J. reine angew. Math. **612** (2007), 1—15
DOI 10.1515/CRELLE.2007.081

Journal für die reine und angewandte Mathematik
© Walter de Gruyter
Berlin · New York 2007

Uniform embeddability of relatively hyperbolic groups

By *Marius Dadarlat* at West Lafayette and *Erik Guentner* at Honolulu

Abstract. Let Γ be a finitely generated group which is hyperbolic relative to a finite family $\{H_1, \ldots, H_n\}$ of subgroups. We prove that Γ is uniformly embeddable in a Hilbert space if and only if each subgroup H_i is uniformly embeddable in a Hilbert space.

1. Introduction

Gromov introduced the notion of uniform embeddability (in Hilbert space), and suggested it should be relevant to the Novikov Conjecture [8], [7]. Subsequently Yu proved the Coarse Baum-Connes Conjecture for bounded geometry discrete metric spaces that are uniformly embeddable; applying a descent principle the Novikov Conjecture followed for groups that, when equipped with a word length metric, are uniformly embeddable [19]. (A condition on finiteness of the classifying space was later removed [17].)

The notion of C^*-exactness of discrete groups was introduced by Kirchberg. It has been extensively studied as a functional analytic property of groups, and developed by many authors. In particular, Ozawa gave a characterization of C^*-exact groups, from which it directly follows that a C^*-exact group is uniformly embeddable [13] (see [9] for the link to uniform embeddability). There is at present no known example of a group that is uniformly embeddable but not C^*-exact. Analogous statements for metric spaces involve Property A of Yu [19], which is equivalent to C^*-exactness for discrete groups.

The classes of metric spaces (and groups) that are uniformly embeddable, or have Property A (are C^*-exact) are the subject of intense study. In this note we introduce a 'gluing' technique for proving uniform embeddability: starting from the assumption that a space is covered in an appropriate way by uniformly embeddable sets we conclude that the space itself is uniformly embeddable. Thus, the individual uniform embeddings of the pieces are 'glued' to give a uniform embedding of the whole. A parallel technique is intro-

The first author was supported in part by NSF Grant #DMS-0500693. The second author was supported in part by NSF Grant #DMS-0349367.

duced for spaces with Property A; it applies to C^*-exact groups. The most primitive gluing result is summarized in the following theorem (compare Theorem 3.2):

Theorem. *Let X be a metric space. Assume that for all $\lambda > 0$ there exists a partition of unity $(\varphi_i)_{i\in I}$ on X such that*

(i) *the associated $\Phi : X \to l^1(I)$ (defined by $\Phi(x)(i) = \varphi_i(x)$) is λ-Lipschitz, and*

(ii) *the subspaces $\big(\mathrm{supp}(\varphi_i)\big)_{i\in I}$ are 'equi' uniformly embeddable.*

Then X is uniformly embeddable.

Our gluing technique is inspired by the work of Bell and Dranishnikov on spaces of finite asymptotic dimension [3], [15], and by the work of Bell on Property A [2]. Our results differ from these works in several ways: first, we treat uniformly embeddable spaces, and allow spaces of unbounded geometry; second, we significantly relax the condition on the 'parameter space'. With these refinements, the gluing technique is extremely versatile—it allows us to give a conceptual treatment of all known permanence properties of the classes of uniformly embeddable spaces and C^*-exact groups (see, for example, [5], [6], [11]).

In this note we develop the basics of our gluing technique and present the above mentioned applications. We also describe an additional application of gluing, proving the following new permanence property of the class of uniformly embeddable groups:

Theorem. *Let Γ be a finitely generated group which is hyperbolic relative to a finite family $\{H_1, \dots, H_n\}$ of subgroups. Then Γ is uniformly embeddable if and only if each subgroup H_i is uniformly embeddable.*

A parallel statement concerning C^*-exactness was recently obtained by Ozawa [14]. Our methods allow us to recover this result; indeed, the C^*-exactness result may be extracted from the combined work of Bell [2] and Osin [12]. Our approach to relative hyperbolicity, which is based on the work of Osin, is quite different from that taken by Ozawa however, who, given amenable H_i-spaces explicitly constructs an amenable Γ-space.

These results should be compared to a recent result of Osin [12]: in the situation described in the theorem, Γ has finite asymptotic dimension if and only if the H_i have finite asymptotic dimension.

2. Preliminaries

Let X and Y be metric spaces, with metrics d_X and d_Y, respectively. A function $F : X \to Y$ is a *uniform embedding* if there exist non-decreasing functions $\rho_\pm : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} \rho_\pm(t) = \infty$ and such that

$$(1) \qquad \rho_-\big(d_X(x,x')\big) \leqq d_Y\big(F(x),F(x')\big) \leqq \rho_+\big(d_X(x,x')\big), \quad \text{for all } x,x' \in X.$$

The space X is (Hilbert space) *uniformly embeddable* if there exists a uniform embedding F of X into a (real) Hilbert space $\mathscr{H}$.

Proposition 2.1. *Let X be a metric space. Then X is uniformly embeddable if and only if for every $R > 0$ and $\varepsilon > 0$ there exists a Hilbert space valued map $\xi : X \to \mathscr{H}$, $(\xi_x)_{x \in X}$, such that $\|\xi_x\| = 1$, for all $x \in X$ and such that*

(i) $d(x, x') \leqq R \Rightarrow \|\xi_x - \xi_{x'}\| \leqq \varepsilon$,

(ii) $\lim_{S \to \infty} \sup\{|\langle \xi_x, \xi_{x'} \rangle| : d(x, x') \geqq S, x, x' \in X\} = 0.$ □

Definition 2.2. A family of metric spaces (X_i, d_i) is called equi-uniformly embeddable if there is a family of Hilbert space valued maps $F_i : X_i \to H_i$ and non-decreasing functions $\rho_\pm : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t \to \infty} \rho_\pm(t) = \infty$ such that

$$\rho_-\big(d_i(x, y)\big) \leqq \|F_i(x) - F_i(y)\| \leqq \rho_+\big(d_i(x, y)\big), \quad \text{for all } i \text{ and all } x, y \in X_i. \tag{2}$$

The proof of Proposition 2.1 given in [5] for arbitrary metric spaces also shows the following characterization of equi-uniform embeddability for families of metric spaces.

