Extensions of Quasidiagonal C^*-algebras and K-theory

Nathanial P. Brown [1]
UC-Berkeley
Berkeley, California 94720
nbrown@math.berkeley.edu

Marius Dadarlat[2]
Purdue University
West Lafayette, Indiana 47907
mdd@math.purdue.edu

Abstract

Let $0 \to I \to E \to B \to 0$ be a short exact sequence of C*-algebras where E is separable, I is quasidiagonal (QD) and B is nuclear, QD and satisfies the UCT. It is shown that if the boundary map $\partial : K_1(B) \to K_0(I)$ vanishes then E must be QD also.

A Hahn-Banach type property for K_0 of QD C^*-algebras is also formulated. It is shown that every nuclear QD C^*-algebra has this K_0-Hahn-Banach property if and only if the boundary map $\partial : K_1(B) \to K_0(I)$ (from above) always completely determines when E is QD in the nuclear case.

1 Introduction

Quasidiagonal (QD) C^*-algebras are those which enjoy a certain finite dimensional approximation property. (See [Vo2], [Br3] for surveys of the theory of QD C^*-algebras.) While these finite dimensional approximations have certainly lead to a better understanding of the structure of QD C^*-algebras, there are a number of very basic open questions. For example, assume that $0 \to I \to E \xrightarrow{\pi} B \to 0$ is a split exact sequence (i.e. there exists a *-homomorphism $\varphi : B \to E$ such that $\pi \circ \varphi = id_B$) where both I and B are QD. It is not known whether E must be QD (and, in fact, it is not even clear what to expect).

In this paper we study the extension problem for QD C^*-algebras and it's relation to some natural questions concerning K-theory of QD C^*-algebras.

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Our techniques rely heavily on Kasparov's theory of extensions and thus we will always need some nuclearity assumptions.

For example, adapting techniques found in [Sp] we will show (Theorem 3.4) that if $0 \to I \to E \to B \to 0$ is short exact where E is separable, I is QD, B is nuclear, QD and satisfies the Universal Coefficient Theorem (UCT) and the boundary map $\partial : K_1(B) \to K_0(I)$ vanishes then E must be QD also. It follows that if $K_1(B) = 0$ then E is always QD, which generalizes work of Eilers, Loring and Pedersen ([ELP]). As another application we observe that in the case that I is the compact operators our result implies that E is QD if and only if the (class of the) extension is in the kernel of the natural map $Ext(B) \to Hom(K_1(B), \mathbb{Z})$, where $Ext(B)$ denotes the classical BDF group (recall that we are assuming B is nuclear and hence $Ext(B)$ is a group). Also, we verify a conjecture of [BK], stating that an asymptotically split extension of NF algebras is NF, under the additional assumption that the quotient algebra satisfies the UCT of [RS].

We then study the general extension problem. Now let $0 \to I \to E \to B \to 0$ be exact where E is separable and nuclear, I is QD and B is QD and satisfies the UCT. Based on previous work of Spielberg ([Sp]) it is reasonable to expect that in this case E will be QD if and only if $\partial(K_1(B)) \cap K_0^+(I) = \{0\}$, where $K_0^+(I) = \{0\}$ denotes the positive cone of $K_0(I)$. Though we are unable to resolve this question we do show that it is equivalent to some other natural questions concerning the K-theory of QD C^*-algebras and that in order to solve the general extension problem it suffices to prove the special case that $B = C(\mathbb{T})$ (see Theorem 4.11).

The first equivalent K-theory question is: If A is nuclear, separable and QD and $G \subset K_0(A)$ is a subgroup such that $G \cap K_0^+(A) = 0$ then can one always find an embedding $\rho : A \hookrightarrow C$ where C is QD and $\rho_*(G) = 0$? The condition $G \cap K_0^+(A) = 0$ is easily seen to be necessary and hence the question is whether or not it is sufficient. The second K-theory question asks whether every nuclear QD C^*-algebra satisfies what we call the *K_0-Hahn-Banach property* (see Definition 4.7). Roughly speaking this K_0-Hahn-Banach property states that if $x \in K_0(A)$ and $\pm x \notin K_0^+(A)$ then one can always find finite dimensional approximate morphisms (i.e. "functionals") which separate x from $K_0^+(A)$. (Due to possible perforation in $K_0(A)$ this statement is not quite correct, but it conveys the main idea.) Determining whether every nuclear QD algebra satisfies the K_0-Hahn-Banach property is of independent interest as our inability to understand how well finite dimensional approximate morphisms read K-theory has been a major obstacle in the classification program.

In section 2 we review the necessary theory of extensions and prove a few simple results needed later. In section 3 we handle the case when $\partial : K_1(B) \to K_0(I)$ vanishes. In section 4 we turn to the general extension problem and show equivalence with the K-theory questions described above.

