with computing homotopy groups of intersections, we use a mapping telescope to establish the long exact sequence (1.1).

As an important application of (1.1), we give an effective criterion for invertibility of E-theory elements: an element in $\mathrm{E}_*(X;A,B)$ is invertible if and only if its image in $\mathrm{E}_*\bigl(A(U),B(U)\bigr)$ is invertible for all $U \in \mathbb{O}(X)$. As a consequence, if all two-sided closed ideals of a separable nuclear C*-algebra A with Hausdorff primitive spectrum X are KK-contractible, then

$$A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong \mathrm{C}_0(X) \otimes \mathcal{O}_2 \otimes \mathbb{K}.$$

This result solves the problem of characterising the trivial continuous fields with fibre $\mathcal{O}_2 \otimes \mathbb{K}$ within the class of strongly purely infinite, stable, continuous fields of C*-algebras. It is worth noting that in general the KK-contractibility of ideals does not follow from the KK-contractibility of the fibres. Indeed, there are examples of separable nuclear continuous fields A

over the Hilbert cube with all fibres isomorphic to $\mathcal{O}_2$ and yet such that $\mathrm{K}_0(A) \neq 0$, see [8].

While (1.1), in principle, reduces the computation of $\mathrm{E}_*(X; A, B)$ for infinite spaces X to the corresponding problem for the finite approximations X_n, this does not yet lead to a Universal Coefficient Theorem. If $\mathrm{E}_*(X_n; A, B)$ is computable by Universal Coefficient Theorems for all $n \in \mathbb{N}$, the latter will usually involve short exact sequences. Thus we have to combine two short exact sequences, as in the computation of the K-theory for crossed products by $\mathbb{Z}^2$ using the Pimsner–Voiculescu exact sequence twice. This can only be carried through if we have some extra information. In terms of the general homological machinery developed in [20], we find that the homological dimension of E-theory over an infinite space X may be one larger than the homological dimensions of the finite approximations X_n. Thus it is usually 2, which does not suffice for classification theorems.

In fact, it is well-known that filtrated K-theory cannot be a complete invariant for C*-algebras over the one-point compactification of $\mathbb{N}$. Here we observe that the counterexample in [10] may be transported easily to any compact Hausdorff space.

The good excision properties of E-theory are particularly useful to study the E-theoretic analogue of the bootstrap class. For a finite space X, the bootstrap class for $\mathrm{KK}(X)$ is studied in [21]. When we replace $\mathrm{KK}(X)$ by $\mathrm{E}(X)$, the technical assumptions in [21] about completely positive sections disappear, so that a C*-algebra A over a finite space X belongs to the E-theoretic bootstrap class over X if and only if all the distinguished ideals $A(U)$ for open subsets $U \subseteq X$ belong to the usual non-equivariant E-theoretic bootstrap class. As we shall see, the latter criterion becomes a useful *definition* of the bootstrap class over an infinite space X. In $\mathrm{KK}(X)$, this condition would not yet be sufficient for a reasonable definition of the bootstrap class.

If X is the Cantor set or, more generally, a totally disconnected metrisable compact space, then we may resolve the counterexamples mentioned above by taking into account coefficients. Our second main result is a Universal *Multi*coefficient Theorem for $\mathrm{E}_*(X; A, B)$ for two C*-algebras A and B over X. It assumes that $A(U)$ belongs to the E-theoretic bootstrap class for all open subsets $U \subseteq X$ and yields a natural exact sequence

$$\mathrm{Ext}_{\mathrm{C}(X,\Lambda)}\big(\underline{\mathrm{K}}(A)[1], \underline{\mathrm{K}}(B)\big) \rightarrowtail \mathrm{E}(X; A, B) \twoheadrightarrow \mathrm{Hom}_{\mathrm{C}(X,\Lambda)}\big(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)\big),$$

where $\underline{\mathrm{K}}$ denotes the K-theory of A with coefficients, viewed as a countable module over the $\mathbb{Z}/2$-graded ring $\mathrm{C}(X, \Lambda)$ of locally constant functions from X to the $\mathbb{Z}/2$-graded ring Λ of Böckstein operations (see [11]). As a consequence, two C*-algebras A and B in the E-theoretic bootstrap class over X are $\mathrm{E}(X)$-equivalent if and only if $\underline{\mathrm{K}}(A)$ and $\underline{\mathrm{K}}(B)$ are isomorphic as $\mathrm{C}(X, \Lambda)$-modules.

2. E-theory for C*-algebras over non-Hausdorff spaces

We recall some definitions from [21] regarding C*-algebras over possibly non-Hausdorff topological spaces and then introduce equivariant E-theory for them. Following the approach of Alain Connes and Nigel Higson in [5], we first describe E-theory concretely using asymptotic morphisms, then abstractly using a universal property. For a locally compact Hausdorff space X, our definition is equivalent to previous ones for $C_0(X)$-algebras by Efton Park and Jody Trout in [24] and by Radu Popescu in [25].

2.1. **C*-algebras over non-Hausdorff spaces.** Here we recall some basic definitions from [21].

For a C*-algebra A, let $\mathrm{Prim}(A)$ denote its primitive ideal space, equipped with the hull–kernel topology, and let $\mathbb{I}(A)$ be the set of ideals in A, partially ordered by inclusion. For a topological space X, let $\mathbb{O}(X)$ be the set of open subsets of X, partially ordered by inclusion. Both $\mathbb{I}(A)$ and $\mathbb{O}(X)$ are complete lattices, that is, any subset has both an infimum and a supremum. It is shown in [13, §3.2] that there is a canonical lattice isomorphism

$$\mathbb{O}\bigl(\mathrm{Prim}(A)\bigr) \cong \mathbb{I}(A), \qquad U \mapsto \bigcap \{\mathfrak{p} : \mathfrak{p} \in \mathrm{Prim}(A) \setminus U\}. \tag{2.1}$$

Definition 2.2. Let X be a topological space.

A C*-*algebra over* X is a C*-algebra A with a continuous map ψ from $\mathrm{Prim}(A)$ to X.

For an open subset U of X, we let $A(U) \in \mathbb{I}(A)$ be the ideal that corresponds to $\psi^{-1}(U) \in \mathbb{O}(\mathrm{Prim}\, A)$ under the isomorphism (2.1).

For a closed subset S of X, we let $A(S) := A/A(X \setminus S)$. For $a \in A$, we write $\|a\|_S$ for the norm of the image of a in the quotient C*-algebra $A(S)$.

More generally, if $S \subseteq X$ is locally closed, that is, $S = U \setminus V$ with open subsets $V \subseteq U \subseteq X$, then we let $A(S) := A(U)/A(V)$. This quotient is independent of the choice of the open sets U and V with $S = U \setminus V$.

Let A and B be C*-algebras over X. A *-homomorphism $f\colon A \to B$ is called X-*equivariant* or a **-homomorphism over* X if f maps $A(U)$ into $B(U)$ for all open subsets U of X.

Let $\mathfrak{C}^*\mathfrak{alg}(X)$ be the category whose objects are the C*-algebras over X and whose morphisms are the *-homomorphisms over X. Let $\mathfrak{C}^*\mathfrak{sep}(X)$ be the full subcategory of *separable* C*-algebras over X with *-homomorphisms over X as morphisms.

We usually drop the map $\mathrm{Prim}(A) \to X$ from our notation and simply call A a C*-algebra over X.

Although the above definition involves X, all that really matters is the lattice $\mathbb{O}(X)$. It is explained in [21] that it is essentially no loss of generality to assume X to be *sober*. In that case, we may recover X from the lattice $\mathbb{O}(X)$ and the map $\mathrm{Prim}(A) \to X$ from the map $\mathbb{O}(X) \to \mathbb{I}(A)$, $U \mapsto A(U)$ (see [21, Lemma 2.25]), which may be any map that commutes with finite infima and arbitrary suprema. Thus if X is a second countable, sober space, a C*-algebra over X is a C*-algebra A endowed with an order preserving map $\mathbb{O}(X) \to \mathbb{I}(A)$, $U \mapsto A(U)$, which satisfies the following conditions:

(1) $A(\emptyset) = 0$, $A(X) = A$,
(2) $A(U_1 \cap U_2) = A(U_1) \cdot A(U_2)$,
(3) $A\big(\bigcup_{n=1}^{\infty} U_n\big) = \overline{\sum_{n=1}^{\infty} A(U_n)}$.

If a C*-algebra A satisfies the conditions (1) and (2) and

(3') $A(U_1 \cup U_2) = A(U_1) + A(U_2)$,

then we say that A is a *quasi* C*-algebra over X. If B is a C*-algebra over X then $\mathrm{C_b}(T, B)$ and $\mathrm{C_b}(T, B) \;/\; \mathrm{C_0}(T, B)$ for $T := [0, \infty)$ become quasi C*-algebras over X, via the maps $U \mapsto \mathrm{C_b}(T, B(U))$ and

$$U \mapsto \mathrm{C_b}(T, B(U)) + \mathrm{C_0}(T, B) \;/\; \mathrm{C_0}(T, B).$$

However, they do not satisfy the condition (3) above.

Let X be a locally compact Hausdorff space and let A be a C*-algebra over X. The continuous map $\mathrm{Prim}(A) \to X$ induces a *-homomorphism

$$\mathrm{C_b}(X) \to \mathrm{C_b}\big(\mathrm{Prim}(A)\big) \cong Z\mathcal{M}(A),$$

where $Z\mathcal{M}(A)$ denotes the centre of the multiplier algebra of A. One verifies that $\mathrm{C_0}(X)A$ is dense in A, so that A becomes a $\mathrm{C_0}(X)$-C*-algebra. This yields an isomorphism of categories between $\mathfrak{C}^*\mathfrak{alg}(X)$ and the category of $\mathrm{C_0}(X)$-C*-algebras with $\mathrm{C_0}(X)$-linear *-homomorphisms as morphisms by [21, Proposition 2.11].

2.2. **Approximately equivariant asymptotic morphisms.** Recall:

Definition 2.3. An *asymptotic morphism* between two C*-algebras A and B is a map $\varphi\colon A \to \mathrm{C_b}(T, B)$ for $T := [0, \infty)$ that induces a *-homomorphism

$$\dot\varphi\colon A \to B_\infty := \mathrm{C_b}(T, B) \;/\; \mathrm{C_0}(T, B).$$

The map φ is equivalent to a family of maps $\varphi_t\colon A \to B$ for $t \in T$ such that $t \mapsto \varphi_t(a)$ is a bounded continuous function from T to B for each $a \in A$. Such a family is an asymptotic morphism if and only if

$$\varphi_t(a^* + \lambda b) - \varphi_t(a)^* - \lambda\varphi_t(b) \quad \text{and} \quad \varphi_t(a \cdot b) - \varphi_t(a) \cdot \varphi_t(b)$$

converge to 0 in the norm topology for $t \to \infty$ for all $a, b \in A$, $\lambda \in \mathbb{C}$.

Two asymptotic morphisms φ and φ' are called *equivalent* if $\dot\varphi = \dot\varphi'$, that is, $\varphi_t(a) - \varphi'_t(a)$ converges to 0 for $t \to \infty$ for all $a \in A$.

Definition 2.4. An asymptotic morphism $\varphi_t\colon A \to B$ from A to B is called *approximately X-equivariant* if, for any open subset $U \subseteq X$,

$$\lim_{t\to\infty} \|\varphi_t(a)\|_{X\setminus U} = 0 \qquad \text{for all } a \in A(U). \tag{2.5}$$

Let $\mathrm{Asymp}(A, B)_X$ be the set of approximately X-equivariant asymptotic morphisms $A \to B$.

Our definition of $\mathrm{Asymp}(A, B)_X$ requires X-equivariance only in the limit, the individual maps φ_t need not be X-equivariant.

Remark 2.6. If φ is equivalent to an approximately X-equivariant asymptotic morphism, then φ itself is approximately X-equivariant.

Lemma 2.7. *An asymptotic morphism φ is approximately X-equivariant if and only if, for all closed subsets S of X,*

$$\limsup_{t\to\infty} \|\varphi_t(a)\|_S \le \|a\|_S \qquad \text{for all } a \in A.$$

Proof. Let $U := X \setminus S$. The lim sup-criterion specialises to the definition of X-equivariance for $a \in A(U)$. Conversely, for any $\varepsilon > 0$ we may split $a \in A$ as $a = a_1 + a_2$ with $a_1 \in A(U)$ and $\|a_2\| < \|a\|_S + \varepsilon$ and estimate

$$\limsup \|\varphi_t(a)\|_S \le \limsup \|\varphi_t(a_1)\|_S + \limsup \|\varphi_t(a_2)\|.$$

The X-equivariance of φ and $a_1 \in A(U)$ imply $\lim \|\varphi_t(a_1)\|_S = 0$, and

$$\limsup \|\varphi_t(a_2)\| = \|\dot\varphi(a_2)\| \le \|a_2\| < \|a\|_S + \varepsilon.$$

Thus $\limsup \|\varphi_t(a)\|_S < \|a\|_S + \varepsilon$ for all $\varepsilon > 0$. □

Let $U \in \mathbb{O}(X)$ and $S := X \setminus U$. The quotient map $\pi_S \colon B \to B(S)$ induces a map $\tilde\pi_S \colon \mathrm{C_b}(T, B) \to \mathrm{C_b}\big(T, B(S)\big)$ whose kernel is $\mathrm{C_b}(T, B(U))$. Condition (2.5) is equivalent to

$$\tilde\pi_S \circ \varphi\big(A(U)\big) \subseteq \mathrm{C_0}\big(T, B(S)\big). \tag{2.8}$$

Lemma 2.9. *An asymptotic morphism φ is approximately X-equivariant if and only if, for all open subsets U of X,*

$$\varphi\big(A(U)\big) \subseteq \mathrm{C_b}\big(T, B(U)\big) + \mathrm{C_0}(T, B). \tag{2.10}$$

Proof. It is clear that (2.10) implies (2.8). To verify the converse, it suffices to prove

$$(\tilde\pi_S)^{-1}\big(\mathrm{C_0}(T, B(S))\big) = \mathrm{C_b}\big(T, B(U)\big) + \mathrm{C_0}(T, B).$$

The Bartle–Graves Theorem provides a continuous section $\gamma \colon B(S) \to B$ of π_S. Any $f \in \mathrm{C_b}(T, B)$ decomposes as $f = g + h$ with $g := f - \gamma \circ \tilde\pi_S(f)$ and $h := \gamma \circ \tilde\pi_S(f)$. We have $g \in \mathrm{C_b}\big(T, B(U)\big)$ and $h \in \mathrm{C_0}(T, B)$ whenever $\tilde\pi_S(f) \in \mathrm{C_0}\big(T, B(S)\big)$ because γ is continuous. □

For Hausdorff spaces X, Park and Trout [24] and Popescu [25] defined an E-theory $\mathcal{R}\mathrm{E}_*(X; A, B)$ for $\mathrm{C_0}(X)$-algebras based on asymptotic morphisms φ that are *asymptotically* $\mathrm{C_0}(X)$*-equivariant* in the sense that $\varphi(fa) - f\varphi(a) \in \mathrm{C_0}(T, B)$ for all $a \in A$ and $f \in \mathrm{C_0}(X)$; equivalently, $\dot\varphi \colon A \to B_\infty$ is $\mathrm{C_0}(X)$-linear.