Proposition 2.3. *A family $(X_i)_{i \in I}$ of metric spaces is equi-uniformly embeddable if and only if for every $R > 0$ and $\varepsilon > 0$ there exists a family $(\xi_i)_{i \in I}$ of Hilbert space valued maps $\xi_i : X_i \to \mathscr{H}$, such that $\|\xi_i(x)\| = 1$, for all $x \in X_i$, and such that*

(i) $\forall i \in I\ \forall x, x' \in X_i\ \ d(x, x') \leqq R \Rightarrow \|\xi_i(x) - \xi_i(x')\| \leqq \varepsilon$,

(ii) $\lim_{S \to \infty} \sup_{i \in I} \sup\{|\langle \xi_i(x), \xi_i(x') \rangle| : d(x, x') \geqq S,\ x, x' \in X_i\} = 0.$ □

Remark 2.4. Let X be a metric space which is uniformly embeddable. Then any family $(X_i)_{i \in I}$ of subspaces of X is equi-uniformly embeddable.

Property A is a condition on metric spaces introduced by Yu [19]. We do not recall the definition of Property A here; rather, we work with the following characterization of Property A obtained by Tu [18].

Proposition 2.5 ([18]). *A discrete metric space X with bounded geometry has Property* A *if and only if for every $R > 0$ and $\varepsilon > 0$ there exist a function $\xi : X \to \ell^1(X)$ and a number $S > 0$ such that for all $x, x' \in X$ we have $\|\xi_x\| = 1$, and*

(i) $d(x, x') \leqq R \Rightarrow \|\xi_x - \xi_{x'}\| \leqq \varepsilon$,

(ii) $\operatorname{supp} \xi_x \subset B(x, S)$.

(*Moreover one can arrange that ξ is nonnegative.*)

Equivalently, for every $R > 0$ and $\varepsilon > 0$ there exists an $S > 0$ and a Hilbert space valued function $\xi : X \to \mathscr{H}$ such that for all $x, x' \in X$ we have $\|\xi_x\| = 1$, (i) *as above and*

(iii) $\exists S > 0$ *such that* $d(x, x') \geqq S \Rightarrow \langle \xi_x, \xi_{x'} \rangle = 0.$ □

Remark 2.6. The existence of ξ satisfying the conditions (i), (ii) (respectively (iii)) of Proposition 2.5 is a consequence of Property A for *arbitrary* metric spaces as shown in [18]. The bounded geometry condition is needed only for the reverse implication.

Let X be a set. A *partition of unity* on X is a family of maps $(\varphi_i)_{i\in I}$, with $\varphi_i : X \to [0,1]$, and such that $\sum_{i\in I} \varphi_i(x) = 1$ for all $x \in X$. If $x \in X$ we do not require that the set $\{i \in I : \varphi_i(x) \neq 0\}$ be finite, although that will be the case in most of our examples. We say that $(\varphi_i)_{i\in I}$ is *subordinated to a cover* $\mathscr{U} = (U_i)_{i\in I}$ of X if each φ_i vanishes outside U_i. Sometimes a partition of unity subordinated to a cover $\mathscr{U}$ will be denoted by $(\varphi_U)_{U\in\mathscr{U}}$.

Definition 2.7. A metric space X is *exact* if for all $R > 0$ and $\varepsilon > 0$ there is a partition of unity $(\varphi_i)_{i\in I}$ on X subordinated to a cover $\mathscr{U} = (U_i)_{i\in I}$ and such that

(i) $\forall x, y \in X$ with $d(x,y) \leqq R$, $\sum_{i\in I} |\varphi_i(x) - \varphi_i(y)| \leqq \varepsilon$,

(ii) the cover $\mathscr{U} = (U_i)_{i\in I}$ is uniformly bounded, i.e. $\sup_{i\in I} \operatorname{diam}(U_i) < \infty$.

Definition 2.8. A family of metric spaces $(X_i)_{i\in I}$ is *equi-exact* if for all $R > 0$ and $\varepsilon > 0$ and for every $i \in I$ there is a partition of unity $(\psi_i^j)_{j\in J_i}$ on X_i subordinated to a cover $\mathscr{U}_i = (U_i^j)_{j\in J_i}$ of X_i and such that

(i) $\forall i \in I$, $\forall x, y \in X_i$ with $d(x,y) \leqq R$, $\sum_{j\in J_i} |\psi_i^j(x) - \psi_i^j(y)| \leqq \varepsilon$,

(ii) the family $(U_i^j)_{i\in I, j\in J_i}$ is uniformly bounded, i.e. $\sup_{i\in I, j\in J_i} \operatorname{diam}(U_i^j) < \infty$.

Remark 2.9. Let X be an exact metric space. Then any family $(X_i)_{i\in I}$ of subspaces of X is equi-exact.

Proposition 2.10. *Let X be a metric space.*

(a) *If X has Property* A *then X is exact.*

(b) *If X is discrete and has bounded geometry then X is exact if and only if it has Property* A.

(c) *If X is exact then X is uniformly embeddable.*

Proof. For (a), assume that X has Property A. Let $R > 0$ and $\varepsilon > 0$ be given. Obtain $\xi : X \to \ell^1(X)$ (which we assume to be non-negative) and $S > 0$ as in Proposition 2.5. Define $(\varphi_z)_{z\in X}$ by $\varphi_z(x) = \xi_x(z)$. If $U_z = \{x \in X : \varphi_z(x) > 0\}$ then $U_z \subset B(z,S)$ since $\operatorname{supp} \xi_x \subset B(x,S)$. It is clear that $\sum_{z\in X} \varphi_z(x) = \|\xi_x\| = 1$ and

$$\sum_{z\in X} |\varphi_z(x) - \varphi_z(y)| = \|\xi_x - \xi_y\| \leqq \varepsilon$$

if $d(x,y) \leqq R$.