The present work is related to work of Salinas [Sa1], [Sa2] and Schochet [Sch]. Those authors study the quasidiagonality of extensions $0 \to I \to E \to B \to 0$ (i.e. the question of whether or not I contains an approximate unit of projections which is quasicentral in E) whereas we study the QD of the C*-algebra E. The two questions are different even if I is the compact operators. Indeed, while the quasidiagonality of $0 \to \mathcal{K} \to E \to B \to 0$ does imply the QD of E, the converse implication is false (see Section 3).

2 Preliminaries and Trivial Extensions.

Most of this section is devoted to reviewing definitions, introducing notation and recalling some standard facts about extensions of C^*-algebras. However, at the end we prove a few simple facts which will be needed later. The main result states that quasidiagonality is preserved in split extensions provided that either the ideal or the quotient is a nuclear C^*-algebra (see Proposition 2.5).

For a comprehensive introduction to the aspects of extension theory which we will need we recommend looking at [Bl, Section 15]. For any C^*-algebra I we will let $M(I)$ be it's multiplier algebra and $Q(I) = M(I)/I$ be it's corona algebra. Let $\pi : M(I) \to Q(I)$ be the quotient map.

If E is any C^*-algebra containing I as an ideal and $B = E/I$ then there exists a unique *-homomorphism $\rho : E \to M(I)$ such that $\rho(I) = I$ and hence an induced *-homomorphism $\gamma : B \to Q(I)$. The map γ is injective if and only if ρ is in injective if and only if I sits as an essential ideal in E. Conversely, given a C^*-algebra B and a *-homomorphism $\gamma : B \to Q(I)$ we can construct the pullback which, by definition, is the C^*-algebra

$$E(\gamma) = \{x \oplus b \in M(I) \oplus B : \pi(x) = \gamma(b)\}.$$

This gives a short exact sequence $0 \to I \to E(\gamma) \to B \to 0$. Moreover, if $B = E/I$ with induced map $\gamma : B \to Q(I)$ then there is an induced

*-isomorphism $\Phi : E \to E(\gamma)$ with commutativity in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\Phi}\downarrow & & \| & & \\ 0 & \longrightarrow & I & \longrightarrow & E(\gamma) & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

Hence there is a one to one correspondence between extensions of I by B and *-homomorphisms $\gamma : B \to Q(I)$. As is standard, we will refer to a *-homomorphism $\gamma : B \to Q(I)$ as a *Busby invariant* and freely use the above correspondence between Busby invariants and extensions.

When I is stable (i.e. $I \cong \mathcal{K} \otimes I$, where $\mathcal{K}$ denotes the compact operators on a separable infinite dimensional Hilbert space) there is a natural way of adding two extensions which we now describe. Any isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$ induces an isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \otimes I \cong \mathcal{K} \otimes I$ which then gives isomorphisms $M_2(\mathbb{C}) \otimes M(\mathcal{K} \otimes I) \cong M(\mathcal{K} \otimes I)$ and $M_2(\mathbb{C}) \otimes Q(\mathcal{K} \otimes I) \cong Q(\mathcal{K} \otimes I)$. Thus if we are given two Busby invariants $\gamma_1, \ \gamma_2 : B \to Q(\mathcal{K} \otimes I)$ we can define a new Busby invariant $\gamma_1 \oplus \gamma_2$ by

$$(\gamma_1 \oplus \gamma_2)(b) = \begin{pmatrix} \gamma_1(b) & 0 \\ 0 & \gamma_2(b) \end{pmatrix} \in M_2(\mathbb{C}) \otimes Q(\mathcal{K} \otimes I) \cong Q(\mathcal{K} \otimes I).$$

Of course the Busby invariant $\gamma_1 \oplus \gamma_2$ constructed in this way will depend on the particular isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$. To remedy this we say that two Busby invariants $\gamma_1, \ \gamma_2$ are *strongly equivalent* if there exists a unitary $u \in M(I)$ such that $\mathrm{Ad}\pi(u)\big(\gamma_1(b)\big) = \pi(u)\gamma_1(b)\pi(u^*) = \gamma_2(b)$, for all $b \in B$, where $\pi : M(I) \to Q(I)$ is the quotient map. Note that if γ_1 and γ_2 are strongly equivalent then $E(\gamma_1)$ and $E(\gamma_2)$ are isomorphic C^*-algebras. Indeed, the map $E(\gamma_1) \to E(\gamma_2)$, $x \oplus b \mapsto uxu^* \oplus b$ is easily seen to be an isomorphism. Since any isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$ is implemented by a unitary we see that $\gamma_1 \oplus \gamma_2$ is unique up to strong equivalence. In particular, the isomorphism class of the C^*-algebra $E(\gamma_1 \oplus \gamma_2)$ does not depend on the choice of isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$.