Proposition 2.11. *Let X be a second countable locally compact Hausdorff space and let A and B be $\mathrm{C_0}(X)$-algebras. Then an asymptotic morphism from A to B is asymptotically $\mathrm{C_0}(X)$-equivariant if and only if it is approximately X-equivariant.*

Proof. Clearly, an asymptotically $C_0(X)$-equivariant asymptotic morphism satisfies (2.10) since $A(U) = C_0(U)A$ and $C_0(U)C_b(T,B) \subseteq C_b(T,B(U))$. Conversely, let φ be approximately X-equivariant. Let $B_\infty^X := C_0(X)\cdot B_\infty \subseteq B_\infty$, this is a $C_0(X)$-algebra. We are going to show that $\dot\varphi\bigl(C_0(U)A\bigr)$ is contained in $C_0(U)\cdot B_\infty^X = C_0(U)\cdot B_\infty$ for all $U \in \mathbb{O}(X)$. This is equivalent to the $C_0(X)$-linearity of $\dot\varphi\colon A \to B_\infty^X$ by [21, Proposition 2.11].

For any $f \in C_0(U)$ and any $\varepsilon > 0$, there are a relatively compact open subset $U_\varepsilon \subseteq \overline{U}_\varepsilon \subseteq U$ and $f_\varepsilon \in C_0(U_\varepsilon)$ with $\|f - f_\varepsilon\| < \varepsilon$. Since A is a $C_0(X)$-C*-algebra, the same approximation applies to all $a \in A(U) = C_0(U)\cdot A$. Therefore, it suffices to prove $\dot\varphi\bigl(A(U')\bigr) \subseteq C_0(U)\cdot B_\infty$ for all relatively compact open subsets U' of U with $\overline{U'} \subseteq U$.

Since there is a function w in $C_0(U)$ with $w(x) = 1$ for all $x \in U'$, we have

$$C_b\bigl(T, B(U')\bigr) \subseteq w\cdot C_b(T,B) \subseteq C_0(U)\cdot C_b(T,B)$$

for all $n \in \mathbb{N}$. Since φ maps $A(U')$ into $C_b\bigl(T,B(U')\bigr) + C_0(T,B)$ by (2.10), $\dot\varphi$ maps $A(U')$ into $C_0(U)\cdot B_\infty$ for all $n \in \mathbb{N}$. □

2.3. **Homotopy of asymptotic morphisms.**

Definition 2.12. A *homotopy* of asymptotic morphisms from A to B is an asymptotic morphism from A to $C([0,1],B)$. Let $[\![A,B]\!]_X$ denote the set of homotopy classes of approximately X-equivariant asymptotic morphisms from A to B.

Equivalent asymptotic morphisms are homotopic.

We do not know whether there is a natural topology on $\mathrm{Asymp}(A,B)_X$ such that $[\![A,B]\!]_X = \pi_0(\mathrm{Asymp}(A,B)_X)$. It is easy to avoid this question by using quasi-topological spaces in the sense of Edwin H. Spanier (see [30]).

Definition 2.13. A *quasi-topological space* is a set W together with distinguished sets of maps $C(Y,W)$ from Y to W for each compact Hausdorff space Y, called *quasi-continuous maps* $Y \to W$. These quasi-continuous maps are required to satisfy the following conditions:

- constant maps are quasi-continuous;
- a function defined on a disjoint union $Y_1 \sqcup Y_2$ is quasi-continuous if and only if its restrictions to Y_1 and Y_2 are quasi-continuous;
- if $f\colon Y_1 \to Y_2$ is a quasi-continuous map and $h\colon Y_2 \to W$ is quasi-continuous, so is $h \circ f$; and, conversely,
- if f is surjective and continuous (so that f is an open surjection), then h is quasi-continuous provided $h \circ f$ is quasi-continuous.

Since W is the set of quasi-continuous functions from the one-point space to W, we may also view a quasi-topological space as a contravariant functor from the category of compact Hausdorff spaces to the category of sets with some additional properties.

We define a quasi-topology on $\mathrm{Asymp}(A,B)_X$ by letting

$$C(Y, \mathrm{Asymp}(A,B)_X) := \mathrm{Asymp}(A, C(Y,B))_X$$

for each compact Hausdorff space Y.

Furthermore, $\mathrm{Asymp}(A,B)_X$ has a canonical base point, the zero map. Thus $\mathrm{Asymp}(A,B)_X$ becomes a pointed quasi-topological space.

Homotopy groups for pointed quasi-topological spaces may be defined as for ordinary topological spaces, using quasi-continuous maps instead of continuous maps. By definition, $[\![A,B]\!]_X = \pi_0(\mathrm{Asymp}(A,B)_X)$.

2.4. **E-theory: Definition and universal property.** The original approach of Alain Connes and Nigel Higson in [5] only works well for separable C*-algebras. The same restriction applies to our equivariant generalisation. Hence we (tacitly) assume all C*-algebras to be separable from now on. For similar reasons, we assume the underlying space X to be second countable, that is, its topology must have a countable basis.

Definition 2.14. Let X be a second countable topological space and let A and B be separable C*-algebras over X. Following [5], we define

$$\mathrm{E}_0(X;A,B) := [\![\mathrm{C}_0(\mathbb{R},A)\otimes\mathbb{K}, \mathrm{C}_0(\mathbb{R},B)\otimes\mathbb{K}]\!]_X.$$

The orthogonal direct sum turns $\mathrm{E}_0(X;A,B)$ into an Abelian group. This also holds for $\mathrm{E}_1(X;A,B) := \mathrm{E}_0(X;\mathrm{C}_0(\mathbb{R},A),B)$.

Proposition 2.15. *The composition of asymptotic morphisms induces a product*

$$[\![A,B]\!]_X \times [\![B,C]\!]_X \to [\![A,C]\!]_X.$$

The proof is similar to the non-equivariant case outlined in [4]. In addition to the arguments from [4, Appendix B of Chapter II], we need the following lemma to take care of approximate X-equivariance.

Recall that an asymptotic morphism φ is called *uniformly continuous* if the map $\varphi\colon A\to \mathrm{C_b}(T,B)$ is continuous. By the Bartle–Graves Theorem, every asymptotic morphism is equivalent to a uniformly continuous one.

Lemma 2.16. *Let X be a second countable topological space, let A, B and C be separable* C**-algebras, and let $\varphi\colon A\to\mathrm{C_b}(T,B)$ and $\psi\colon B\to \mathrm{C_b}(T,C)$ be uniformly continuous, approximately X-equivariant asymptotic morphisms. Let A_0 be a σ-compact dense *-subalgebra of A. There is an increasing, continuous map $r_0\colon T\to T$ such that for any other increasing, continuous map $r\colon T\to T$ with $r(t)\geq r_0(t)$ for all $t\in T$, there is an approximately X-equivariant asymptotic morphism $\theta\colon A\to\mathrm{C_b}(T,C)$ such that* $\lim_{t\to\infty}\|\theta_t(a)-\psi_{r(t)}\circ\varphi_t(a)\| = 0$ *for all $a\in A_0$.*

Proof. Let $(U_i)_{i=1}^\infty$ be a basis of open sets for the topology of X. Choose a dense sequence $(a_{ij})_{j=1}^\infty$ in $A(U_i)$ for each $i\geq 1$. We will find a map r_0 such that, for all $r\geq r_0$,

(i) $(\psi_{r(t)}\varphi_t)$ is a bounded asymptotic morphism from A_0 to C, and
(ii) $\lim_{t\to\infty}\|\psi_{r(t)}\circ\varphi_t(a_{ij})\|_{X\setminus U_i} = 0$ for all i,j.

Then $\psi_{r(t)} \circ \varphi_t$ defines a bounded *-homomorphism $A_0 \to C_\infty$ by (i). It extends to a *-homomorphism $\dot\theta$ on A. Let $\theta\colon A \to \mathrm{C_b}(T, C)$ be a lifting of $\dot\theta$. Then θ is approximately X-equivariant by (ii).

It remains to construct r_0. By the usual non-equivariant case, there is a continuous map r_{00} such that (i) holds for all $r \geq r_{00}$. Since $\varphi\bigl(A(U_i)\bigr) \subseteq \mathrm{C_b}\bigl(T, B(U_i)\bigr) + \mathrm{C_0}(T, B)$, there are $f_{ij} \in \mathrm{C_b}\bigl(T, B(U_i)\bigr)$ and $g_{ij} \in \mathrm{C_0}(T, B)$ such that $\varphi(a_{ij}) = f_{ij} + g_{ij}$ for all $i, j \geq 1$. Consider the following countable families of compact sets:

$$K_n := \bigcup_{i,j=1}^{n} f_{ij}[1, n+1] \cup g_{ij}[1, n+1] \subseteq B,$$

$$L_{i,n} := \bigcup_{j=1}^{n} f_{ij}[1, n+1] \subseteq B(U_i).$$

Since ψ is a uniformly continuous asymptotic morphism, we can inductively construct an increasing sequence $(s_n)_n$ such that for any $s \geq s_n$

$$\|\psi_s(x+y) - \psi_s(x) - \psi_s(y)\| < 1/n, \qquad \text{for all } x, y \in K_n, \tag{2.17}$$

$$\|\psi_s(x)\| < \|x\| + 1/n, \qquad \text{for all } x \in K_n. \tag{2.18}$$

Since ψ is approximately X-equivariant and $L_{i,n} \subseteq B(U_i)$, for each i there is an increasing sequence $(r_{i,n})_n$ such that

$$\|\psi_s(x)\|_{X \setminus U_i} < 1/n, \qquad \text{for all } x \in L_{i,n} \text{ and all } s \geq r_{i,n}. \tag{2.19}$$

Choose an increasing continuous map $r_0\colon T \to T$ with $r_0(t) \geq r_{00}(t)$ and $r_0(n) \geq \max\{s_n, r_{1,n}, r_{2,n}, \dots, r_{n,n}\}$ for all $n \geq 1$. We claim that any increasing, continuous function $r \geq r_0$ satisfies (ii). This will finish the proof.

Fix i, j and $\varepsilon > 0$. Choose n such that $n \geq i$, $n \geq j$ and $1/n < \varepsilon/3$. We shall show that for any $t \geq n$,

$$\|\psi_{r(t)} \circ \varphi_t(a_{ij})\|_{X \setminus U_i} < \varepsilon + \|g_{ij}(t)\|.$$

This will conclude the proof since $\lim_{t\to\infty} g_{ij}(t) = 0$ by construction. If $t \geq n$, then there is an integer $m \geq n$ such that $m \leq t < m+1$. Therefore $f_{ij}(t)$ and $g_{ij}(t)$ are in K_m and $r(t) \geq r(m) \geq s_m$. Equation (2.17) yields

$$\|\psi_{r(t)}(f_{ij}(t) + g_{ij}(t)) - \psi_{r(t)}(f_{ij}(t)) - \psi_{r(t)}(g_{ij}(t))\| < 1/m < \varepsilon/3. \tag{2.20}$$

Since $i, j \leq n \leq m$ and $t < m+1$, we have $f_{ij}(t) \in L_{i,m}$ and $r(t) \geq r(m) \geq r_{i,m}$. Inequality (2.19) yields

$$\|\psi_{r(t)}(f_{ij}(t))\|_{X \setminus U_i} < 1/m < \varepsilon/3. \tag{2.21}$$

Similarly, (2.18) yields

$$\|\psi_{r(t)}(g_{ij}(t))\| \leq \|g_{ij}(t)\| + 1/m < \|g_{ij}(t)\| + \varepsilon/3. \tag{2.22}$$

Putting together (2.20), (2.21) and (2.22), we get

$$\begin{aligned}\|\psi_{r(t)}\varphi_t(a_{ij})\|_{X\setminus U_i} &\le \|\psi_{r(t)}(f_{ij}(t)+g_{ij}(t)) - \psi_{r(t)}(f_{ij}(t)) - \psi_{r(t)}(g_{ij}(t))\| \\ &\qquad + \|\psi_{r(t)}(f_{ij}(t))\|_{X\setminus U_i} + \|\psi_{r(t)}(g_{ij}(t))\| \\ &< \varepsilon + \|g_{ij}(t)\|. \end{aligned}$$ □

For any extension of separable C*-algebras $I \rightarrowtail A \overset{p}{\twoheadrightarrow} B$, there is a canonical asymptotic morphism from $\mathrm{C}_0\big((0,1),B\big)$ to I. If A is a C*-algebra over X, then I and B become C*-algebras over X in a unique natural way, such that the given extension is an extension of C*-algebras over X. Specifically, $I(U) = I \cap A(U)$ and $B(U) = p(A(U))$ for all U open in X.

Proposition 2.23. *Let $I \rightarrowtail A \twoheadrightarrow B$ be an extension of* C*-*algebras over X. Then the associated asymptotic morphism from* $\mathrm{C}_0\big((0,1),B\big)$ *to I is approximately X-equivariant.*

Proof. Having an extension of C*-algebras over X means that we have C*-algebra extensions

$$I(U) \rightarrowtail A(U) \twoheadrightarrow B(U)$$

for all open subsets U of X. Since the map $B(U) \to B$ is injective, this implies $I(U) = I \cap A(U) = I \cdot A(U)$.

We fix a positive and contractive continuous approximate unit $(u_t)_{t\in T}$ of I which is quasi-central in A. The canonical asymptotic morphism

$$\gamma\colon SB := \mathrm{C}_0\big((0,1),B\big) \to \mathrm{C_b}(T,I)$$

is defined in two steps. First, we define a homomorphism

$$\gamma'\colon SA \to \mathrm{C_b}(T,I) \big/ \mathrm{C}_0(T,I), \qquad \gamma'_t(f\otimes a) := f(u_t)\cdot a.$$

Secondly, since the restriction of γ' to SI is equivalent to the null asymptotic morphism, γ' induces an asymptotic morphism from SB to I. Clearly, γ' is approximately X-equivariant because $I \cdot A(U) \subseteq I(U)$. This is inherited by γ because $\dot\gamma \circ p = \dot\gamma'$, where $p\colon A \to B$ is the quotient map. □

Let $I \rightarrowtail B \overset{p}{\twoheadrightarrow} C$ be an extension of C*-algebras over X. Let A be a C*-algebra over X and let $\varphi\colon A \to C$ be an X-equivariant *-homomorphism. Let E be the C*-algebra defined by the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \\ 0 & \longrightarrow & I & \longrightarrow & B & \overset{p}{\longrightarrow} & C & \longrightarrow & 0, \end{array}$$

that is, $E = \{(a,b) \in A \oplus B : \varphi(a) = p(b)\}$. For $U \in \mathbb{O}(X)$, set $E(U) := E \cap \big(A(U) \oplus B(U)\big)$.