For (b), assume that X is exact. It suffices to find a Hilbert space valued function on X satisfying the conditions from the second part of Proposition 2.5. Let $R > 0$ and $\varepsilon > 0$ be given. Let $(\varphi_i)_{i \in I}$ be as in Definition 2.7. Define $\xi : X \to \ell^2(I)$ by $\xi_x(i) = \varphi_i(x)^{1/2}$. Then $\|\xi_x\|^2 = \sum_{i \in I} \varphi_i(x) = 1$. Using the inequality $|a^{1/2} - b^{1/2}|^2 \leqq |a - b|$ we see that

$$\|\xi_x - \xi_y\|^2 = \sum_{i \in I} |\varphi_i(x)^{1/2} - \varphi_i(y)^{1/2}|^2 \leqq \sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leqq \varepsilon$$

if $d(x, y) \leqq R$. Finally we note that

$$\langle \xi_x, \xi_y \rangle = \sum_{i \in I} \varphi_i(x)^{1/2} \varphi_i(y)^{1/2} = 0$$

whenever $d(x, y) > \sup_{i \in I} \operatorname{diam}(U_i)$.

The proof of (c) is similar to that of (b), but one applies Proposition 2.1. □

Remark 2.11. If $f : X \to Y$ is a uniform embedding of metric spaces and Y is exact then X is exact. Therefore exactness of a metric space is a coarse invariant. This remark is generalized in Corollary 3.3.

3. Gluing spaces using partitions of unity

Let X be a metric space and let $\mathscr{U} = (U_i)_{i \in I}$ be a cover of X. Denote by $\mathscr{U}_R$ the cover obtained by enlarging the sets in $\mathscr{U}$ by taking their R-closed neighborhoods:

$$\mathscr{U}_R = \{U_i(R) : i \in I\}, \quad U_i(R) = \{x \in X : d(x, U_i) \leqq R\}.$$

One verifies immediately that if the family $\mathscr{U}$ (with the metric structure induced from X) is equi-uniformly embeddable (or equi-exact) then so is $\mathscr{U}_R$.

Theorem 3.1. *Let X be a metric space. Suppose that for all $R > 0$ and $\varepsilon > 0$ there is a partition of unity $(\varphi_i)_{i \in I}$ on X such that*

(i) *$\forall x, y \in X$ with $d(x, y) \leqq R$, $\sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leqq \varepsilon$, and*

(ii) *$(\varphi_i)_{i \in I}$ is subordinated to an equi-exact cover $(U_i)_{i \in I}$ of X.*

Then X is exact.

Proof. Let $R > 0$ and $\epsilon > 0$ be given. We construct a partition of unity as required by Definition 2.7. By assumption there is a cover $\mathscr{U} = (U_i)_{i \in I}$ of X which is equi-exact and there is a partition of unity $(\varphi_i)_{i \in I}$ subordinated to $\mathscr{U}$ such that $\forall x, y \in X$ with $d(x, y) \leqq R$,

$$\sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leqq \varepsilon/2.$$

Since $\mathscr{U}$ is equi-exact so is $\mathscr{U}_R = \{U_i(R) : i \in I\}$. Therefore for each $U_i \in \mathscr{U}$ there is a cover $\mathscr{V}_i = (V_i^j)_{j \in J_i}$ of $U_i(R)$ such that the cover $\{V_i^j : i \in I, j \in J_i\}$ of X is uniformly bounded.

Moreover for each $U_i \in \mathcal{U}$ there is a partition of unity (ψ_i^j) on $U_i(R)$ subordinated to $\mathcal{V}_i$ such that $\forall x, y \in U_i(R)$ with $d(x, y) \leqq R$,

$$\sum_{j \in J_i} |\psi_i^j(x) - \psi_i^j(y)| \leqq \varepsilon/2.$$

It is useful to extend ψ_i^j to X by setting it equal to zero outside $U_i(R)$. Define $\theta_{i,j} = \varphi_i \psi_i^j$. Then $(\theta_{i,j})$ is a partition of unity on X subordinated to a uniformly bounded cover. Moreover,

$$\sum_{i,j} |\theta_{i,j}(x) - \theta_{i,j}(y)| \leqq \sum_i \varphi_i(x) \sum_j |\psi_i^j(x) - \psi_i^j(y)| + \sum_i |\varphi_i(x) - \varphi_i(y)| \sum_j \psi_i^j(y).$$

Assume now that $d(x, y) \leqq R$. If $\varphi_i(x) \neq 0$ then $x \in U_i$ hence $y \in U_i(R)$ as $d(x, y) \leqq R$. Therefore

$$\sum_i \varphi_i(x) \sum_j |\psi_i^j(x) - \psi_i^j(y)| \leqq \varepsilon/2.$$

Since $\sum_j \psi_i^j(y)$ equals 1 for $y \in U_i(R)$ and 0 for $y \notin U_i(R)$,

$$\sum_i |\varphi_i(x) - \varphi_i(y)| \sum_j \psi_i^j(y) \leqq \sum_i |\varphi_i(x) - \varphi_i(y)| \leqq \varepsilon/2.$$

Combining the above estimates we obtain that

$$\sum_{i,j} |\theta_{i,j}(x) - \theta_{i,j}(y)| \leqq \varepsilon$$

whenever $x, y \in X$ and $d(x, y) \leqq R$. □

Theorem 3.2. *Let X be a metric space. Suppose that for all $R > 0$ and $\varepsilon > 0$ there is a partition of unity $(\varphi_i)_{i \in I}$ on X such that*

(i) $\forall x, y \in X$ *with* $d(x, y) \leqq R$, $\sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leqq \varepsilon$,

(ii) $(\varphi_i)_{i \in I}$ *is subordinated to an equi-uniformly embeddable cover* $(U_i)_{i \in I}$ *of* X.

Then X is uniformly embeddable.