A Busby invariant τ is called *trivial* if it lifts to a *-homomorphism $\varphi : B \to M(I)$ (i.e. $\pi \circ \varphi = \gamma$). A Busby invariant $\gamma : B \to Q(\mathcal{K} \otimes I)$ is called *absorbing* if $\gamma \oplus \tau$ is strongly equivalent to γ for every trivial τ. Note that if γ is absorbing then so is $\tilde{\gamma} \oplus \gamma$ for any $\tilde{\gamma}$. In particular if γ is absorbing then γ is injective. Note also that if τ_1 and τ_2 are both trivial and absorbing then τ_1, $\tau_1 \oplus \tau_2$ and τ_2 are all strongly equivalent. Thus we get the following fact.

Lemma 2.1 *If $\tau_1,\ \tau_2 : B \to Q(\mathcal{K} \otimes I)$ are both trivial and absorbing then $E(\tau_1) \cong E(\tau_2)$.*

Another simple fact we will need is the following.

Lemma 2.2 *If $\gamma,\ \tau : B \to Q(\mathcal{K} \otimes I)$ are Busby invariants with τ trivial then there is a natural embedding $E(\gamma) \hookrightarrow E(\gamma \oplus \tau)$.*

Proof. Let $\varphi : B \to M(I)$ be a lifting of τ. Define a map $E(\gamma) \to E(\gamma \oplus \tau)$ by

$$x \oplus b \mapsto \begin{pmatrix} x & 0 \\ 0 & \varphi(b) \end{pmatrix} \oplus b.$$

Evidently this map is an injective $*$-homomorphism. □

The following generalization of Voiculescu's Theorem, which is due to Kasparov, will be crucial in what follows.

Theorem 2.3 *([Bl, Thm. 15.12.4]) Assume that B is separable, I is σ-unital and either B or I is nuclear. Let $\rho : B \to B(H)$ be a faithful representation such that H is separable, $\rho(B) \cap \mathcal{K}(H) = \{0\}$ and the orthogonal complement of the nondegeneracy subspace of $\rho(B)$ (i.e. $H \ominus \overline{\rho(B)H}$) is infinite dimensional. Regarding $B(H) \cong B(H) \otimes 1 \subset M(\mathcal{K} \otimes I)$ as scalar operators we get a short exact sequence*

$$0 \to \mathcal{K} \otimes I \to \rho(B) \otimes 1 + \mathcal{K} \otimes I \to B \to 0.$$

If τ denotes the induced Busby invariant then τ is both trivial and absorbing.

We define an equivalence relation on the set of Bubsy invariants $B \to Q(\mathcal{K} \otimes I)$ by saying γ is related to $\tilde{\gamma}$ if there exist trivial Busby invariants $\tau, \tilde{\tau}$ such that $\gamma \oplus \tau$ is strongly equivalent to $\tilde{\gamma} \oplus \tilde{\tau}$. Taking the quotient by this relation yields the semigroup $Ext(B, \mathcal{K} \otimes I)$. The image of a map $\gamma : B \to Q(\mathcal{K} \otimes I)$ in $Ext(B, \mathcal{K} \otimes I)$ is denoted $[\gamma]$. Note that all trivial Busby invariants give rise to the same class denoted by $0 \in Ext(B, \mathcal{K} \otimes I)$ and this class is a neutral element (i.e. identity) for the semigroup. Note also that if $[\gamma] = 0 \in Ext(B, \mathcal{K} \otimes I)$ then it does not follow that γ is trivial. However, it does follow that if τ is a trivial absorbing Busby invariant then so is $\gamma \oplus \tau$.

We are almost ready to prove the main result of this section. We just need one more definition.

Definition 2.4 *If* $0 \to I \to E \to B \to 0$ *is an exact sequence with Busby invariant* γ *then we let* $\gamma^s : \mathcal{K} \otimes B \to Q(\mathcal{K} \otimes I)$ *denote the stabilization of* γ*. That is,* γ^s *is the Busby invariant of the exact sequence* $0 \to \mathcal{K} \otimes I \to \mathcal{K} \otimes E \to \mathcal{K} \otimes B \to 0$*.*

Note that there is always an embedding $E \cong E(\gamma) \hookrightarrow E(\gamma^s)$.