Lemma 2.24. *E is a* C*-*algebra over X and $I \rightarrowtail E \twoheadrightarrow A$ is an extension of* C*-*algebras over X. The same conclusions hold if B and C are only quasi* C*-*algebras over X.*

Proof. Recall that for a quasi C*-algebra B over X, the map $U \mapsto B(U)$ preserves only finite suprema in general. The map $U \mapsto E(U)$ is obviously order-preserving. The conditions $E(\emptyset) = 0$, $E(X) = E$ and $E(U_1 \cap U_2) = E(U_1) \cap E(U_2)$ are easily verified. Let us show that $E(U_1 \cup U_2) \subset E(U_1) + E(U_2)$, the reverse inclusion being obvious. Let $(a, b) \in E(U_1 \cup U_2)$. Then $a \in A(U_1 \cup U_2) = A(U_1) + A(U_2)$ and hence there are $a_i \in A(U_i)$, $i = 1, 2$ such that $a = a_1 + a_2$. Since φ is X-equivariant, $\varphi(a_i) \in C(U_i)$ and hence there are $b_i \in B(U_i)$ such that $p(b_i) = \varphi(a_i)$, $i = 1, 2$. It follows that $b_1 + b_2 - b \in B(U_1 \cup U_2)$ and $p(b_1 + b_2 - b) = \varphi(a_1) + \varphi(a_2) - \varphi(a) = 0$. Therefore, $b_1 + b_2 - b \in I \cap B(U_1 \cup U_2) = I(U_1 \cup U_2) = I(U_1) + I(U_2)$. This shows that there are $x_i \in I(U_i)$, $i = 1, 2$, such that $b_1 + b_2 - b = x_1 + x_2$. It follows that $(a_i, b_i - x_i) \in E(U_i)$ and $(a, b) = (a_1, b_1 - x_1) + (a_2, b_2 - x_2)$.

It remains to show that $E\bigl(\bigcup U_n\bigr)$ is the closure of $\bigcup E(U_n)$ for any increasing sequence (U_n) in $\mathbb{O}(X)$. The sequence of C*-algebras

$$I(U) \rightarrowtail E(U) \twoheadrightarrow A(U)$$

is exact for each open set U. Since A and I are C*-algebras over X,

$$A(U) = \overline{\bigcup A(U_n)} = \varinjlim A(U_n),$$
$$I(U) = \overline{\bigcup I(U_n)} = \varinjlim I(U_n).$$

Since the C*-algebra inductive limit functor is exact, we get another extension of C*-algebras

$$I(U) \rightarrowtail \overline{\bigcup E(U_n)} \twoheadrightarrow A(U)$$

because $\varinjlim E(U_n) = \overline{\bigcup E(U_n)}$. This implies that $E(U)$ is the supremum of $\bigl\{E(U_n)\bigr\}$, so that E is a C*-algebra over X. □

Theorem 2.25. *The equivariant* E*-theory defined above carries a composition product and hence yields a category* $\mathfrak{E}(X)$*. The canonical functor from the category* $\mathfrak{C}^*\mathfrak{sep}(X)$ *of separable* C**-algebras over* X *to* $\mathfrak{E}(X)$ *is the universal half-exact,* C**-stable homotopy functor.*

Proof. The composition product is described in Proposition 2.15. The same argument as in the non-equivariant case shows that it is associative. The functor $\mathfrak{C}^*\mathfrak{sep}(X) \to \mathfrak{E}(X)$ is a C*-stable homotopy functor by definition. Next we check its exactness.

Let $I \rightarrowtail E \overset{p}{\twoheadrightarrow} Q$ be an extension of C*-algebras over X. The cone

$$C_p := \{(f, a) \in \mathrm{C}_0((0, 1], Q) \oplus E : f(1) = p(a)\},$$
$$C_p(U) := \bigl(\mathrm{C}_0((0, 1], Q(U)) \oplus E(U)\bigr) \cap C_p \qquad \text{for } U \in \mathbb{O}(X),$$

is a C*-algebra over X by Lemma 2.24. The asymptotic morphism $\gamma_t \colon SC_p \to SI$ induced by the extension $SI \rightarrowtail CE \twoheadrightarrow C_p$ is approximately X-equivariant. There is a natural X-equivariant inclusion $i \colon I \to C_p$, $i(a) = (0, a)$. The proof of [7, Theorem 13] with no essential change yields that γ is a homotopy inverse of Si, that is, $[\![\gamma \circ Si]\!]_X = [\![\mathrm{id}_{SI}]\!]_X$ and $[\![Si \circ \gamma]\!]_X = [\![\mathrm{id}_{SC_p}]\!]_X$. As in the non-equivariant case, this excision result and Proposition 2.15 show that $\mathrm{E}_0(X;A,B) := [\![SA\otimes\mathbb{K}, SB\otimes\mathbb{K}]\!]_X$ is a periodic exact functor in both variables A and B, that is, if $I \rightarrowtail E \twoheadrightarrow Q$ is an extension in $\mathfrak{C}^*\mathfrak{sep}(X)$ and B is a separable C*-algebra over X, then there are six-term exact sequences

$$\begin{array}{ccccc} \mathrm{E}_0(X;Q,B) & \longrightarrow & \mathrm{E}_0(X;E,B) & \longrightarrow & \mathrm{E}_0(X;I,B) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ \mathrm{E}_1(X;I,B) & \longleftarrow & \mathrm{E}_1(X;E,B) & \longleftarrow & \mathrm{E}_1(X;Q,B) \end{array}$$

and

$$\begin{array}{ccccc} \mathrm{E}_0(X;B,I) & \longrightarrow & \mathrm{E}_0(X;B,E) & \longrightarrow & \mathrm{E}_0(X;B,Q) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ \mathrm{E}_1(X;B,Q) & \longleftarrow & \mathrm{E}_1(X;B,E) & \longleftarrow & \mathrm{E}_1(X;B,I). \end{array}$$

The horizontal maps in both exact sequences are induced by the given maps $I \to E \to Q$, and the vertical maps are, up to signs, products with the class of the approximately X-equivariant asymptotic morphism associated to the extension as in Proposition 2.23.

It remains to verify universality. Again this is similar to the proof of the non-equivariant case in [2, Theorem 25.6.1], using Lemma 2.26 below as a substitute for [2, Proposition 25.6.2]. □

Lemma 2.26. *Any element of* $\mathrm{E}_0(X;A,B)$ *may be written as* $[\rho]\circ[\pi]^{-1}$ *for* X*-equivariant* **-homomorphisms* ρ *and* π*.*

Proof. Let $\varphi\colon A \to \mathrm{C_b}(T,B)$ be an approximately X-equivariant asymptotic morphism. We shall use Lemma 2.24 to show that the C*-algebra

$$E := \{(a,b) \in A \oplus \mathrm{C_b}(T,B) : \varphi(a) - b \in \mathrm{C}_0(T,B)\},$$

becomes a C*-algebra over X by

$$E(U) := E \cap \bigl(A(U) \oplus \mathrm{C_b}(T,B(U))\bigr).$$

As a consequence of the Bartle–Graves Theorem, for any two closed two-sided ideals J_1 and J_2 in a C*-algebra D, $\mathrm{C}_b\bigl(T,J_1+J_2\bigr) = \mathrm{C}_b\bigl(T,J_1\bigr) + \mathrm{C}_b\bigl(T,J_2\bigr)$. From this we see that

$$\mathrm{C}_0\bigl(T,B\bigr) \rightarrowtail \mathrm{C}_b\bigl(T,B\bigr) \twoheadrightarrow \mathrm{C}_b\bigl(T,B\bigr)/\mathrm{C}_0\bigl(T,B\bigr) = B_\infty$$

is an extension of quasi C*-algebras over X. By Lemma 2.24, its pullback under the X-equivariant *-homomorphism $\dot\varphi\colon A \to B_\infty$ is an extension of C*-algebras over X:

$$\mathrm{C}_0(T,B) \rightarrowtail E \overset{\pi}{\twoheadrightarrow} A$$

with $\pi(a,b) := \varphi(a)$. The map π becomes an isomorphism in $\mathfrak{E}(X)$ because $\mathrm{C}_0(T,B)$ is contractible over X. Let $\rho'\colon E \to \mathrm{C_b}(T,B)$ be the *-homomorphism $\rho'(a,b) = b$. When regarded as an asymptotic morphism from E to B, ρ' is homotopic to the constant asymptotic morphism $\rho(a,b) = b(0)$. We have

$[\varphi]\circ[\pi] = [\rho']$ because $\varphi\bigl(\pi(a,b)\bigr) - \rho'(a,b) \in \mathrm{C}_0(T,B)$ for all $(a,b)\in E$. Hence $[\varphi] = [\rho]\circ[\pi]^{-1}$. □

2.5. **Further properties of E-theory.** Like the category $\mathfrak{KK}(X)$, the category $\mathfrak{E}(X)$ carries the additional structure of a triangulated category (see [19, 23]). As in KK-theory, the translation automorphism is the suspension functor $A \mapsto SA := \mathrm{C}_0\bigl((0,1),A\bigr)$, and a triangle is exact if it is isomorphic to the mapping cone triangle of some X-equivariant *-homomorphism.

Theorem 2.27. *The category $\mathfrak{E}(X)$ is triangulated.*

Proof. The argument is essentially the same as in the appendix of [19]. The only axiom that requires a different treatment is the one that requires each $\varphi\in \mathrm{E}_0(X;A,B)$ to embed in an exact triangle. Here we use the factorisation $\varphi = [\rho]\circ[\pi]^{-1}$ of Lemma 2.26 with X-equivariant *-homomorphisms $\rho\colon E\to B$ and $\pi\colon E\to A$. Since $[\pi]$ is invertible in E-theory, the mapping cone triangle

$$SB \to C_\rho \to E \xrightarrow{\rho} B$$

is isomorphic to an exact triangle $SB \to C_\rho \to A \xrightarrow{[\varphi]} B$. □

The proof that E-theory is exact shows that any extension $I \rightarrowtail E \twoheadrightarrow Q$ of C*-algebras over X gives rise to an exact triangle $SQ\to I\to E\to Q$, where the map $SQ\to I$ is the Connes–Higson construction (see Proposition 2.23) and the maps $I\to E\to Q$ are the given ones. Such triangles are called *extension triangles*. This works for all extensions, so that we need no admissibility assumption as in $\mathfrak{KK}(X)$.

Since there is no admissibility hypothesis, several constructions in Kasparov theory simplify in E-theory. For instance, the colimit $\varinjlim (A_n,\varphi_n)$ of any inductive system $\varphi_n\colon A_n\to A_{n+1}$, $n\in\mathbb{N}$, in $\mathfrak{C}^*\mathfrak{sep}(X)$ is also a homotopy colimit in $\mathfrak{E}(X)$, by the argument in [19, Section 2.4].

Proposition 2.28. *If A is the inductive limit of an inductive system (A_n,φ_n) in $\mathfrak{C}^*\mathfrak{sep}(X)$, then there is a natural short exact sequence*

$$0\to \varprojlim{}^1 \mathrm{E}_1(X;A_n,B)\to \mathrm{E}(X;A,B)\to \varprojlim \mathrm{E}(X;A_n,B)\to 0.$$

Proof. The functor $A\mapsto \mathrm{E}(X;A,B)$ is seen to be countably additive as in the proof of [15, Proposition 7.1]. Then we follow the standard argument based on mapping telescopes in [2, Section 21.3.2]. □

For locally compact Hausdorff spaces, we may compare our definition of equivariant E-theory with previous ones in [24, 25]. Since we use the original Connes–Higson model of E-theory instead of iterated asymptotic algebras, this does not yet follow directly from Proposition 2.11 and [25].

Proposition 2.29. *Let X be Hausdorff and locally compact and let A and B be C*-algebras over X. Then $\mathrm{E}_*(X;A,B)$ is naturally isomorphic to $\mathcal{R}\mathrm{E}_*(X;A,B)$.*

Proof. Both theories satisfy the same universal property. Alternatively, the statement follows from Proposition 2.11 and [24]. □

Recall that for a compact Hausdorff space X, there is a canonical isomorphism

$$\mathrm{KK}_*(X; \mathrm{C}(X, A), B) \cong \mathrm{KK}_*(A, B)$$

for any C*-algebra A and any C*-algebra B over X. The same isomorphism holds in E-theory as well:

Lemma 2.30. *Let X be a compact Hausdorff space. Then*

$$\mathrm{E}_*(X; \mathrm{C}(X, A), B) \cong \mathrm{E}_*(A, B)$$

for any C*-*algebra A and any* C*-*algebra B over X.*

Proof. We may view $\mathrm{C}(X, A)$ as a C*-algebra over X using the obvious map $\operatorname{Prim} \mathrm{C}(X, A) \to X$, so that $\mathrm{C}(X, A)(U) := \mathrm{C}_0(U, A)$ for $U \in \mathbb{O}(X)$. We have to show that the functor

$$\mathfrak{C} \to \mathfrak{C}(X), \qquad A \mapsto \mathrm{C}(X, A),$$

is left adjoint to the functor

$$\mathfrak{C}(X) \to \mathfrak{C}, \qquad B \mapsto B(X).$$

First of all, both maps on objects clearly induce functors on E-theory categories because of the universal properties. For the adjointness, we have to furnish the unit and counit of adjunction and verify the two zigzag equations (see [18]). The unit is the X-equivariant *-homomorphism $\mathrm{C}(X, B) = \mathrm{C}(X) \otimes B(X) \to B$ that comes from viewing a C*-algebra B over X as a $\mathrm{C}(X)$-C*-algebra. The counit is the embedding $A \to \mathrm{C}(X, A)(X) = \mathrm{C}(X, A)$, $a \mapsto 1 \otimes a$, as constant functions. The zigzag equations are trivial to verify and hold already on the level of *-homomorphisms. □

Proposition 2.31. *Let $Y \subseteq X$ be a locally closed subset. Then there exists a natural restriction functor $\mathrm{E}_*(X; A, B) \to \mathrm{E}_*\big(Y; r_X^Y(A), r_X^Y(B)\big)$ for* C*-*algebras A and B over X.*

Proof. The restriction functor $\mathfrak{C}^*\mathfrak{sep}(X) \to \mathfrak{C}^*\mathfrak{sep}(Y)$ is defined in [21] by $r_X^Y A(Z) := A(Y \cap Z)$ for $Z \in \mathbb{O}(Y)$. It evidently maps extensions again to extensions and commutes with stabilisation. Hence it induces a functor on E-theory by the universal property. □

3. Approximation by finite spaces

Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a countable basis for the topology of X. For each $n \in \mathbb{N}$, let τ_n be the topology generated by the open subsets $U_1, \dots, U_n$. That is, the subsets U_j are a subbasis for τ_n, so that the intersections

$$U_F := \bigcap_{i\in F} U_i$$

for $F \subseteq \{1, \dots, n\}$ are a basis for τ_n.

Since the topology τ_n is finite, it is pulled back from a finite T_0-space X_n; namely, we equip X with the equivalence relation

$$x \sim_n y \iff \{1 \le j \le n : x \in U_j\} = \{1 \le j \le n : y \in U_j\}$$

for $x, y \in X$. We may view τ_n as a topology on the finite set $X/{\sim_n}$. A point in $X/{\sim_n}$ is parametrised by the set $\{1 \le j \le n : x \in U_j\}$.

Remark 3.1. The minimal open neighbourhood in X_n that contains the point corresponding to $F \subseteq \{1, \dots, n\}$ is the image in X_n of $U_F := \bigcap_{i \in F} U_i$.