Proof. Let $R > 0$ and $\epsilon > 0$ be given. We construct a Hilbert space valued function η on X satisfying the conditions in Proposition 2.1. By assumption there is a cover $\mathcal{U} = (U_i)_{i \in I}$ of X which is equi-uniformly embeddable and there is a partition of unity $(\varphi_i)_{i \in I}$ subordinated to $\mathcal{U}$ such that $\forall x, y \in X$ with $d(x, y) \leqq R$,

$$\sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leqq \varepsilon^2/4.$$

Since $\mathcal{U}$ is equi-uniformly embeddable so is $\mathcal{U}_R = \{U_i(R) : i \in I\}$ where as above $U_i(R) = \{x \in X : d(x, U_i) \leqq R\}$. Therefore there exist Hilbert space valued maps $\xi_i : U_i(R) \to \mathscr{H}_i$, with $\|\xi_i(x)\| = 1$ for all $x \in U_i(R)$ and such that

(iv) $\sup\{\|\xi_i(x) - \xi_i(y)\| : d(x, y) \leqq R, x, y \in U_i(R)\} \leqq \varepsilon/2$ for all $i \in I$,

(v) $\lim_{S \to \infty} \sup_{i \in I} \sup\{|\langle \xi_i(x), \xi_i(y) \rangle| : d(x, y) \geqq S, x, y \in U_i(R)\} = 0.$

We extend each ξ_i to X by setting $\xi_i(x) = 0$ for $x \in X \setminus U_i(R)$. Define $\eta : X \to \mathscr{H} = \bigoplus_{i \in I} \mathscr{H}_i$, $\eta(x) = \big(\eta_i(x)\big)_{i \in I}$ by

$$\eta_i(x) = \varphi_i(x)^{1/2} \xi_i(x). \tag{3}$$

One verifies that $\|\eta(x)\| = 1$, $\forall x \in X$. Let $x, y \in X$ with $d(x, y) \leqq R$. Consider $\alpha(x, y), \beta(x, y) \in \mathscr{H}$ with components

$$\alpha_i(x, y) = \varphi_i(x)^{1/2}\big(\xi_i(x) - \xi_i(y)\big),$$
$$\beta_i(x, y) = \big(\varphi_i(x)^{1/2} - \varphi_i(y)^{1/2}\big)\xi_i(y).$$

Note that $\alpha(x, y)$ and $\beta(x, y)$ are well-defined because of the following norm estimates.

$$\|\alpha(x, y)\|^2 = \sum_{i \in I} \varphi_i(x) \|\xi_i(x) - \xi_i(y)\|^2,$$

where the summation is done for those i with $x \in U_i$. If $d(x, y) \leqq R$ and $x \in U_i$ then $y \in U_i(R)$, so that using (iv) we obtain $\|\alpha(x, y)\| \leqq \varepsilon/2$. Since $|a^{1/2} - b^{1/2}|^2 \leqq |a - b|$ we have

$$\begin{aligned}\|\beta(x, y)\|^2 &= \sum_{i \in I} \big\| \big(\varphi_i(x)^{1/2} - \varphi_i(y)^{1/2}\big)\xi_i(y)\big\|^2 \leqq \sum_{i \in I} |\varphi_i(x)^{1/2} - \varphi_i(y)^{1/2}|^2 \\ &\leqq \sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leqq \varepsilon^2/4,\end{aligned}$$

hence $\|\beta(x, y)\| \leqq \varepsilon/2$. Therefore

$$\|\eta(x) - \eta(y)\| = \|\alpha(x, y) + \beta(x, y)\| \leqq \|\alpha(x, y)\| + \|\beta(x, y)\| \leqq \varepsilon$$

whenever $d(x, y) \leqq R$. In order to prove the support condition (ii) of Proposition 2.1 we use that φ_i vanishes outside U_i and the Cauchy-Schwarz inequality. Thus for any $x, y \in X$ with $d(x, y) \geqq S$ we have:

$$\begin{aligned}|\langle \eta(x), \eta(y) \rangle| &\leqq \sum_{i \in I} \varphi_i(x)^{1/2} \varphi_i(y)^{1/2} |\langle \xi_i(x), \xi_i(y) \rangle| \\ &\leqq \sup_{i \in I} \sup\{ |\langle \xi_i(x'), \xi_i(y') \rangle| : d(x', y') \geqq S, x', y' \in U_i \}.\end{aligned}$$

In view of (v), this concludes the proof. □

Corollary 3.3. *Let $p : X \to Y$ be a map of metric spaces with the property that $\forall R > 0\ \exists S > 0$ such that $d\big(p(x), p(x')\big) \leqq S$ whenever $d(x, x') \leqq R$. Suppose that Y is exact. If for each uniformly bounded cover $(U_i)_{i \in I}$ of Y, the family $\big(p^{-1}(U_i)\big)_{i \in I}$ of subspaces of X is equi-uniformly embeddable* (*respectively, equi-exact*), *then X is* (*respectively, exact*).

Proof. Let $R > 0$ and $\varepsilon > 0$ be given and let $S > 0$ be as in the statement. Since Y is exact, we find a uniformly bounded cover $(U_i)_{i \in I}$ of Y, together with a partition of unity $(\varphi_i)_{i \in I}$ as in Definition 2.7 with S playing the role of R. Then $(\varphi_i \circ p)_{i \in I}$ is a partition of unity on X subordinated to $\big(p^{-1}(U_i)\big)_{i \in I}$ and satisfying the assumptions of Theorem 3.2 (respectively, 3.1). □

An action of a group Γ (as always, countable and discrete) on a metric space Y by isometries is *cobounded* if there exists a bounded subset $B \subset Y$ such that $\Gamma \cdot B = Y$. Transitive actions and, more generally, actions with finitely many orbits are cobounded.

Corollary 3.4. *Let $p : X \to Y$ be a Lipschitz map of metric spaces. Assume that a group Γ acts by isometries on both X and Y, that the action on Y is cobounded and that p is Γ-equivariant. Assume Y is exact. If there exists $y_0 \in Y$ such that for every $n \in \mathbb{N}$ the inverse image $p^{-1}\big(B(y_0, n)\big)$ is uniformly embeddable* (*respectively*, *exact*) *then X is uniformly embeddable* (*respectively*, *exact*).

Proof. This follows immediately from Corollary 3.3. Let $(U_i)_{i \in I}$ be a uniformly bounded cover of Y and let $y_0 \in Y$ be as in the statement. Let $B \subset Y$ be a bounded set such that every orbit of the Γ-action on Y intersects B; since B is bounded and the cover $(U_i)_{i \in I}$ is uniformly bounded

$$n = \sup_{i \in I} \operatorname{diam}(U_i) + \sup_{z \in B} d(y_0, z) < \infty.$$

We claim that for all $i \in I$ there exists $s_i \in \Gamma$ such that $s_i U_i \subset B(y_0, n)$; indeed, if $s_i \in \Gamma$ is such that $s_i U_i$ intersects B we have, for all $y \in U_i$,

$$d(s_i y, y_0) \leqq d(s_i y, z) + d(z, y_0) = d(y, s_i^{-1} z) + d(z, y_0) \leqq n,$$

where $z \in s_i U_i \cap B$ and we use the fact that Γ acts isometrically. Finally, since $s_i p^{-1}(U_i) = p^{-1}(s_i U_i) \subset p^{-1}\big(B(y_0, n)\big)$, we see that the family $\big(p^{-1}(U_i)\big)_{i \in I}$ is isometric to a family of subspaces of $p^{-1}\big(B(y_0, n)\big)$. Since the latter space is uniformly embeddable (exact), we conclude that the family $\big(p^{-1}(U_i)\big)_{i \in I}$ is equi-uniformly embeddable (equi-exact). □

4. Partitions of unity coming from the combinatorics of asymptotic dimension

Let $\mathcal{U}$ be a cover of X. A *Lebesgue number* for $\mathcal{U}$ is a number $L > 0$ with the property that any subset $B \subset X$ of diameter less than L is contained in some $U \in \mathcal{U}$. A cover $\mathcal{U}$ of X has *multiplicity at most k* if any $x \in X$ belongs to at most k members of $\mathcal{U}$. One way to construct partitions of unity with Lipschitz properties is given by the following proposition.

Proposition 4.1. *Let $\mathcal{U}$ be a cover of a metric space X with multiplicity at most $k + 1$,* $(k \geqq 0)$ *and Lebesgue number $L > 0$. For $U \in \mathcal{U}$ define*

$$\varphi_U(x) = \frac{d(x, X \backslash U)}{\sum_{V \in \mathcal{U}} d(x, X \backslash V)}.$$

Then $(\varphi_U)_{U \in \mathcal{U}}$ is a partition of unity on X subordinated to the cover $\mathcal{U}$. Moreover each φ_U satisfies

$$|\varphi_U(x) - \varphi_U(y)| \leqq \frac{2k + 3}{L} d(x, y), \quad \forall x, y \in X, \tag{4}$$

and the family $(\varphi_U)_{U \in \mathscr{U}}$ *satisfies*

$$\sum_{U \in \mathscr{U}} |\varphi_U(x) - \varphi_U(y)| \leqq \frac{(2k+2)(2k+3)}{L} d(x,y), \quad \forall x, y \in X. \tag{5}$$

Proof. This is folklore. See Bell's paper for a proof of (4). Note that (5) follows from (4) since any point in X belongs to at most $k+1$ distinct elements of the cover $\mathscr{U}$. □

A metric space X has *asymptotic dimension* $\leqq k$ if for every $R > 0$ there exists a uniformly bounded cover $\mathscr{U}$ of X such that every ball of radius R in X meets at most $k+1$ elements of $\mathscr{U}$.

In the context of non-uniformly bounded covers we require several closely related properties. Let $\mathscr{V}$ be a family of nonempty subsets of the metric space X. The *multiplicity* of $\mathscr{V}$ is the maximum number of elements of $\mathscr{V}$ that contain a common point of X; the *R-multiplicity* of $\mathscr{V}$ is the maximum number of elements of $\mathscr{V}$ that meet a common ball of radius R in X. If $d(U,V) > L$ for all $U, V \in \mathscr{V}$ with $U \neq V$ then $\mathscr{V}$ is *L-separated* $(L > 0)$. Note that if $\mathscr{V}$ consists of just one set then $\mathscr{V}$ is L-separated (vacuously) for every $L > 0$. A cover $\mathscr{U}$ of X is (k, L)-*separated* $(k \geqq 0$ and $L > 0)$ if there is a partition of $\mathscr{U}$ into $k+1$ families

$$\mathscr{U} = \mathscr{U}_0 \cup \cdots \cup \mathscr{U}_k$$

such that each family $\mathscr{U}_i$ is L-separated. In particular $\mathscr{U}$ has multiplicity at most $k+1$.

We make two observations, which we will apply to covers that are not necessarily uniformly bounded. First, a $(k, 2R)$-separated cover has R-multiplicity $\leqq k+1$. Second, if $\mathscr{U}$ is a cover of X with L-multiplicity $\leqq k+1$ then L is a Lebesgue number for the cover $\mathscr{U}_L$ obtained by enlarging the sets in $\mathscr{U}$ by taking their L-neighborhoods;

$$\mathscr{U}_L = \{U(L) : U \in \mathscr{U}\}, \quad U(L) = \{x \in X : d(x, U) \leqq L\}.$$

Further, in this case, the cover $\mathscr{U}_L$ has multiplicity $\leqq k+1$.

We summarize the previous discussion in the following form.

Lemma 4.2. *A* $(k, 2L)$*-separated cover of a metric space has L-multiplicity* $\leqq k+1$. *If a cover* $\mathscr{U}$ *of a metric space has L-multiplicity* $\leqq k+1$ *then the enlarged cover* $\mathscr{U}_L$ *has multiplicity* $\leqq k+1$ *and Lebesgue number L.* □

The following result was proven by Higson and Roe in the case of discrete bounded geometry metric spaces [10], Lemma 4.3; in our more general setting it follows immediately from Proposition 4.1 and Definition 2.7.

Proposition 4.3. *A metric space of finite asymptotic dimension is exact.* □

We now prove a natural generalization of this result, where uniform boundedness of the cover is replaced by the appropriate uniform versions of uniform embeddability and Property A, defined earlier. We also provide the proper setting to generalize the 'union theorems' of Bell and Dranishnikov [2], [3].

Theorem 4.4. *Let X be a metric space. Assume that for every $\delta > 0$ there is a (k, L)-separated cover $\mathcal{U}$ of X with $k^2 + 1 \leqq L\delta$ and such that the family $\mathcal{U}$ is equi-uniformly embeddable (where each $U \in \mathcal{U}$ is given the induced metric from X). Then X is uniformly embeddable. If instead we assume that the family $\mathcal{U}$ is equi-exact then X is exact.*

Proof. The statement concerning uniform embeddability follows from Lemma 4.2, Proposition 4.1 and Theorem 3.2; for exactness use Theorem 3.1 instead of 3.2.