In the following, we view C*-algebras over X as C*-algebras over (X, τ_n) or, equivalently, over $X_n := (X/{\sim_n}, \tau_n)$ by forgetting most of the distinguished ideals.

Theorem 3.2. *Let A and B be* C**-algebras over X, viewed as* C**-algebras over $X_n := (X/{\sim_n}, \tau_n)$ for $n \in \mathbb{N}$. Then there is a natural extension of $\mathbb{Z}/2$-graded Abelian groups*

$$\varprojlim{}^1 \mathrm{E}_{*+1}(X_n; A, B) \rightarrowtail \mathrm{E}_*(X; A, B) \twoheadrightarrow \varprojlim \mathrm{E}_*(X_n; A, B).$$

Proof. Recall the description of $[\![A, B]\!]_X$ as the zeroth homotopy group of a quasi-topological space $\mathrm{Asymp}(A, B)_X$ in Section 2.3. This also applies to E-theory: we have $\mathrm{E}_0(X; A, B) \cong \pi_0(\Gamma_X)$ with

$$\Gamma_X := \mathrm{Asymp}(\mathrm{C}_0(\mathbb{R}, A) \otimes \mathbb{K}, \mathrm{C}_0(\mathbb{R}, B) \otimes \mathbb{K})_X.$$

The same definitions for X_n yield quasi-topological spaces $\Gamma_n := \Gamma_{X_n}$ for $n \in \mathbb{N}$ with $\mathrm{E}_0(X_n; A, B) \cong \pi_0(\Gamma_n)$. The quasi-topological spaces Γ_n form a projective system because approximate X_{n+1}-equivariance implies approximate X_n-equivariance.

We claim that

$$\Gamma_X = \bigcap_{n \in \mathbb{N}} \Gamma_n, \qquad \mathrm{C}(Y, \Gamma_X) = \bigcap_{n \in \mathbb{N}} \mathrm{C}(Y, \Gamma_n)$$

for each compact Hausdorff space Y, where $\mathrm{C}(Y, \Gamma_n)$ denotes the space of quasi-continuous maps $Y \to \Gamma_n$.

The inclusion $\mathrm{C}(Y, \Gamma_X) \subseteq \bigcap \mathrm{C}(Y, \Gamma_n)$ is evident. The intersection of $\mathrm{C}(Y, \Gamma_n)$ consists of those asymptotic morphisms that satisfy (2.10) for all $U \in \mathcal{U}$. Since the set of open subsets for which (2.10) holds is closed under arbitrary unions and $\mathcal{U}$ is a basis for the topology of X, this implies (2.10) for all open subsets of X, proving the claim.

The claim above shows that Γ_X is the inverse limit of the projective system Γ_n. The homotopy groups of inverse limits of ordinary topological spaces are computed by an exact sequence of the desired form if the maps $\Gamma_{n+1} \to \Gamma_n$ have the *homotopy covering property*, see [32]. It is easy to see that this carries over to quasi-topological spaces; but in our case the maps $\Gamma_{n+1} \to \Gamma_n$ are injective and therefore cannot have the homotopy covering property. Nevertheless, we can get the desired result by following part of the argument in [32].

First we observe that [32, Theorem C], which computes the homotopy groups of homotopy equalisers remains true for quasi-topological spaces. Let $f,g\colon A \rightrightarrows B$ be two base point preserving quasi-continuous maps between pointed quasi-topological spaces. The *homotopy equaliser* of f,g is the quasi-topological space $D(f,g)$ defined so that, for all Y compact Hausdorff,

$$\mathrm{C}\bigl(Y,D(f,g)\bigr) = \{(a,b)\in \mathrm{C}(Y,A)\times \mathrm{C}(Y\times I,B) \mid \\ f\circ a = b(\text{\textvisiblespace},0),\quad g\circ a = b(\text{\textvisiblespace},1)\}.$$

Let Y be a compact Hausdorff space. Then there is an exact sequence of pointed sets

$$* \to T \to [Y,D(f,g)] \to K \to * \tag{3.3}$$

where $[Y,X]$ denotes homotopy classes of quasi-continuous maps $Y\to X$, $K := \{a\in[Y,A] \mid f_*(a) = g_*(a)\}$, and T is the orbit space for a certain canonical action of $[Y\times\mathbb{S}^1,A]_*$ on $[Y\times\mathbb{S}^1,B]_*$, where $[Y\times\mathbb{S}^1,\text{\textvisiblespace}]_*$ means that we restrict attention to quasi-continuous maps and homotopies that map $Y\times\{1\}\subseteq Y\times\mathbb{S}^1$ to the base point.

Next we apply (3.3) to the pair of maps

$$\mathrm{Id}, f\colon \prod_{n=0}^{\infty}\Gamma_n \rightrightarrows \prod_{n=0}^{\infty}\Gamma_n,$$

where f is the shift map from the definition of the projective limit. Letting $\gamma_{n+1}^n\colon \Gamma_{n+1}\to\Gamma_n$ denote the maps in the projective system, we have $f\bigl((x_n)_{n\in\mathbb{N}}\bigr) := \bigl(\gamma_{n+1}^n(x_{n+1})\bigr)_{n\in\mathbb{N}}$. The homotopy equaliser of (id,f) is quasi-homeomorphic to the quasi-topological space Γ_∞ defined by

$$\mathrm{C}(Y,\Gamma_\infty) := \Bigl\{(f_n)_{n=0}^{\infty}\in\prod_{n\in\mathbb{N}}\mathrm{C}([0,1]\times Y,\Gamma_n) \Bigm| \\ f_n(1) = \gamma_{n+1}^n\bigl(f_{n+1}(0)\bigr) \text{ for all } n\in\mathbb{N}\Bigr\}.$$

This is a familiar mapping telescope construction. The quasi-topological version of [32, Theorem C] shows that the homotopy groups of Γ_∞ are computed by an exact sequence of exactly the desired form.

To finish the proof of the theorem, it remains to show that the homotopy limit Γ_∞ and the limit Γ_X of the projective system (Γ_n) have isomorphic π_0. Lacking the homotopy covering property used in [32], we do this by hand.

Let us describe the homotopy limit Γ_∞ more concretely. The maps $\gamma_{n+1}^n\colon\Gamma_{n+1}\to\Gamma_n$ are just the inclusion maps. It is convenient to identify $\mathrm{C}(Y,\Gamma_\infty)$ with

$$\mathrm{C}(Y,\Gamma_\infty) = \Bigl\{(f_n)_{n=0}^{\infty}\in\prod_{n\in\mathbb{N}}\mathrm{C}([n,n+1]\times Y,\Gamma_n) \Bigm| \\ f_n(n+1) = f_{n+1}(n+1) \text{ for all } n\in\mathbb{N}\Bigr\}.$$

We view each f_n as an approximately X_n-equivariant asymptotic morphism from A' to $\mathrm{C}([n, n+1] \times Y, B')$, where $A' := \mathrm{C}_0(\mathbb{R}, A) \otimes \mathbb{K}$ and $B' := \mathrm{C}_0(\mathbb{R}, B) \otimes \mathbb{K}$. We may piece together these asymptotic morphisms to a single family of maps $\varphi_{s,t} \colon A' \to \mathrm{C}(Y, B')$, $s, t \in T$, where $\varphi_{s,t}|_{s \in [n,n+1]}$ is f_n. That is, $\varphi_{s,t}$ is an asymptotic morphism for fixed s, uniformly for $s \in [n, n+1]$ for all n, and hence uniformly for s in compact subsets of T; furthermore, this asymptotic morphism is (uniformly) approximately X_n-equivariant for $s \in [n, n+1]$ and hence for s in compact subsets of $[n, \infty)$.

We map Γ_X to Γ_∞ by taking a constant family of asymptotic morphisms. It remains to show that this map $\Gamma_X \to \Gamma_\infty$ induces an isomorphism on homotopy classes.

Let $(\varphi_{s,t}) \in \Gamma_\infty$ and let $A_0 \subseteq A'$ be a compactly generated dense subalgebra. The same considerations as in the construction of the product of asymptotic homomorphisms show that there is an increasing continuous function $h_0 \colon T \to T$ such that $\varphi_{t,h(t)} \colon A_0 \to B'$ extends to an X-equivariant asymptotic morphism for all continuous $h \geq h_0$. Here we use that an asymptotic morphism is X-equivariant once it satisfies (2.10) for all $U \in \mathcal{U}$. Furthermore, we may choose h_0 such that the convex homotopies $\varphi_{s,rh(t)+(1-r)t}$ from $\varphi_{s,t}$ to $\varphi_{s,h(t)}$ and $\varphi_{rt+(1-r)s,h(t)}$ from $\varphi_{s,h(t)}$ to $\varphi_{t,h(t)}$ are homotopies in Γ_∞ for $h \geq h_0$. We discuss this in detail below. Thus $(\varphi_{s,t})$ is homotopic to the constant family of asymptotic morphism $(\varphi_{t,h(t)})$ in Γ_∞, so that the map $\pi_0(\Gamma_X) \to \pi_0(\Gamma_\infty)$ is surjective. A similar argument may be applied to homotopies in Γ_∞ and shows that two elements of $\Gamma_X(A, B)$ that become homotopic in Γ_∞ are already homotopic in Γ_X.

Let us now show how to construct the function h_0 for given $(\varphi_{s,t}) \in \Gamma_\infty$. The first homotopy from $\varphi_{s,t}$ to $\varphi_{s,h(t)}$ is a homotopy of asymptotic morphisms provided $h(t) \geq t$, for obvious reasons. Thus it only remains to study the second homotopy. Let $A_0 = \{a_1, a_2, \dots\} \subseteq A'$ be a countable dense *-subalgebra. Let $\{\lambda_1, \lambda_2, \dots\}$ be a sequence dense in $\mathbb{C}$. Let $(U_i)_{i=1}^\infty$ be a basis of open sets for the topology of X. Choose a dense sequence $(a_{ij})_{j=1}^\infty$ in $A'(U_i)$ for each $i \geq 1$.

For each integer $m \geq 1$ choose $\alpha_m > 0$ such that for all $1 \leq i, j, k \leq m$ and all $t \geq \alpha_m$,

$$\sup_{s \in [0,m+1]} \|\varphi_{s,t}(a_i^* + \lambda_k a_j) - \varphi_{s,t}(a_i)^* - \lambda_k \varphi_{s,t}(a_j)\| < 1/m, \tag{3.4}$$

$$\sup_{s \in [0,m+1]} \|\varphi_{s,t}(a_i a_j) - \varphi_{s,t}(a_i)\varphi_{s,t}(a_j)\| < 1/m. \tag{3.5}$$

For each integer $n \geq 1$ we construct a sequence $(\tau_{m,n})_{m=1}^\infty$ such that

$$\sup_{s \in [n,m+1]} \|\varphi_{s,t}(a_{ij})\|_{X \setminus U_i} < 1/m, \tag{3.6}$$

for all $1 \leq i \leq n$, $1 \leq j \leq m$ and all $t \geq \tau_{m,n}$. Moreover, once the sequence $(\tau_{m,n})_{m=1}^\infty$ is constructed, we construct the next sequence $(\tau_{m,n+1})_{m=1}^\infty$ such that $\tau_{m,n+1} \geq \tau_{m,n}$ for all $m \geq 1$. Let $h_0 \colon T \to T$ be a continuous increasing function with $h_0(m) \geq \max\{\alpha_m, \tau_{m,m}\}$ and $\lim_{t \to \infty} h_0(t) = \infty$.

Let $h \geq h_0$ be a continuous function. The homotopy $\varphi_{rt+(1-r)s,h(t)}$ is defined by an element H in

$$\mathrm{C}([0,1]\times Y,\Gamma_\infty) = \Big\{(H_n)_{n=0}^\infty \in \prod_{n\in\mathbb{N}} \mathrm{C}([0,1]\times[n,n+1]\times Y,\Gamma_n) \;\Big|\; H_n(n+1) = H_{n+1}(n+1) \text{ for all } n\in\mathbb{N}\Big\},$$

where for $r\in[0,1]$, $(H_n)_r := (\varphi_{rt+(1-r)s,h(t)})_{s\in[n,n+1],t\in T}$.

In order to verify that H is an element of $\mathrm{C}([0,1]\times Y,\Gamma_\infty)$, it is sufficient to show that for all $i,j,k\geq 1$

$$\lim_{t\to\infty}\sup_{s\in[n,n+1],\ r\in[0,1]} \|\varphi_{rt+(1-r)s,h(t)}(a_i^*+\lambda_k a_j) - \varphi_{rt+(1-r)s,h(t)}(a_i)^* - \lambda_k\,\varphi_{rt+(1-r)s,h(t)}(a_j)\| = 0, \tag{3.7}$$

$$\lim_{t\to\infty}\sup_{s\in[n,n+1],\ r\in[0,1]} \|\varphi_{rt+(1-r)s,h(t)}(a_i a_j) - \varphi_{rt+(1-r)s,h(t)}(a_i)\ \varphi_{rt+(1-r)s,h(t)}(a_j)\| = 0, \tag{3.8}$$

and that for all $1\leq i\leq n$, $j\geq 1$

$$\lim_{t\to\infty}\sup_{s\in[n,n+1],\ r\in[0,1]} \|\varphi_{rt+(1-r)s,h(t)}(a_{ij})\|_{X\setminus U_i} = 0. \tag{3.9}$$

We deal first with (3.7) and (3.8). Let $i,j,k\geq 1$ and $\varepsilon>0$ be given. We claim that for any $t\geq \max\{n,i,j,k,1/\varepsilon\}+1$, the quantities whose limits are taken in (3.7) and (3.8) are smaller than ε. If m is the integer part of t, then $\max\{n,i,j,k,1/\varepsilon\} < m \leq t < m+1$. Moreover, for any $s\in[n,n+1]$ and $r\in[0,1]$, $rt+(1-r)s\in[0,m+1]$ and $h(t)\geq h_0(t)\geq h_0(m)\geq \alpha_m$. Since $1/m<\varepsilon$ our claim follows now from (3.4) and (3.5).