More precisely, given $R > 0$ and $\varepsilon > 0$ we fix a number δ, $0 < \delta < 1/20R$. Then

$$k^2 + 1 \geqq 2(2k+2)(2k+3)R\delta,$$

for all integers $k \geqq 0$. By assumption there is a $(k, 2L)$-separated cover $\mathcal{U}$ of X such that $\mathcal{U}$ is equi-uniformly embeddable and $k^2 + 1 \leqq 2L\delta\varepsilon$. By Lemma 4.2 the cover $\mathcal{U}_L$ has multiplicity $\leqq k + 1$ and Lebesgue number L. Proposition 4.1 provides a partition of unity subordinated to $\mathcal{U}_L$ with the following property: for all $x, y \in X$, if $d(x, y) \leqq R$ then

$$\sum_{U(L) \in \mathcal{U}_L} |\varphi_{U(L)}(x) - \varphi_{U(L)}(y)| \leqq \frac{(2k+2)(2k+3)R}{L} \leqq \frac{k^2+1}{2L\delta} \leqq \varepsilon.$$

Since the cover $\mathcal{U}$ is equi-uniformly embeddable so is the cover $\mathcal{U}_L$. We conclude the proof by applying Theorem 3.2. □

Corollary 4.5. *If a metric space X is a union of finitely many uniformly embeddable subspaces then X is uniformly embeddable. A similar result is true for exact spaces.*

Proof. By assumption there is a finite cover $\mathcal{U}$ of X with each $U \in \mathcal{U}$ uniformly embeddable. Let δ be given. Let $k + 1$ denote the cardinality of $\mathcal{U}$ and choose L such that $k^2 + 1 \leqq L\delta$. Then $\mathcal{U}$ is a (k, L)-separated cover of X and $\mathcal{U}$ is equi-uniformly embeddable. The result follows now from Theorem 4.4. □

Corollary 4.6. *If a metric space X is the union of an equi-uniformly embeddable family of subspaces $\mathcal{U}$ with the property that for every $L > 0$ there is a uniformly embeddable subspace $Y \subset X$ such that the family $\{U \backslash Y : U \in \mathcal{U}\}$ is L-separated then X is uniformly embeddable. A similar result is true for exact spaces.*

Proof. Given $\delta > 0$ we fix $L \geqq 2/\delta$. Let $\mathcal{U}$ and Y (depending on L) be as in the statement. We apply Theorem 4.4 using the $(1, L)$-separated cover of X given by the families of sets $\mathcal{U}_0 \cup \mathcal{U}_1$, where $\mathcal{U}_0 = \{Y\}$ and $\mathcal{U}_1 = \{U \backslash Y : U \in \mathcal{U}\}$. □

Corollary 4.7. *Let $p : X \to Y$ be a Lipschitz map of metric spaces. Assume that a group Γ acts by isometries on both X and Y, that the action on Y is cobounded and that p is Γ-equivariant. Assume Y has finite asymptotic dimension. If there exists $y_0 \in Y$ such that for every $n \in \mathbb{N}$ the inverse image $p^{-1}(B(y_0, n))$ is uniformly embeddable, then X is uniformly embeddable.*

Proof. This follows from Proposition 4.3 in conjunction with Corollary 3.4. □

Remark 4.8. In the previous corollary, if we assume instead that $p^{-1}(B_Y(y_0, n))$ is exact then we conclude that X is exact. The result so obtained is closely related to a result of Bell concerning Property A for groups acting on spaces of finite asymptotic dimension (compare [2], Theorem 1).

5. Extensions and free products

This section and the next are concerned with countable discrete groups. In earlier work we proved that the class of uniformly embeddable groups is closed under the formation of free products (both with and without amalgam) and certain extensions [5], [4]. In this section we briefly indicate how these permanence results follow quite easily from the general gluing technique presented above. This approach to permanence results originated with the work of Bell and Dranishnikov [2], [3]. See also Roe [15].

A countable discrete group admits a proper length function; further, any two (left invariant) metrics associated to proper length functions are coarsely equivalent. Since uniform embeddability and C^*-exactness (Property A) are coarse invariants we are free to equip our groups with conveniently chosen metrics from this class. For generalities, definitions, and a fuller discussion of these points see [5].

Theorem 5.1. *Let Γ be an extension with kernel H and quotient G, and assume that G is C^*-exact. If H is C^*-exact then Γ is C^*-exact; if H is uniformly embeddable then Γ is uniformly embeddable.*

The result concerning uniform embeddability is [5], Theorem 4.1. Our original proof was inspired by the proof of Anantharaman-Delaroche and Renault that the class of countable exact groups is closed under extensions [1], a result due to Kirchberg and Wassermann [11]. The present method provides a different and geometric (as opposed to functional analytic) proof of this result, which for this reason is included in our statement.

Outline of Proof (compare [15]). Equip the groups H, Γ and G with proper length functions and (left invariant) metrics as in [5], Section 4. Precisely, fix a proper length function ℓ_Γ on Γ and define

$$\ell_H(s) = \ell_\Gamma(s), \quad \text{for all } s \in H,$$
$$\ell_G(x) = \min\{\ell_\Gamma(g) : g \in \Gamma \text{ and } \dot{g} = x\}, \quad \text{for all } x \in G,$$

where $g \mapsto \dot{g}$ denotes the map $\Gamma \to G$. With respect to the associated metrics $H \hookrightarrow \Gamma$ is an isometry and $\Gamma \to G$ is contractive. Observe that Γ acts transitively by isometries on G. Further, by assumption G is C^*-exact, hence exact in the sense of Definition 2.7. Therefore, to apply Corollary 3.4 it suffices to show that for every $n \in \mathbb{N}$ the set

$$B(n) = \{h \in \Gamma : \ell_G(\dot{h}) \leqq n\} \subset \Gamma$$

is uniformly embeddable (respectively, exact) when equipped with the metric inherited from Γ. But, the isometry $H \hookrightarrow B(n)$ is n-dense in the sense that for every $h \in B(n)$ there exists $s \in H$ such that $d_\Gamma(h,s) \leqq n$; indeed, it suffices to put $s = hg^{-1}$ where g achieves the minimum in the definition of $\ell_G(\dot{h})$. It follows that $H \hookrightarrow B(n)$ is a coarse equivalence, indeed a quasi-isometry. □

Remark 5.2. If H and Γ are finitely generated we could, as an alternative, use metrics associated to finite, symmetric generating sets chosen as follows: a fixed generating set for H is extended to one for Γ, the image of which is a generating set for G. Having done so $\Gamma \to G$ is contractive and, although not necessarily an isometry, $H \hookrightarrow \Gamma$ is a contractive uniform embedding. Again, $H \hookrightarrow B(n)$ is a coarse equivalence.