Let us now check (3.9). Let $1\leq i\leq n$, $j\geq 1$ and $\varepsilon>0$ be given and suppose that $t\geq\max\{n,j,1/\varepsilon\}+1$. Then there is an integer m such that $\max\{n,j,1/\varepsilon\}<m\leq t<m+1$. Observe that for any $s\in[n,n+1]$ and $r\in[0,1]$, $rt+(1-r)s\in[n,m+1]$ and $h(t)\geq h_0(t)\geq h_0(m)\geq\tau_{m,m}\geq\tau_{m,n}$. Since $1/m<\varepsilon$, it follows from (3.6) that the quantity whose limit is taken in (3.9) is smaller than ε whenever $t\geq\max\{n,j,1/\varepsilon\}+1$. □

Theorem 3.10. *Let X be a second countable topological space. An element in $\mathrm{E}_*(X;A,B)$ is invertible if and only if its image in $\mathrm{E}_*\big(A(U),B(U)\big)$ is invertible for all $U\in\mathbb{O}(X)$.*

Proof. The necessity of the condition is trivial. Next we sketch why the condition is sufficient if X is a finite space. The proof is similar to the proof of a similar statement in KK-theory in [21, Proposition 4.9]. If X is finite, any point $x\in X$ is contained in a minimal open subset U_x. For a C^*-algebra A, let i_xA be A viewed as a C^*-algebra over X concentrated at

$x \in X$, that is, $i_x(A)(U) = A$ for $x \in U$ and $i_x(A)(U) = 0$ for $x \notin U$. An argument similar to the proof of [21, Proposition 3.13] yields

$$\mathrm{E}_*(X; i_x(A), B) \cong \mathrm{E}_*\bigl(A, B(U_x)\bigr)$$

for $x \in X$, a C*-algebra A and a C*-algebra B over X. An argument similar to the proof of [21, Proposition 4.7] shows that objects of the form $i_x(A)$ generate $\mathfrak{E}(X)$, that is, no proper triangulated subcategory of $\mathfrak{E}(X)$ contains $i_x(A)$ for all A (see also Proposition 4.5 below). Hence a map in $\mathrm{E}_*(X; A, B)$ is invertible if the induced map $\mathrm{E}_*(X; i_x(D), A) \to \mathrm{E}_*(X; i_x(D), B)$ is invertible for all $x \in X$ and all D. By the isomorphism above, this is equivalent to invertibility of the induced map $\mathrm{E}_*\bigl(D, A(U_x)\bigr) \to \mathrm{E}_*\bigl(D, B(U_x)\bigr)$, which is equivalent to invertibility in $\mathrm{E}_*\bigl(A(U_x), B(U_x)\bigr)$ for all x. This finishes the argument for finite X.

If X is infinite, let $\mathcal{U}$ be a countable basis for its topology and let X_n be the resulting finite approximations to X. Theorem 3.2 shows that an arrow in $\mathfrak{E}(X)$ is invertible if and only if its image in $\mathfrak{E}(X_n)$ is invertible for all $n \in \mathbb{N}$. (The naturality of the extension in Theorem 3.2 implies that the kernel $\varprojlim^1 \dots$ is nilpotent.) This reduces the general case to the finite case already settled. □

Theorem 3.11. *Let A be a separable nuclear* C*-*algebra with Hausdorff primitive spectrum X. Suppose that each two-sided closed ideal of A is* KK-*contractible. Then*

$$A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong \mathrm{C}_0(X) \otimes \mathcal{O}_2 \otimes \mathbb{K}.$$

Proof. By a result of Fell, A is a continuous $\mathrm{C}_0(X)$-algebra with nonzero simple fibres. Set $B := \mathrm{C}_0(X) \otimes \mathcal{O}_2$. Then $0 \in \mathrm{E}(X; A, B)$ is an $\mathrm{E}(X)$-equivalence by Theorem 3.10. Theorem 5.4 yields $\mathrm{E}_*(X; C, D) \cong \mathrm{KK}_*(X; C, D)$ for $C, D \in \{A, B\}$ because A and B are nuclear and continuous $\mathrm{C}_0(X)$-algebras. Hence $0 \in \mathrm{KK}(X; A, B)$ is a $\mathrm{KK}(X)$-equivalence, and we may apply the main result of [17] to conclude that $A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathbb{K}$. □

4. The E-theoretic bootstrap category

Recall that the bootstrap class $\mathcal{B}$ in $\mathfrak{KK}$ is the localising subcategory of the triangulated category $\mathfrak{KK}$ that is generated by the object $\mathbb{C}$. Similarly, we define the E-theoretic bootstrap class $\mathcal{B}_{\mathrm{E}} \subseteq \mathfrak{E}$ as the localising subcategory of $\mathfrak{E}$ generated by $\mathbb{C}$. This is the class of all separable C*-algebras A for which $\mathrm{E}_*(A, B)$ fulfills the Universal Coefficient Theorem for all B.

For a finite topological space X, a bootstrap class $\mathcal{B}(X)$ in $\mathfrak{KK}(X)$ is defined in [21] along similar lines. Here we follow a different approach:

Definition 4.1. Let $\mathcal{B}_{\mathrm{E}}(X) \subseteq \mathfrak{E}(X)$ for a second countable topological space X be the class of all separable C*-algebras A over X with $A(U) \in \mathcal{B}_{\mathrm{E}}$ for all $U \in \mathbb{O}(X)$.

Since the functors $\mathfrak{E}(X) \to \mathfrak{E}$, $A \mapsto A(U)$, are triangulated and commute with direct sums and $\mathcal{B}_{\mathrm{E}}$ is a localising subcategory of $\mathfrak{E}$, $\mathcal{B}_{\mathrm{E}}(X)$ is a localising subcategory of $\mathfrak{E}(X)$. Furthermore, if $A \in \mathcal{B}_{\mathrm{E}}(X)$, then $A(Y) \in \mathcal{B}_{\mathrm{E}}$ for all locally closed subsets $Y \subseteq X$ because of the extension $A(U) \rightarrowtail A(V) \twoheadrightarrow A(Y)$ with $Y = V \setminus U$ and suitable open subsets U and V in X.

Proposition 4.2. *Let X be a finite topological space and let A be a separable* C^**-algebra over X. Then $A \in \mathcal{B}_{\mathrm{E}}(X)$ if and only if $A(\overline{\{x\}}) \in \mathcal{B}_{\mathrm{E}}$ for all $x \in X$.*

If A is tight, that is, the map $\mathrm{Prim}(A) \to X$ is a homeomorphism, then the C^*-algebras $A(\overline{\{x\}})$ for $x \in X$ are precisely the *prime quotients* of A.

Proof. Since $\mathcal{B}_{\mathrm{E}}$ is triangulated, the class Good of locally closed subsets Y of X with $A(Y) \in \mathcal{B}_{\mathrm{E}}$ has the following property: if $Y \subseteq Z$ and if two of $Y, Z, Z \setminus Y$ belong to Good, then so does the third. We are going to prove that a set Good of subsets must contain all locally closed subsets if it has this two-out-of-three property and contains all point closures $\overline{\{x\}}$. The proof is by induction on the length of the subspace $\overline{Y}$, that is, the length of the largest chain $x_0 \prec x_1 \prec \cdots \prec x_\ell$ in the specialisation preorder on the closure $\overline{Y}$. If $\ell = 0$, the subspace Y is a set of closed points of X, and the assertion is easy.

Let Y be a locally closed subset of X of length ℓ. Then $Y = \overline{Y} \setminus \partial Y$, so that it suffices to prove $\overline{Y}, \partial Y \in$ Good. Therefore, we may assume without loss of generality that Y is closed. Let $Z \subseteq Y$ be the set of all open points of Y. The difference $Y \setminus Z$ has length $\ell - 1$ and is therefore good by induction assumption. If $x \in Z$, then the closure $\overline{\{x\}}$ is good by assumption, and $\overline{\{x\}} \setminus \{x\}$ is good because its length is at most $\ell - 1$. Hence $\{x\}$ is good for all $x \in Z$. Since Z is discrete, it follows that Z is good. Hence so is Y. ☐

Similarly, if X is finite, then $A \in \mathcal{B}_{\mathrm{E}}(X)$ if and only if $A(U_x) \in \mathcal{B}_{\mathrm{E}}$ for all $x \in X$, where U_x denotes the minimal open subset of X containing x.

Proposition 4.2 remains true for some infinite spaces X as well. For instance, let X be a finite-dimensional, compact, metrisable Hausdorff space. It is proved in [8] that a continuous, separable and nuclear $\mathrm{C}(X)$-algebra A lies in the bootstrap class $\mathcal{B}$ if all its fibres $A(x) = A(\overline{\{x\}})$ are in $\mathcal{B}$. Applying this to all closed subsets of X, we get $A \in \mathcal{B}_{\mathrm{E}}(X)$ under the same assumptions.

For finite spaces X, we may also describe the bootstrap class in terms of generators. For $x \in X$ and a C^*-algebra A, let $i_x A$ be A viewed as a C^*-algebra over X concentrated over $x \in X$, that is, $i_x(A)(U) = A$ for $x \in U$ and $i_x(A)(U) = 0$ for $x \notin U$. This C^*-algebra over X satisfies

$$\mathrm{KK}_*(X; i_x(A), B) \cong \mathrm{KK}_*\bigl(A, B(U_x)\bigr)$$

for all B by [21, Proposition 3.13]. The same argument with E-theory instead of KK-theory yields

$$\mathrm{E}_*(X; i_x(A), B) \cong \mathrm{E}_*\bigl(A, B(U_x)\bigr) \tag{4.3}$$

for $x \in X$, a C*-algebra A and a C*-algebra B over X. Here U_x denotes the minimal open neighbourhood of x, which exists because X is finite. Furthermore,

$$\mathrm{E}_*\bigl(X; A, i_x(B)\bigr) \cong \mathrm{E}_*\bigl(A\bigl(\overline{\{x\}}\bigr), B\bigr) \tag{4.4}$$

as in [21], even for infinite X, but we will not use this in the following.

Proposition 4.5. *Let X be a finite topological space. Then $\mathcal{B}_{\mathrm{E}}(X)$ is the localising subcategory of $\mathfrak{E}(X)$ that is generated by $i_x\mathbb{C}$ for all $x \in X$. The whole category $\mathfrak{E}(X)$ is generated by* C*-*algebras of the form i_xA for separable* C*-*algebras A and $x \in X$.*

Proof. It is clear that $i_x\mathbb{C} \in \mathcal{B}_{\mathrm{E}}(X)$ and that $\mathcal{B}_{\mathrm{E}}(X)$ is localising, so that it contains the localising subcategory generated by $i_x\mathbb{C}$ for $x \in X$. The same proof as for [21, Proposition 4.7] shows that a C*-algebra A over X belongs to the localising subcategory of $\mathfrak{E}(X)$ generated by $i_x\bigl(A(x)\bigr)$ for all $x \in X$. The admissibility assumptions in [21] are only needed for KK, they become automatic in E-theory. In particular, this shows that $\mathfrak{E}(X)$ is generated by C*-algebras of the form i_xA, while $\mathcal{B}_{\mathrm{E}}(X)$ is generated by i_xA with $A \in \mathcal{B}_{\mathrm{E}}$. Since $\mathcal{B}_{\mathrm{E}}$ is generated by $\mathbb{C}$, we may replace the set of i_xA with $A \in \mathcal{B}_{\mathrm{E}}(X)$ by $i_x\mathbb{C}$ here. □

Theorem 4.6. *Let X be a second countable topological space and let A and B belong to $\mathcal{B}_{\mathrm{E}}(X)$. An element in $\mathrm{E}_*(X; A, B)$ is invertible if and only if it induces invertible maps $\mathrm{K}_*\bigl(A(U)\bigr) \to \mathrm{K}_*\bigl(B(U)\bigr)$ for all $U \in \mathbb{O}(X)$.*

Proof. It is well-known that an element in $\mathrm{KK}_*(A, B)$ that induces an isomorphism on K-theory is invertible in KK provided A and B belong to the bootstrap category. The same argument applies to E-theory. Finally, apply Theorem 3.10 and the definition of $\mathcal{B}_{\mathrm{E}}(X)$. □

5. Comparing KK- and E-theory

In the definition of E-theory, we may restrict attention to asymptotic morphisms φ for which the maps φ_t are all completely positive contractions. It is shown by Houghton-Larsen and Thomsen [16] that the resulting variant of E-theory agrees with Kasparov's KK. A corresponding result for equivariant KK- and E-theory is established by Thomsen in [31]. It is a routine exercise to show that the same works in our situation.

Definition 5.1. Let $[\![A, B]\!]_X^{\mathrm{cp}}$ denote the space of homotopy classes of X-equivariant, completely positive, linear, contractive asymptotic morphisms φ from A to B, where homotopy is defined using X-equivariant, completely positive, linear, contractive asymptotic morphisms $A \to \mathrm{C_b}\bigl(T, \mathrm{C}([0,1], B)\bigr)$. X-equivariance means $\varphi\bigl(A(U)\bigr) \subseteq \mathrm{C_b}\bigl(T, B(U)\bigr)$ for all $U \in \mathbb{O}(X)$.

The map $\varphi\colon A \to \mathrm{C_b}(T, B)$ is an X-equivariant, completely positive, linear contraction if and only if all the individual maps $\varphi_t\colon A \to B$ are X-equivariant, completely positive, linear contractions.

Theorem 5.2. *There is a natural isomorphism*

$$\mathrm{KK}_0(X; A, B) \cong [\![\mathrm{C}_0(\mathbb{R}, A) \otimes \mathbb{K}, \mathrm{C}_0(\mathbb{R}, B) \otimes \mathbb{K}]\!]^{\mathrm{cp}}_X.$$

Proof. Copy the proofs of the corresponding assertions for non-equivariant Kasparov theory and equivariant Kasparov theory for group actions in [16, 31]. The main point is to go through the proof of the universal property of E-theory and to check that the variant $[\![\mathrm{C}_0(\mathbb{R}, A) \otimes \mathbb{K}, \mathrm{C}_0(\mathbb{R}, B) \otimes \mathbb{K}]\!]^{\mathrm{cp}}_X$ satisfies an analogous universal property, but with exactness only for extensions of C^*-algebras over X with a completely positive contractive section over X. Since $\mathfrak{KK}(X)$ satisfies the same universal property, the two theories must be naturally isomorphic.

Our case is somewhat closer to case of non-equivariant KK in [16] because some issues like Hilbert space representations of groups and equivariance of approximate units do not occur. □

Corollary 5.3. *Let X be a second countable topological space and let A be a C^*-algebra over X which is* $\mathrm{KK}(X)$*-equivalent to a C^*-algebra over X, A' such that any extension $I \rightarrowtail E \twoheadrightarrow \mathrm{C}_0(\mathbb{R}, A') \otimes \mathbb{K}$ of C^*-algebras over X has an X-equivariant completely positive contractive linear section. Then the canonical map $\mathrm{KK}_0(X; A, B) \to \mathrm{E}_0(X; A, B)$ is an isomorphism for any C^*-algebra B over X.*

Proof. We may assume that $A = A'$. Any asymptotic morphism is equivalent to one with $\varphi_0 = 0$ – multiply pointwise with a suitable scalar-valued function. Hence it makes no difference whether we assume this for the definition of $[\![A, B]\!]_X$ and $[\![A, B]\!]^{\mathrm{cp}}_X$. An asymptotic morphism from A to B with $\varphi_0 = 0$ generates an extension $\mathrm{C}_0(T, B) \rightarrowtail E \twoheadrightarrow A$ with $E = \varphi(A) + \mathrm{C}_0(T, B) \subseteq \mathrm{C}_\mathrm{b}(T, B)$, and two asymptotic morphisms generate the same extension if and only if they are equivalent. The asymptotic morphism itself is a section for this extension. The assumption of the corollary therefore implies $[\![\mathrm{C}_0(\mathbb{R}, A) \otimes \mathbb{K}, D]\!]^{\mathrm{cp}}_X = [\![\mathrm{C}_0(\mathbb{R}, A) \otimes \mathbb{K}, D]\!]_X$ for all D. □

Theorem 5.4. *Let X be a second countable locally compact Hausdorff space, let A be a nuclear and continuous C^*-algebra over X, and let B be any separable C^*-algebra over X. Then the canonical map $\mathrm{KK}_0(X; A, B) \to \mathrm{E}_0(X; A, B)$ is an isomorphism.*

Proof. The result follows from [24, Theorem 4.7]. Alternatively, we may argue that A is $\mathrm{C}_0(X)$-nuclear by [1, Theorem 7.2], so that it satisfies the assumptions of Corollary 5.3. □

Theorem 5.5. *Let X be a finite topological space and let (A, ψ_A) and (B, ψ_B) be C^*-algebras over X. The canonical map*

$$\mathrm{KK}_*(X; A, B) \to \mathrm{E}_*(X; A, B)$$

is an isomorphism if A belongs to the bootstrap class in $\mathfrak{KK}(X)$ defined in [21]. In particular, this applies if the C^*-algebra $A(X)$ is nuclear.

Proof. If A belongs to the bootstrap class of [21], then we may compute $\mathrm{KK}_*(X; A, B)$ by a spectral sequence whose first page only involves the K-theory groups of $A(U)$ and $B(U)$ for minimal open subsets U in X. The arguments in [21] only use the universal property of $\mathfrak{KK}(X)$ and work equally well for $\mathfrak{E}(X)$, with some simplifications because we do not have to worry about equivariant completely positive sections of various extensions. Thus there is an analogous spectral sequence computing $\mathrm{E}_*(X; A, B)$, and it has the same first page as the spectral sequence computing $\mathrm{KK}_*(X; A, B)$. The canonical map $\mathfrak{KK}(X) \to \mathfrak{E}(X)$ provides a morphism between these spectral sequences, which is an isomorphism on the first page and thus on all later pages. Hence the two spectral sequences are isomorphic, so that $\mathrm{KK}_*(X; A, B) \cong \mathrm{E}_*(X; A, B)$. □

Example 5.6. We exhibit an extension of nuclear C*-algebras over $[0, 1]$ which is not excisive for $\mathrm{KK}([0, 1]; \text{␣}, B)$. Consider the extension of C*-algebras over $[0, 1]$

$$0 \to \mathrm{C}_0[0, 1) \to \mathrm{C}[0, 1] \xrightarrow{\pi} \mathbb{C} \to 0,$$

where $\pi(f) = f(1)$. We claim that the mapping cone C_π is not $\mathrm{KK}([0, 1])$-equivalent to $\ker(\pi) = \mathrm{C}_0[0, 1)$ and that

$$\mathrm{KK}([0, 1]; S\mathbb{C}, \mathrm{C}_0[0, 1)) \neq \mathrm{E}([0, 1]; S\mathbb{C}, \mathrm{C}_0[0, 1)).$$

Here $S\mathbb{C}$ is regarded as a $\mathrm{C}[0, 1]$-algebra via the multiplication $f \cdot g = f(1)g$ for $f \in \mathrm{C}[0, 1]$ and $g \in S\mathbb{C}$. Let us address first the second part of the claim. It is convenient to work with asymptotic morphisms parametrised by $t \in [0, 1)$. For each such t consider the map $\nu_t \colon [0, 1] \to [0, 1]$,

$$\nu_t(s) = \begin{cases} 0 & \text{if } 0 \leq s < t, \\ \frac{s-t}{1-t} & \text{if } t \leq s \leq 1. \end{cases}$$

Define a continuous family of *-homomorphisms $\varphi_t \colon S\mathbb{C} \to \mathrm{C}_0[0, 1)$, $t \in [0, 1)$ by $\varphi_t(\exp(2\pi\mathrm{i}s) - 1) := \exp(2\pi\mathrm{i}\nu_t(s)) - 1$. It is easily verified that the asymptotic homomorphism (φ_t) is asymptotically $[0, 1]$-equivariant since $\exp(2\pi\mathrm{i}\nu_t(s)) - 1$ is suported on $[t, 1)$. Set $A = S\mathbb{C}$ and $B = \mathrm{C}_0[0, 1)$. We observe that the class of (φ_t) in $\mathrm{E}([0, 1]; A, B)$ is non-zero since its image in $\mathrm{Hom}\big(\mathrm{K}_1(A(0, 1)), \mathrm{K}_1(B(0, 1))\big) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ is equal to $\mathrm{id}_\mathbb{Z}$. On the other hand, $\mathrm{KK}_*([0, 1]; A, B) = \mathrm{KK}_*\big(S\mathbb{C}, \bigcap_n B((1 - 1/n, 1])\big) = \mathrm{KK}_*(S\mathbb{C}, \{0\}) = 0$, by [21, Proposition 3.13].

Let us verify now the first part of the claim. The Puppe sequence for $\mathrm{KK}([0, 1]; \text{␣}, B)$ associated to the map π yields $\mathrm{KK}([0, 1], C_\pi, B) = 0$ since $\mathrm{KK}_*([0, 1], \mathrm{C}[0, 1], B) = \mathrm{KK}_*(\mathbb{C}, B) = 0$ and $\mathrm{KK}_*([0, 1]; \mathbb{C}, B) = 0$ as argued above. At the same time, $\mathrm{KK}_*([0, 1]; B, B) \neq 0$ since the natural map $\mathrm{KK}_*([0, 1]; B, B) \to \mathrm{Hom}(\mathrm{K}_1(B(0, 1)), \mathrm{K}_1(B(0, 1)) \cong \mathbb{Z}$ sends $[\mathrm{id}_B]$ to 1.

6. A universal coefficient theorem for C*-algebras over totally disconnected spaces

In this section, we study C*-algebras over a totally disconnected compact metrisable space X. Our goal is to construct a Universal Coefficient Theorem that computes $\mathrm{E}_*(X;A,B)$ for $A,B\in\mathcal{B}_{\mathrm{E}}(X)$. For this purpose, we use filtrated K-theory *with coefficients* and obtain a Universal Coefficient exact sequence that generalises the Multicoefficient Theorem of [11]. In order to explain the key role of filtrated K-theory with coefficients, we also revisit an example from [10] showing that the spectral sequence generated by filtrated K-theory does not degenerate to an exact sequence.

In this section, all C*-algebras are assumed separable and all groups countable.

Let $\mathcal{P}\subseteq\mathbb{N}$ be the set consisting of 0 and all prime powers. The relevance of the set $\mathcal{P}$ in the Universal Multicoefficient Theorem is that the groups $\mathbb{Z}/p$ for $p\in\mathcal{P}$ are exactly the indecomposable Abelian groups.

For $p\in\mathcal{P}$ let $\mathbb{I}_p$ be the mapping cone of the unital *-homomorphism $\mathbb{C}\to\mathbb{M}_p(\mathbb{C})$. For $p=0$, we let $\mathbb{I}_0:=\mathbb{C}$. It is convenient to denote $\mathbb{I}_p$ by $\mathbb{I}_p^0$ and its suspension $S\mathbb{I}_p$ by $\mathbb{I}_p^1$. Then for a C*-algebra A:

$$\mathrm{K}_i(A;\mathbb{Z}/p):=\mathrm{KK}_i(\mathbb{I}_p,A)\cong\mathrm{KK}(\mathbb{I}_p^i,A),\quad i=0,1.$$

Let us set $\mathbb{I}:=\bigoplus_{p\in\mathcal{P}}\mathbb{I}_p$ and consider the ring $\mathrm{KK}_*(\mathbb{I},\mathbb{I})$ with multiplication given by the Kasparov product. The non-unital subring

$$\Lambda=\bigoplus_{p,q\in\mathcal{P}}\mathrm{KK}_*(\mathbb{I}_p,\mathbb{I}_q)$$

of $\mathrm{KK}_*(\mathbb{I},\mathbb{I})$ is called the ring of *Böckstein operations.* It consists of matrices indexed by $\mathcal{P}\times\mathcal{P}$ with only finitely many non-zero entries $\lambda_{pq}\in\mathrm{KK}_*(\mathbb{I}_p,\mathbb{I}_q)$. The Kasparov product

$$\mathrm{KK}_*(\mathbb{I}_p,\mathbb{I}_q)\times\mathrm{KK}_*(\mathbb{I}_q,A)\to\mathrm{KK}_*(\mathbb{I}_p,A)$$

induces a natural Λ-module structure on the $\mathbb{Z}/2\times\mathcal{P}$-graded group

$$\underline{\mathrm{K}}(A)=\bigoplus_{p\in\mathcal{P}}\mathrm{K}_*(A;\mathbb{Z}/p).$$

The KK-class x_p^i of $\mathrm{id}_{\mathbb{I}_p^i}$ generates the group $\mathrm{K}_i(\mathbb{I}_p^i,\mathbb{Z}/p)\cong\mathrm{KK}(\mathbb{I}_p^i,\mathbb{I}_p^i)$. We shall work with $\mathbb{Z}/2\times\mathcal{P}$-graded Λ-modules $M=(M_p^i)$ such that for $\lambda\in\mathrm{KK}_j(\mathbb{I}_q,\mathbb{I}_k)$ and $m\in M_p^i$, $\lambda m\in M_q^{j+i}$ if $k=p$ and $\lambda m=0$ if $k\neq p$. We also ask that x_p^i acts as the identity automorphism on M_p^i. In particular, this implies that $pM_p^i=0$. These assumptions are modelled on the case $M=\underline{\mathrm{K}}(A)$ where $M_p^i=\mathrm{KK}_*(\mathbb{I}_p^i,A)$.

Definition 6.1. A Λ-module isomorphic to $\underline{\mathrm{K}}(\mathbb{I}_p^i)$ for some $(i,p)\in\mathbb{Z}/2\times\mathcal{P}$ is called *basic.*

Lemma 6.2. *For all $(i,p) \in \mathbb{Z}/2 \times \mathcal{P}$, $\underline{\mathrm{K}}(\mathbb{I}_p^i) = \Lambda \cdot x_p^i$. The basic Λ-modules are projective in the category of $\mathbb{Z}/2 \times \mathcal{P}$-graded Λ-modules.*

Proof. The first part follows because $\mathrm{KK}_*(\mathbb{I}_p, \mathbb{I}_p) \cong \mathrm{KK}_*(\mathbb{I}_p^i, \mathbb{I}_p^i)$ and $x_p^i = [\mathrm{id}_{\mathbb{I}_p^i}]$ is idempotent. For the second part we observe that if $\lambda x_p^i = 0$ for some $\lambda \in \mathrm{KK}_*(\mathbb{I}_q, \mathbb{I}_k)$ then either $k \neq p$ or $\lambda = 0$. This shows that if $\pi \colon B \to C$ is a surjective morphism of Λ-modules, then any morphism $\varphi \colon \Lambda x_p^i \to C$ lifts to a morphism $\Phi \colon \Lambda x_p^i \to B$ defined by $\Phi(\lambda x_p^i) = \lambda b_p^i$, $\lambda \in \Lambda$, where b_p^i is some lifting of $\varphi(x_p^i)$. □

We give a very concise proof of the following result from [11].

Proposition 6.3. *Let A and B be separable* C*-*algebras and suppose that A is in the bootstrap class $\mathcal{B}$ with $\mathrm{K}_*(A)$ finitely generated. Then $\mathrm{KK}(A,B) \cong \mathrm{Hom}_\Lambda\big(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)\big)$.*

Proof. Both sides are additive in the first variable. Thus by the UCT we may assume that $A = \mathbb{I}_p^i$ for some $(i,p) \in \mathbb{Z}/2 \times \mathcal{P}$. Let us observe that any element $h \in \mathrm{Hom}_\Lambda\big(\Lambda x_p^i, \underline{\mathrm{K}}(B)\big)$ is completely determined by $h(x_p^i) \in \mathrm{K}_i(B;\mathbb{Z}/p) \cong \mathrm{KK}(\mathbb{I}_p^i, B)$. Moreover, the image of $h(x_p^i)$ under the map $\mathrm{KK}(\mathbb{I}_p^i, B) \to \mathrm{Hom}_\Lambda\big(\underline{\mathrm{K}}(\mathbb{I}_p^i), \underline{\mathrm{K}}(B)\big)$ is precisely h. Indeed, the Kasparov product $\mathrm{KK}(\mathbb{I}_p^i, \mathbb{I}_p^i) \times \mathrm{KK}(\mathbb{I}_p^i, B) \to \mathrm{KK}(\mathbb{I}_p^i, B)$ gives $[\mathrm{id}_{\mathbb{I}_p^i}] \times \alpha = \alpha$. □

If A is a separable C*-algebra over a zero-dimensional space X, then $\underline{\mathrm{K}}(A)$ has a natural structure of module over the ring $\mathrm{C}(X,\Lambda)$ of locally constant functions from X to Λ. This is easily seen by observing that $A \cong \bigoplus_{k=1}^n A(U_k)$ for any clopen partition $(U_k)_{k=1}^n$ of X. A C*-algebra over X is called *elementary* if it is isomorphic to $\bigoplus_{k=1}^n \mathrm{C}(U_k, A_k)$, where $(U_k)_{k=1}^n$ is a clopen partition of X, each A_k is a separable C*-algebra in the bootstrap class, and $\mathrm{K}_*(A_k)$ is finitely generated. If A is elementary, then the $\mathrm{C}(X,\Lambda)$-module $\underline{\mathrm{K}}(A)$ is isomorphic to $\bigoplus_{k=1}^n \mathrm{C}(U_k, \underline{\mathrm{K}}(A_k))$. Since $\mathrm{K}_*(A_k)$ is finitely generated, it follows from the UCT that A_k is KK-equivalent to a finite direct sum of $\mathbb{I}_p^i$s, so that $\underline{\mathrm{K}}(A_k)$ is Λ-projective by Lemma 6.2. It follows easily that the $\mathrm{C}(X,\Lambda)$-module $\underline{\mathrm{K}}(A_k)$ is projective.