Theorem 5.3. *Let A and B be countable discrete groups and let C be a common subgroup. If both A and B are uniformly embeddable, then the amalgamated free product $\Gamma = A *_C B$ is uniformly embeddable.*

The analogous statement for C^*-exactness is due to Dykema [6]; alternate proofs were subsequently given by Tu [18] and Bell [2] (see also [3]). Whereas our original proof of Theorem 5.3 (see [5], Theorem 5.1) was motivated by Tu's proof our current proof follows the general scheme of Bell. Whereas our original proof was rather technical some aspects of it were quite intuitive and have been incorporated into the present exposition. We recall briefly the relevant ideas; see [5], Section 5 and Appendix, for details.

Let T be the Bass-Serre tree of Γ [16]; the vertex and edge sets of T are given by

$$V = \Gamma/A \cup \Gamma/B,$$
$$E = \Gamma/C,$$

and the endpoint maps are the quotient maps $\Gamma/C \to \Gamma/A$ and $\Gamma/C \to \Gamma/B$. Observe that the vertices and edges are themselves subsets of Γ and that the vertex v is an endpoint of the edge e if and only if $e \subset v$. The action of Γ on T has two orbits on vertices (and one on edges).

Fix an integer-valued proper length function and associated left invariant metric on Γ. Let $\mathscr{X}$ be the associated tree of metric spaces; $\mathscr{X} = \{X_v, X_e\}$ is the family of metric spaces indexed by the vertices and edges of T given by

$$X_v = v \subset \Gamma, \quad X_e = e \subset \Gamma,$$

each metrized as a subspace of Γ; if the vertex v is an endpoint of the edge e the structure map $X_e \to X_v$ is the (isometric) inclusion $e \subset v$. Let X be the total space of $\mathscr{X}$. Precisely, X is the disjoint union of the vertex spaces equipped with an appropriate metric; if, for notational convenience, we denote $x \in X_v \subset X$ by x_v the metric is the largest satisfying

$$d(x_v, y_w) \leqq \begin{cases} d_\Gamma(x, y), & \text{if } v = w, \text{ so that } x = x_v,\ y = y_v \in X_v, \\ 1, & \text{if } v \neq w \text{ but } x = y \in \Gamma \end{cases}$$

(the pair (x_v, x_w) for $v \neq w$ is an adjacency). An action of Γ on X by isometries is defined by $g \cdot x_v = (gx)_{g \cdot v}$. The map $p : X \to T$ defined by $p(x_v) = v$ is equivariant; according to the formula for the metric on X in [5], Proposition 5.5, it is contractive. According to the same formula, the inclusions $X_v \to X$ are isometric, so that each X_v, with the metric from X, is isometric to one of the subgroups A or B, with the metric from Γ. In particular, the X_v are equi-uniformly embeddable.

Outline of Proof. As in [5] it suffices to show that the space X is uniformly embeddable. It is well-known that the tree T, hence also its vertex set, has finite asymptotic dimension; for a proof see [15]. In light of the observations above and Corollary 4.7 it suffices to show, fixing a vertex $v_0 \in T$, that for all $n \in \mathbb{N}$ the set

$$B(n) = \bigcup \{X_v : d_T(v_0, v) \leqq n\} \subset X,$$

where $d_T(v_0, v)$ denotes the distance in the Bass-Serre tree, is uniformly embeddable when equipped with the metric inherited from X. We proceed by induction on n. Certainly $B(0) = X_{v_0}$ is uniformly embeddable. For the induction step assume that $B(n-1)$ is uniformly embeddable. We apply the infinite union theorem (Corollary 4.6) to the decomposition of $B(n)$ as a disjoint union

$$B(n) = B(n-1) \cup \bigcup_{d_T(v_0, v) = n} X_v$$

of equi-uniformly embeddable subsets. Let $L > 0$. For every v such that $d_T(v_0, v) = n$ let $Y_v \subset X_v$ be the set of points at a distance not greater than $L/2$ from $B(n-1)$ and let

$$Y = B(n-1) \cup \bigcup_{d_T(v_0, v) = n} Y_v.$$

By the manner in which Y is defined, the isometric inclusion $B(n-1) \hookrightarrow Y$ is $L/2$-dense, so that Y is coarsely equivalent, indeed quasi-isometric to $B(n-1)$. Thus Y is uniformly embeddable. Further, the family $\{X_v \backslash Y : d_T(v_0, v) = n\}$ is L-separated. Indeed, if $v \neq w$ are each a distance n from v_0 in T, $x_v \in X_v \backslash Y$ and $y_w \in X_w \backslash Y$ then $d(x_v, y_w) \geqq L$, again by the formula for the metric on X in [5], Proposition 5.5. □

6. Relatively hyperbolic groups

Let Γ be a finitely generated group which is hyperbolic relative to a finite family $\{H_1, \ldots, H_n\}$ of subgroups. We prove that Γ is uniformly embeddable if and only if each subgroup H_i is uniformly embeddable. There are two analogous results in the literature: Osin proved an analogous statement for finite asymptotic dimension [12] and Ozawa proved an analogous statement for exactness [14]. We rely heavily on Osin's method (Ozawa's method is completely different), and are indebted to Ozawa for alerting us to Osin's paper.