Lemma 6.4. *Suppose that M is isomorphic to the inductive limit of an inductive system (M_j) of projective $\mathrm{C}(X,\Lambda)$-modules. Then for any $\mathrm{C}(X,\Lambda)$-module N there is a natural isomorphism*

$$\varprojlim{}^1 \mathrm{Hom}_{\mathrm{C}(X,\Lambda)}(M_j, N) \cong \mathrm{Ext}_{\mathrm{C}(X,\Lambda)}(M,N).$$

Proof. Set $R = \mathrm{C}(X,\Lambda)$. The extension

$$0 \to \bigoplus_{j\in\mathbb{N}} M_j \xrightarrow{\mathrm{Id}-S} \bigoplus_{j\in\mathbb{N}} M_j \to M \to 0,$$

where S is the natural shift map, is a projective resolution of M. Since $\bigoplus_{j\in\mathbb{N}} M_j$ is projective, we have an exact sequence

$$\operatorname{Hom}_R\Bigl(\bigoplus_{j\in\mathbb{N}} M_j, N\Bigr) \xrightarrow{(\mathrm{id}-S)^*} \operatorname{Hom}_R\Bigl(\bigoplus_{j\in\mathbb{N}} M_j, N\Bigr) \to \operatorname{Ext}_R(M,N) \to 0,$$

where the first map identifies with the first map of the exact sequence

$$\prod_{j\in\mathbb{N}} \operatorname{Hom}_R(M_j,N) \to \prod_{j\in\mathbb{N}} \operatorname{Hom}_R(M_j,N) \to \varprojlim{}^1 \operatorname{Hom}_R(M_j,N) \to 0$$

that defines $\varprojlim^1$. Thus the two maps have isomorphic cokernels. □

Proposition 6.5. *Let A be a separable nuclear continuous* C^**-algebra over a totally disconnected compact metrisable space X. Suppose that each fibre of A belongs to the bootstrap class $\mathcal{B}$. Then A is* $\mathrm{KK}(X)$*-equivalent to the inductive limit of an inductive system of elementary* $\mathrm{C}(X)$*-algebras.*

Proof. [8, Theorem 2.5] shows that A is $\mathrm{KK}(X)$-equivalent to a unital continuous $\mathrm{C}(X)$-algebra $A^\sharp$ whose fibres are Kirchberg algebras. Thus we may assume that $A = A^\sharp$. By [12, Theorem 3.6], there is a sequence $(A_n)_{n=1}^\infty$ of elementary unital $\mathrm{C}(X)$-subalgebras of A which is exhausting A in the sense that for every finite subset F of A, $\lim_{n\to\infty} \operatorname{dist}(F, A_n) = 0$. Since A_n is locally trivial and its fibres are weakly semiprojective ([9, Section 3]) each inclusion map $\gamma_n\colon A_n \hookrightarrow A$ can be perturbed to some $\mathrm{C}(X)$-linear unital *-monomorphism $\gamma_{n,n+k}\colon A_n \to A_{n+k}$ with $\|\gamma_n(a) - \gamma_{n,n+k}(a)\| < 1/2^n$ for a in a prescribed finite subset of A_n. It follows that after passing to a subsequence of (A_n) we can represent A as the inductive limit of a system $(A_{n_k}, \gamma_{n_k,n_{k+1}})$ of elementary $\mathrm{C}(X)$-algebras. □

Lemma 6.6. *Let A and B be separable* $\mathrm{C}(X)$*-algebras over a totally disconnected compact metrisable space X and suppose that A is elementary. Then* $\mathrm{KK}(X;A,B) \cong \operatorname{Hom}_{\mathrm{C}(X,\Lambda)}\bigl(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)\bigr)$.

Proof. Write $A = \bigoplus_{i=1}^k \mathrm{C}(U_i, D_i)$ where $U_1,\dots,U_k$ is a clopen partition of X and each D_i is in the bootstrap class with $\mathrm{K}_*(D_i)$ finitely generated. We have $\mathrm{KK}(X;A,B) \cong \bigoplus_{i=1}^k \mathrm{KK}(U_i; A(U_i), B(U_i))$ and

$$\operatorname{Hom}_{\mathrm{C}(X,\Lambda)}\bigl(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)\bigr) \cong \bigoplus_{i=1}^k \operatorname{Hom}_{\mathrm{C}(U_i,\Lambda)}\bigl(\underline{\mathrm{K}}(A(U_i)), \underline{\mathrm{K}}(B(U_i))\bigr).$$

Thus we may assume that $A = \mathrm{C}(X,D)$. In this case, the assertion follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{KK}(X;\mathrm{C}(X,D),B) & \longrightarrow & \operatorname{Hom}_{\mathrm{C}(X,\Lambda)}\bigl(\underline{\mathrm{K}}(\mathrm{C}(X,D)), \underline{\mathrm{K}}(B)\bigr) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathrm{KK}(D,B) & \xrightarrow{\cong} & \operatorname{Hom}_\Lambda(\underline{\mathrm{K}}(D), \underline{\mathrm{K}}(B)) \end{array}$$

The bottom horizontal map of the diagram is bijective by Proposition 6.3, the left vertical may by Lemma 2.30. The right vertical map is bijective because

$$\underline{\mathrm{K}}\big(\mathrm{C}(X,D)\big) \cong \mathrm{C}\big(X,\underline{\mathrm{K}}(D)\big) \cong \mathrm{C}(X,\mathbb{Z}) \otimes \underline{\mathrm{K}}(D) \cong \mathrm{C}(X,\mathbb{Z}) \otimes \Lambda \otimes_\Lambda \underline{\mathrm{K}}(D) \cong \mathrm{C}(X,\Lambda) \otimes_\Lambda \underline{\mathrm{K}}(D)$$

and

$$\mathrm{Hom}_{\mathrm{C}(X,\Lambda)}\big(\mathrm{C}(X,\Lambda) \otimes_\Lambda \underline{\mathrm{K}}(D), \underline{\mathrm{K}}(B)\big) \cong \mathrm{Hom}_\Lambda\big(\underline{\mathrm{K}}(D), \underline{\mathrm{K}}(B)\big). \qquad \square$$

Lemma 6.7. *Any separable* $\mathrm{C}(X)$*-algebra over a totally disconnected compact metrisable space* X *is isomorphic to the inductive limit of a sequence of locally trivial separable* $\mathrm{C}(X)$*-algebras.*

Proof. Let A be a separable $\mathrm{C}(X)$-algebra over X. If $\mathcal{U}$ is a finite clopen cover of X we denote by $A_\mathcal{U}$ the locally trivial continuous $\mathrm{C}(X)$-algebra $\bigoplus_{U\in\mathcal{U}} \mathrm{C}(U) \otimes A(U)$. For each $x \in U$ the fibre $A_\mathcal{U}(x)$ is $A(U)$. There is a natural morphism of $\mathrm{C}(X)$-algebras $\alpha_\mathcal{U}\colon A_\mathcal{U} \to A$ which maps $(f_U \otimes a_U)_{U\in\mathcal{U}}$ to $\sum_{U\in\mathcal{U}} f_U a_U$.

If V is a closed subset of U we have a natural restriction homomorphism $\mathrm{C}(U) \otimes A(U) \to \mathrm{C}(V) \otimes A(V)$, which maps $f \otimes a$ to $f|_V \otimes \pi_V(a)$. Therefore, if $\mathcal{V}$ is a finite clopen cover of X which refines $\mathcal{U}$, there is a natural morphism of $\mathrm{C}(X)$-algebras $\alpha_\mathcal{U}^\mathcal{V}\colon A_\mathcal{U} \to A_\mathcal{V}$ such that $\alpha_\mathcal{V} \circ \alpha_\mathcal{U}^\mathcal{V} = \alpha_\mathcal{U}$.

Let $(\mathcal{U}_n)_n$ be an infinite sequence of finite clopen covers of X, with $\mathcal{U}_{n+1}$ refining $\mathcal{U}_n$, and such that $\mathrm{diam}(\mathcal{U}_n) \to 0$ with respect to some metric inducing the topology of X. Set $A_n = A_{\mathcal{U}_n}$, $\alpha_n = \alpha_{\mathcal{U}_n}$ and $\alpha_n^m = \alpha_{\mathcal{U}_n}^{\mathcal{U}_m}$. We claim that the natural morphism $\varinjlim(A_n, \alpha_n^m) \to A$ is an isomorphism. This morphism is surjective since each α_n is surjective. To prove its injectivity, it suffices to show that if $F \in A_n$ satisfies $\alpha_n(F) = 0$, then for any $\varepsilon > 0$ there is $m > n$ such that $\|\alpha_n^m(F)\| \le \varepsilon$. By localising at each element of $\mathcal{U}_n$, we may assume that $A_n = \mathrm{C}(X) \otimes A(X)$ and regard F as a continuous function $F\colon X \to A(X)$. Since F is continuous, each $x \in X$ has a neighbourhood V_x such that $\|F(x) - F(y)\| < \varepsilon/2$ for all $y \in V_x$. Since $A(X)$ is a $\mathrm{C}(X)$-algebra, for each $a \in A(X)$, the map $x \mapsto \|\pi_x(a)\|$ is upper semi-continuous. The assumption $\alpha_n(F) = 0$ implies that $\pi_x(F(x)) = 0$ for all $x \in X$. Thus, after shrinking each V_x if necessary, we may arrange that $\|\pi_z(F(x))\| < \varepsilon/2$ for all $z \in V_x$. It follows that for any $y, z \in V_x$,

$$\|\pi_z(F(y))\| \le \|\pi_z(F(y) - F(x))\| + \|\pi_z(F(x))\| < \varepsilon.$$

Extract now a finite cover $V_{x_1}, \dots, V_{x_r}$ of X. Since $\mathrm{diam}(\mathcal{U}_m) \to 0$ there is $m > n$ such that each element of $\mathcal{U}_m$ is contained in some V_{x_i}. It follows that $\|\alpha_n^m(F)\| \le \varepsilon$. $\square$

Proposition 6.8. *Any separable* $\mathrm{C}(X)$*-algebra over a totally disconnected compact metrisable space* X *is* $\mathrm{E}(X)$*-equivalent to a continuous separable* $\mathrm{C}(X)$*-algebra.*

Proof. For a given $\mathrm{C}(X)$-algebra A, let (A_n, α_n^m) be the corresponding inductive system constructed as in the proof of the previous lemma. Let

$$T(A_n, \alpha_n^m) = \left\{ (f_n) \in \bigoplus_{n\in\mathbb{N}} \mathrm{C}([n, n+1], A_n) : f_{n+1}(n+1) = \alpha_n^{n+1}(f_n(n+1)) \right\}$$

be the associated mapping telescope. Since the mapping telescope construction is functorial, there is a natural $\mathrm{C}(X)$-linear *-homomorphism

$$\alpha\colon T(A_n, \alpha_m^n) \to T(A, \mathrm{id}_A) \cong SA.$$

Arguing as in the paragraphs following the proof of [19, Proposition 2.6], it follows that α is an $\mathrm{E}(X)$-equivalence. Indeed, let $\tilde{T}(A_m, \alpha_m^n)$ be the variant of $T(A_m, \alpha_m^n)$ where we require $\lim_{t\to\infty} \alpha_m(f_m(t))$ to exist in A instead of $\lim f_m(t) = 0$. The algebra $\tilde{T}(A_m, \alpha_m^n)$ is contractible over X in a natural way. There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(A_m, \alpha_m^n) & \longrightarrow & \tilde{T}(A_m, \alpha_m^n) & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \| & & \\
0 & \longrightarrow & T(A, \mathrm{id}) & \longrightarrow & \tilde{T}(A, \mathrm{id}) & \longrightarrow & A & \longrightarrow & 0,
\end{array}$$

whose rows are short exact sequences. Since the algebras in the middle are contractible, it follows that α induces an $\mathrm{E}(X)$-equivalence. We conclude by observing that $T(A_m, \alpha_m^n)$ is a continuous $\mathrm{C}(X)$-algebra since it is a $\mathrm{C}(X)$-subalgebra of a direct sum of continuous $\mathrm{C}(X)$-algebras. □

Proposition 6.9. *A separable and nuclear* $\mathrm{C}(X)$*-algebra* A *over a totally disconnected compact metrisable space* X *belongs to the bootstrap class* $\mathcal{B}_{\mathrm{E}}(X)$ *if and only if all its fibres are in the bootstrap call* $\mathcal{B}_{\mathrm{E}}$*.*

Proof. By Propositions 6.8 and 2.29, we may assume that A is a continuous $\mathrm{C}(X)$-algebra. By a result of [8] a separable nuclear continuous $\mathrm{C}(X)$-algebra over a finite-dimensional compact metrisable space X belongs to $\mathcal{B}$ if and only if all its fibres belong to $\mathcal{B}$. This concludes the proof, since a nuclear C^*-algebra belongs to $\mathcal{B}$ if and only if it belongs to $\mathcal{B}_{\mathrm{E}}$. □

Proposition 6.10. *Let* A *be a separable* $\mathrm{C}(X)$*-algebra over a totally disconnected compact metrisable space* X*. If* $A(U)$ *is* E*-equivalent to a separable nuclear C*-algebra for each clopen set* $U \subset X$*, then* A *is* $\mathrm{E}(X)$*-equivalent to a separable, continuous, nuclear* $\mathrm{C}(X)$*-algebra.*

Proof. The proposition applies for instance when A belongs to the bootstrap class $\mathcal{B}_{\mathrm{E}}(X)$. It was shown in [8, Lemma 2.2] that A is $\mathrm{KK}(X)$-equivalent to a $\mathrm{C}(X)$-algebra A' such that $A' \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong A'$ and that A' contains a full projection. Thus we may assume that A itself has these properties. Let (A_n, α_n^m) be the inductive system constructed in the proof of Lemma 6.7, that is, A_n is of the form $\bigoplus_{k=1}^{r(n)} \mathrm{C}(U_k) \otimes A(U_k)$ with a partition into clopen sets U_k. It is clear that $A(U_k) \cong A(U_k) \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ and that $A(U_k)$ contains a full projections. By assumption, each C*-algebra $A(U_k)$ is E-equivalent to some nuclear separable C*-algebra and hence it is E-equivalent to some stable Kirchberg algebra D_k. For each k, Kirchberg's Classification Theorem [29, Theorem 8.3.3] yields a *-homomorphism $\eta_k\colon D_k \to A(U_k)$ which lifts the given E-equivalence. Moreover, we may arrange that η_k decomposes as $\eta_k = \mu_k \oplus \theta_k$, where θ_k is a full *-monomorphism that factors through the stable Cuntz algebra $\mathcal{O}_2 \otimes \mathbb{K}$. Extending the η_k by $\mathrm{C}(X)$-linearity and taking their direct sum, we get a $\mathrm{C}(X)$-linear monomorphism $\varphi_n\colon B_n \to A_n$, where $B_n := \bigoplus_{k=1}^{r(n)} \mathrm{C}(U_k) \otimes D_k$. Moreover, each φ_n induces an equivalence in $\mathfrak{E}(X)$. Another application of [29, Theorem 8.3.3] yields $\mathrm{C}(X)$-linear *-monomorphisms $\beta_n^{n+1}\colon B_n \to B_{n+1}$ such that for each n the diagram

$$\begin{array}{ccc} A_n & \xrightarrow{\alpha_n^{n+1}} & A_{n+1} \\ {\scriptstyle\varphi_n}\uparrow & & \uparrow{\scriptstyle\varphi_{n+1}} \\ B_n & \xrightarrow[\beta_n^{n+1}]{} & B_{n+1} \end{array}$$

commutes in $\mathfrak{E}(X)$ and hence in the category $\mathfrak{KK}(X)$ (since each D_k is nuclear). The uniqueness part of [29, Theorem 8.3.3] shows that we may arrange that the diagram above commutes up to unitary homotopy. By [6, Section 2] this gives a $\mathrm{C}(X)$-linear *-homomorphism $\varphi\colon B \to \mathrm{C}_b(T, A)/\mathrm{C}_0(T, A)$, where B is the limit of the inductive system (B_n, β_n^{n+1}), such that the diagram

$$\begin{array}{ccc} A_n & \longrightarrow & A \\ {\scriptstyle\varphi_n}\uparrow & & \uparrow{\scriptstyle\varphi} \\ B_n & \longrightarrow & B \end{array}$$

commutes in $\mathfrak{E}(X)$. By Proposition 2.28, for any separable $\mathrm{C}(X)$-algebra D there is a commutative diagram with exact rows

$$\begin{array}{ccccc} \varprojlim^1 \mathrm{E}_1(X; A_i, D) & \rightarrowtail & \mathrm{E}(X; A, D) & \twoheadrightarrow & \varprojlim \mathrm{E}(X; A_i, D) \\ \downarrow{\scriptstyle\varphi_n^*} & & \downarrow{\scriptstyle\varphi^*} & & \downarrow{\scriptstyle\varphi_n^*} \\ \varprojlim^1 \mathrm{E}_1(X; B_i, D) & \rightarrowtail & \mathrm{E}(X; B, D) & \twoheadrightarrow & \varprojlim \mathrm{E}(X; B_i, D). \end{array}$$