If A is a symmetric set of generators of Γ, we denote by d_A the corresponding left-invariant metric on Γ. If B is another such set with $A \subset B$ the identity map $p : (\Gamma, d_A) \to (\Gamma, d_B)$ is equivariant and $d_B\big(p(x), p(y)\big) \leqq d_A(x, y)$. Let S be a a finite symmetric set generating Γ. Let

$$\mathscr{H} = \bigcup_k (H_k \backslash e).$$

Let d_S and $d_{S \cup \mathscr{H}}$ be the left invariant metrics on Γ induced by S and $S \cup \mathscr{H}$, respectively. For $n \geqq 1$, let

$$B(n) = \{g \in \Gamma : d_{S \cup \mathscr{H}}(g, e) \leqq n\}.$$

We always view $B(n)$ as a subspace of Γ equipped with the metric d_S. The following useful recursive decomposition of $B(n)$ is contained in the proof of [12], Lemma 12:

$$B(1) = S \cup \left(\bigcup_k H_k \right), \tag{6}$$

$$B(n) = \left(\bigcup_k B(n-1)H_k \right) \cup \left(\bigcup_{x \in S} B(n-1)x \right), \tag{7}$$

$$B(n-1)H_k = \bigsqcup_{g \in R(n-1)} gH_k, \tag{8}$$

where the final equality represents a partition of $B(n-1)H_k$ into disjoint cosets according to a fixed set of coset representatives, $R(n-1) \subset B(n-1)$.

Proposition 6.1 (Osin). *For every $L > 0$ there exists $\kappa(L) > 0$ such that if*

$$Y = \{x \in \Gamma : d_S(x, B(n-1)) \leqq \kappa(L)\}$$

then for each k

$$B(n-1)H_k \subset Y \cup \left(\bigcup_{g \in R(n-1)} gH_k \backslash Y \right) \tag{9}$$

and the subspaces $gH_k \backslash Y$, $g \in R(n-1)$ of (Γ, d_S) are L-separated.

Proof. The statement is implicit in the proof of [12], Lemma 12. □

Proposition 6.2. *If each H_k is uniformly embeddable so is $B(n)$. A similar statement is true for exactness.*

Proof. The proof is by induction. For the basis, observe that $B(1)$ is uniformly embeddable by (6) and the finite union theorem (Corollary 4.5). For the induction step, assume that $B(n-1)$ is uniformly embeddable. Using again the finite union theorem and (7) we are reduced to verifying that each $B(n-1)H_k$ is uniformly embeddable. This follows from the infinite union theorem (Corollary 4.6) and Proposition 6.1.

The proof for exactness is analogous (compare [2]). □

Osin also proves the following result [12] (although not explicitly stated, the result is the content of Lemmas 17, 18 and 19):

Proposition 6.3 (Osin). *The metric space $(\Gamma, d_{S \cup \mathscr{H}})$ has finite asymptotic dimension.* □

Theorem 6.4. *Let Γ be a finitely generated group which is hyperbolic relative to a finite family $\{H_1, \dots, H_n\}$ of subgroups. Then Γ is uniformly embeddable in a Hilbert space if and only if each subgroup H_i is uniformly embeddable in a Hilbert space.*

Proof. If Γ is uniformly embeddable, then so are its subgroups, the H_k. For the converse we apply Corollary 4.7 to the isometric actions of Γ on the metric spaces $X = (\Gamma, d_S)$,

$Y = (\Gamma, d_{S\cup\mathscr{H}})$, where p is the identity map and $y_0 = e$. Then $B(n) = p^{-1}(B_Y(e,n))$ which is uniformly embeddable by Proposition 6.2. □

It is clear that the analogous result for C^*-exact groups, due to Ozawa [14], can be recovered arguing as above.

References

[1] *C. Anantharaman-Delaroche* and *J. Renault*, Amenable groupoids, Enseign. Math. **36** (2000), with a foreword by Georges Skandalis and Appendix B by E. Germain.

[2] *G. Bell*, Property A for groups acting on metric spaces, Topol. Appl. **130** (2003), 239–251.

[3] *G. Bell* and *A. Dranishnikov*, On asymptotic dimension of groups, Alg. Geom. Topol. **1** (2001), 57–71.

[4] *X. Chen*, *M. Dadarlat*, *E. Guentner*, and *G. Yu*, Uniform embeddings into Hilbert space and free products of groups, J. Funct. Anal. **205** (2003), 168–179.

[5] *M. Dadarlat* and *E. Guentner*, Constructions preserving Hilbert space uniform embeddability of discrete groups, Trans. Amer. Math. Soc. **355** (2003), 3253–3275.

[6] *K. Dykema*, Exactness of reduced amalgamated free product C^*-algebras, preprint 1999.

[7] *S. Ferry*, *A. Ranicki*, and *J. Rosenberg* (eds.), Novikov conjectures, index theorems and rigidity, London Math. Soc. Lect. Notes **226**, **227**, Cambridge University Press, 1995.

[8] *M. Gromov*, Asymptotic invariants of infinite groups, Geometric Group Theory, A. Niblo and M. Roller, eds., London Math. Soc. Lect. Notes **182**, Cambridge University Press (1993), 1–295.

[9] *E. Guentner* and *J. Kaminker*, Exactness and the Novikov conjecture, and addendum, Topology **41** (2002), no. 2, 411–418, 419–420.

[10] *N. Higson* and *J. Roe*, Amenable actions and the Novikov conjecture, J. reine angew. Math. **519** (2000), 143–153.

[11] *E. Kirchberg* and *S. Wassermann*, Permanence properties of C^*-exact groups, Docum. Math. **4** (1999), 513–558.

[12] *D. Osin*, Asymptotic dimension of relatively hyperbolic groups, preprint GR/0411585, 2004.

[13] *N. Ozawa*, Amenable actions and exactness for discrete groups, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 8, 691–695.

[14] *N. Ozawa*, Boundary amenability of relatively hyperbolic groups, preprint 2004.

[15] *J. Roe*, Lectures on coarse geometry, Univ. Lect. Ser. **31**, American Mathematical Society, Providence, RI, 2003.

[16] *J. P. Serre*, Trees, Springer, New York 1980, Translation from French of "Arbres, Amalgames, SL_2", Astérisque **46**.

[17] *G. Skandalis*, *J. L. Tu*, and *G. Yu*, Coarse Baum-Connes conjecture and groupoids, Topology **41** (2002), 807–834.

[18] *J. L. Tu*, Remarks on Yu's Property A for discrete metric spaces and groups, Bull. Soc. Math. France **129** (2001), 115–139.

[19] *G. Yu*, The Coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Invent. Math. **139** (2000), 201–240.

Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067
e-mail: mdd@math.purdue.edu

Mathematics Department, University of Hawai'i, Mānoa, 2565 McCarthy Mall, Honolulu, HI 96822
e-mail: erik@math.hawaii.edu

Eingegangen 7. Juli 2005