Since the maps φ_n^* are bijective by construction, we conclude that A is $\mathrm{E}(X)$-equivalent to the nuclear continuous $\mathrm{C}(X)$-algebra B. □

Theorem 6.11. *Let A and B be separable $\mathrm{C}(X)$-algebras over a totally disconnected compact metrisable space X. If A is in the bootstrap class $\mathcal{B}_{\mathrm{E}}(X)$, then there is an exact sequence*

$$\mathrm{Ext}_{\mathrm{C}(X,\Lambda)}\big(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(SB)\big) \rightarrowtail \mathrm{E}(X; A, B) \twoheadrightarrow \mathrm{Hom}_{\mathrm{C}(X,\Lambda)}\big(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)\big).$$

Proof. By Proposition 6.10 we may assume that A is a continuous nuclear $\mathrm{C}(X)$-algebra with all its fibres in the bootstrap class $\mathcal{B}$. Then $\mathrm{E}(X; A, B) \cong$

$\mathrm{KK}(X;A,B)$ by Theorem 5.4. By Proposition 6.5 we may also assume $A \cong \varinjlim A_n$ for an increasing sequence $(A_n)_{n=1}^{\infty}$ of elementary C^*-subalgebras of A. Then we can apply the $\varprojlim^1$-sequence for nuclear continuous $\mathrm{C}(X)$-algebras and $\mathrm{KK}(X;\text{␣},\text{␣})$ to obtain the following exact sequence:

$$\varprojlim{}^1 \mathrm{KK}_1(X;A_n,B) \rightarrowtail \mathrm{KK}(X;A,B) \twoheadrightarrow \varprojlim \mathrm{KK}(X;A_n,B).$$

By Lemma 6.6

$$\begin{aligned}\varprojlim \mathrm{KK}(X;A_n,B) &\cong \varprojlim \mathrm{Hom}_{\mathrm{C}(X,\Lambda)}\bigl(\underline{\mathrm{K}}(A_n),\underline{\mathrm{K}}(B)\bigr)\\ &\cong \mathrm{Hom}_{\mathrm{C}(X,\Lambda)}\bigl(\varinjlim \underline{\mathrm{K}}(A_n),\underline{\mathrm{K}}(B)\bigr) \cong \mathrm{Hom}_{\mathrm{C}(X,\Lambda)}\bigl(\underline{\mathrm{K}}(A),\underline{\mathrm{K}}(B)\bigr).\end{aligned}$$

Using again Lemma 6.6 and Lemma 6.4, we get

$$\begin{aligned}\varprojlim{}^1 \mathrm{KK}_1(X;A_n,B) &\cong \varprojlim{}^1 \mathrm{Hom}_{\mathrm{C}(X,\Lambda)}\bigl(\underline{\mathrm{K}}(A_n),\underline{\mathrm{K}}(SB)\bigr)\\ &\cong \mathrm{Ext}_{\mathrm{C}(X,\Lambda)}\bigl(\underline{\mathrm{K}}(A),\underline{\mathrm{K}}(SB)\bigr).\end{aligned}$$ □

Remark 6.12. If A is a separable nuclear continuous $\mathrm{C}(X)$-algebra with all the fibres in $\mathcal{B}$, then $A \in \mathcal{B}_{\mathrm{E}}(X)$, and Theorem 5.4 shows that the exact sequence from Theorem 6.11 holds with $\mathrm{KK}(X;A,B)$ replacing $\mathrm{E}(X;A,B)$.

For abelian groups G and H, $\mathrm{PExt}_{\mathbb{Z}}(G,H)$ denotes the subgroup of $\mathrm{Ext}_{\mathbb{Z}}(G,H)$ generated by pure extensions, that is, extensions $H \rightarrowtail E \twoheadrightarrow G$ whose restrictions to all finitely generated subgroups of G split. Theorem 6.11 is a generalisation of the main result of [11], which corresponds to the case when X reduces to a point.

Proposition 6.13. *Let A and B be separable C^*-algebras. If $A \in \mathcal{B}$, there is a natural isomorphism* $\mathrm{Ext}_{\Lambda}\bigl(\underline{\mathrm{K}}(A),\underline{\mathrm{K}}(B)\bigr) \cong \mathrm{PExt}_{\mathbb{Z}}\bigl(\mathrm{K}_*(A),\mathrm{K}_*(B)\bigr)$.

Proof. Consider the natural restriction map

$$\eta\colon \mathrm{Ext}_{\Lambda}\bigl(\underline{\mathrm{K}}(A),\underline{\mathrm{K}}(B)\bigr) \to \mathrm{Ext}_{\mathbb{Z}}\bigl(\mathrm{K}_*(A),\mathrm{K}_*(B)\bigr).$$

Let $\underline{\mathrm{K}}(B) \rightarrowtail \underline{M} \twoheadrightarrow \underline{\mathrm{K}}(A)$ be an extension of Λ-modules. We claim that its η-image $\mathrm{K}_*(B) \rightarrowtail M_* \twoheadrightarrow \mathrm{K}_*(A)$ is pure. Purity follows if any element x in $\mathrm{K}_i(A)$ of order $n \in \mathcal{P}_{\geq 1}$ lifts to an element in M_i of the same order. Since x has order n, there is an element $y \in \mathrm{K}_{i+1}(A;\mathbb{Z}/n)$ such that $\beta_n(y) = x$, because of the exactness of the sequence

$$\mathrm{K}_{i+1}(A;\mathbb{Z}/n) \xrightarrow{\beta_n} \mathrm{K}_i(A) \xrightarrow{n} \mathrm{K}_i(A),$$

where $\beta_n \in \Lambda$. Let $\hat{y} \in M_n^{i+1}$ be a lifting of y. Then the image $\hat{x} := \beta_n(\hat{y}) \in M_0^i$ of $\hat{y}$ is a lifting of x of order n. Thus the image of η is contained in $\mathrm{PExt}_{\mathbb{Z}}\bigl(\mathrm{K}_*(A),\mathrm{K}_*(B)\bigr)$.

Conversely, if $\mathrm{K}_*(B) \rightarrowtail G_* \twoheadrightarrow \mathrm{K}_*(A)$ is a pure extension of $\mathbb{Z}/2$-graded abelian groups, then the UCT provides a separable C^*-algebra E and an extension of C^*-algebras $B \otimes K \rightarrowtail E \twoheadrightarrow A$ such that $\mathrm{K}_*(B) \rightarrowtail \mathrm{K}_*(E) \twoheadrightarrow \mathrm{K}_*(A)$ is isomorphic to the given extension. We claim that $\underline{\mathrm{K}}(B) \rightarrowtail \underline{\mathrm{K}}(E) \twoheadrightarrow \underline{\mathrm{K}}(A)$ is an extension of Λ-modules. Purity yields extensions

$$\mathrm{Tor}_q(\mathrm{K}_*(B),\mathbb{Z}/n) \rightarrowtail \mathrm{Tor}_q(\mathrm{K}_*(E),\mathbb{Z}/n) \twoheadrightarrow \mathrm{Tor}_q(\mathrm{K}_*(A),\mathbb{Z}/n)$$

for any $n \in \mathcal{P}$ and for $q = 0, 1$. Furthermore, there is a natural extension

$$\mathrm{Tor}_0(\mathrm{K}_*(A), \mathbb{Z}/n) \rightarrowtail \mathrm{K}_*(A; \mathbb{Z}/n) \twoheadrightarrow \mathrm{Tor}_1(\mathrm{K}_*(A), \mathbb{Z}/n),$$

and the same for E and B. Now a diagram chase shows that $\mathrm{K}_*(B; \mathbb{Z}/n) \rightarrowtail \mathrm{K}_*(E; \mathbb{Z}/n) \twoheadrightarrow \mathrm{K}_*(A; \mathbb{Z}/n)$ is an extension.

Having identified the image of η as $\mathrm{PExt}_{\mathbb{Z}}\bigl(\mathrm{K}_*(A), \mathrm{K}_*(B)\bigr)$, it remains to show that η is injective. We may assume that A is nuclear. Suppose that the extension $\mathrm{K}_*(B) \rightarrowtail \mathrm{K}_*(E) \twoheadrightarrow \mathrm{K}_*(A)$ splits. By the UCT, the class of the extension $B \otimes \mathbb{K} \rightarrowtail E \twoheadrightarrow A$ in $KK_1(A, B)$ is zero. It follows that the extension $B \otimes \mathbb{K} \rightarrowtail E \twoheadrightarrow A$ is stably split, so that the extension $\underline{\mathrm{K}}(B) \rightarrowtail \underline{\mathrm{K}}(E) \twoheadrightarrow \underline{\mathrm{K}}(A)$ is trivial. □

The following example adapted from [10] shows that the map $\mathrm{E}(X; A, B) \to \mathrm{Hom}_{\mathrm{C}(X,\mathbb{Z})}\bigl(\mathrm{K}_*(A), \mathrm{K}_*(B)\bigr)$ is not always surjective.

Example 6.14. Let $X = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$. We shall exhibit two separable continuous $\mathrm{C}(X)$-algebras E_k and $E_{k'}$ with all fibres isomorphic to Kirchberg algebras in the bootstrap category such that E_k and $E_{k'}$ have isomorphic filtrated K-theory but non-isomorphic filtrated K-theory with coefficients.

Let A be a Kirchberg algebra in the bootstrap category with $\mathrm{K}_0(A) = 0$ and $\mathrm{K}_1(A) = \mathbb{Z}/n$ for $n \geq 2$. For $k \in \mathbb{Z}/n$ let $\varphi_k \colon A \to \mathcal{O}_\infty$ be a *-homomorphism such that $[\varphi_k] = k \in \mathrm{KK}(A, \mathcal{O}_\infty) \cong \mathbb{Z}/n$. Consider the $\mathrm{C}(X)$-algebra

$$E_k = \{(f, a) \in \mathrm{C}(X, \mathcal{O}_\infty) \oplus A : f(\infty) = \varphi_k(a)\}.$$

We note that $\mathrm{K}_*(E_k) \cong \mathrm{K}_*(E_{k'})$ as $\mathrm{C}(X, \mathbb{Z})$-modules for any k, k', and we claim that if $k\mathbb{Z}/n \neq k'\mathbb{Z}/n$, then $\underline{\mathrm{K}}(E_k) \ncong \underline{\mathrm{K}}(E_{k'})$ as $\mathrm{C}(X, \Lambda)$-modules. Indeed, $\mathrm{K}_0(E_k) = \mathrm{K}_0(E_{k'}) = \mathrm{C}_0(X, \mathbb{Z})$ with $\mathrm{C}(X, \mathbb{Z})$ acting by pointwise multiplication and $\mathrm{K}_1(E_k) = \mathrm{K}_1(E_{k'}) = \mathbb{Z}/n$ with $\mathrm{C}(X, \mathbb{Z})$-module structure $fm = f(\infty)m$ for $m \in \mathbb{Z}/n$. On the other hand,

$$\mathrm{K}_0(E_k; \mathbb{Z}/n) = \{(f, r) \in \mathrm{C}(X, \mathbb{Z}/n) \oplus \mathbb{Z}/n : f(\infty) = kr\}.$$

The coefficient map $\rho \colon \mathrm{K}_0(E_k) \to \mathrm{K}_0(E_k; \mathbb{Z}/n)$ is $g \mapsto (\dot{g}, 0)$. The Böckstein map $\beta \colon \mathrm{K}_0(E_k; \mathbb{Z}/n) \to \mathrm{K}_1(E_k)$ is $\beta(f, r) = r$.

Suppose that $\alpha \colon \underline{\mathrm{K}}(E_k) \to \underline{\mathrm{K}}(E_{k'})$ is an isomorphism of $\mathrm{C}(X, \Lambda)$-modules. Then α must act on K_0 by multiplication by a function $u \colon X \to \{-1, 1\}$. Since α is $\mathrm{C}(X, \mathbb{Z})$-linear and commutes with ρ and β, there is a unit $v \in \mathbb{Z}/n$ such that $\alpha \colon \mathrm{K}_0(E_k; \mathbb{Z}/n) \to \mathrm{K}_0(E_{k'}; \mathbb{Z}/n)$ is given by $\alpha(f, r) = (uf, vr)$. Choose f such that $(f, 1) \in \mathrm{K}_0(E_k)$. It follows that for all sufficiently large i we have $u(i)f(i) = k'v$ and hence $\pm kr = k'v$. Thus $k\mathbb{Z}/n = k'\mathbb{Z}/n$.

Next we generalise the previous example, constructing a suitable continuous $\mathrm{C}(X)$-algebra over any compact Hausdorff space X.

Example 6.15. Let X be an infinite metrisable compact space. We shall exhibit two unital separable continuous $\mathrm{C}(X)$-algebras F and F' with all fibres isomorphic to Kirchberg algebras in the bootstrap category such that

F and F' have isomorphic filtrated K-theory but non-isomorphic filtrated K-theory with coefficients.

Using the assumption on X we find a sequence $(x_i)_{i=1}^{\infty}$ of distinct elements of X which converges to some $x_\infty \in X$. Fix an embedding $\mathcal{O}_\infty \subset \mathcal{O}_2$. For each $k \in \mathbb{Z}/n$ let A and $\varphi_k \colon A \to \mathcal{O}_\infty$ be as in Example 6.14. Consider the $\mathrm{C}(X)$-algebra

$$F_k := \{(f, a) \in \mathrm{C}(X, \mathcal{O}_2) \oplus A \mid f(x_i) \in \mathcal{O}_\infty \text{ for all } i \in \mathbb{N},\ f(x_\infty) = \varphi_k(a)\}.$$

Choose $k, k' \in \mathbb{Z}/n$ such that $k\mathbb{Z}/n \neq k'\mathbb{Z}/n$ and set $F = F_k$ and $F' = F_{k'}$. Then F and F' have non-isomorphic filtrated K-theory with coefficients since their restrictions to the subspace $Y := \{x_\infty\} \cup \{x_i : i \in \mathbb{N}\}$ are isomorphic to the $\mathrm{C}(Y)$-algebras E_k and $E_{k'}$ from Example 6.14, respectively. At the same time, we have an exact sequence of $\mathrm{C}(X)$-algebras $G \rightarrowtail F_k \twoheadrightarrow E_k$ with $G = \mathrm{C}_0(X \setminus Y, \mathcal{O}_2)$. Since $\mathrm{K}_*(\mathcal{O}_2) = 0$, we see that $\mathrm{K}_*\bigl(G(T \setminus Y)\bigr) = 0$ for all locally closed subsets T of X. It follows that the filtrated K-theory of F is isomorphic to the filtrated K-theory of F' since we have seen that E_k and $E_{k'}$ have this property.

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