Proposition 2.5 *Let* $0 \to I \to E \to B \to 0$ *be exact with Busby invariant* γ*. If both* I *and* B *are QD,* B *is separable,* I *is* σ*-unital, either* I *or* B *is nuclear and* $[\gamma^s] = 0 \in Ext(\mathcal{K} \otimes B, \mathcal{K} \otimes I)$ *then* E *is also QD.*

Proof. Since quasidiagonality passes to subalgebras, it suffices to show that if $\tau : \mathcal{K} \otimes B \to Q(\mathcal{K} \otimes I)$ is a trivial absorbing Busby invariant (which exists by Theorem 2.3) then $E(\tau)$ is QD. Indeed, by Lemmas 2.1, 2.2 and the definition of $Ext(\mathcal{K} \otimes B, \mathcal{K} \otimes I)$ we have the inclusions

$$E \hookrightarrow E(\gamma^s) \hookrightarrow E(\gamma^s \oplus \tau) \cong E(\tau).$$

To prove that $E(\tau)$ is QD we may assume (again by Lemma 2.1) that τ arises from the particular extension described in Theorem 2.3. However for that extension it is easy to see that $E(\tau) \hookrightarrow (\rho(B) + \mathcal{K}) \otimes \tilde{I}$, where $\tilde{I}$ is the unitization of I. But since $\rho(B) \cap \mathcal{K} = \{0\}$ it follows that $\rho(B) + \mathcal{K}$ is QD ([Br3, Thm. 3.11]). Hence $(\rho(B) + \mathcal{K}) \otimes \tilde{I}$ is also QD as a minimal tensor product QD C*-algebras ([Br3, Prop. 7.5]). □

Note that the above proposition covers the case of split extensions (i.e. when γ is trivial).

3 When $\partial : K_1(B) \to K_0(I)$ is zero.

The main result of this section (Theorem 3.4) states that if the boundary map $\partial : K_1(B) \to K_0(I)$ coming from an exact sequence $0 \to I \to E \to B \to 0$ is zero then E will be QD whenever I is QD and B is nuclear, QD and satisfies the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet ([RS]). The main ideas in the proof are inspired by work of Spielberg ([Sp]). We also discuss a few consequences of our result, including generalization of work of Eilers-Loring-Pedersen ([ELP]) and a partial solution to a conjecture of Blackadar and Kirchberg [BK].

Definition 3.1 An embedding $I \hookrightarrow J$ is called *approximately unital* if it takes an approximate unit of I to an approximate unit of J.

In this case there is a natural inclusion $M(I) \hookrightarrow M(J)$ which induces an inclusion $Q(I) \hookrightarrow Q(J)$ [Pe, 3.12.12]. Hence for any Busby invariant $\gamma : B \to Q(I)$ there is an induced Busby invariant $\eta : B \to Q(J)$ with commutativity in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E(\gamma) & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & J & \longrightarrow & E(\eta) & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

Moreover, the two vertical maps on the left are injective.

There are two ways of producing approximately unital embeddings which we will need. The first is $I \hookrightarrow I \otimes A$, for some *unital* C^*-algebra A. If $\{e_\lambda\}$ is an approximate unit of I then, of course, $e_\lambda \otimes 1_A$ will be an approximate unit of $I \otimes A$. The other is to start with an arbitrary embedding $I \hookrightarrow J'$ and define J to be the hereditary subalgebra in J' generated by I. That is, define J to be the closure of $\cup_\lambda e_\lambda J' e_\lambda$. One easily checks that J is then a hereditary subalgebra of J' and the embedding $I \hookrightarrow J$ is approximately unital.

In the theory of separable QD C^*-algebras there are some *nonseparable* algebras which play a key role. The first is the direct product $\Pi_i M_{n_i}(\mathbb{C})$ for some sequence of integers $\{n_i\}$. This algebra is the multiplier algebra of the direct sum $\oplus_i M_{n_i}(\mathbb{C})$. If H is any separable Hilbert space then we can always find a decomposition $H = \oplus_i \mathbb{C}^{n_i}$ and then we have natural inclusions $\oplus_i M_{n_i}(\mathbb{C}) \hookrightarrow \mathcal{K}(H)$, $\Pi_i M_{n_i}(\mathbb{C}) \hookrightarrow B(H)$ and $Q(\oplus_i M_{n_i}(\mathbb{C})) \hookrightarrow Q(\mathcal{K}(H))$. Another algebra which we will need is $\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$.

Lemma 3.2 *Let $J \subset \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ be a hereditary subalgebra containing $\mathcal{K}(H)$. Then $K_1(J) = 0$.*

Proof. Letting $\pi : B(H) \to Q(H)$ be the quotient map we have that $\pi(J)$ is a hereditary subalgebra of $Q(\oplus_i M_{n_i}(\mathbb{C}))$ (use the fact that if $0 \le a \in J, b \in Q(\oplus_i M_{n_i}(\mathbb{C}))$ and $0 \le b \le \pi(a)$ then there exists $0 \le c \in \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ such that $c \le a$ and $\pi(c) = b$; [Da, Cor. IX.4.5]. Also, the exact sequence $0 \to \mathcal{K}(H) \to J \to \pi(J) \to 0$ is a *quasidiagonal extension* (i.e. $\mathcal{K}(H)$ contains an approximate unit *of projections which is quasicentral in* J). Hence [BD, Thm. 8], states that we have a short exact sequence

$$0 \to K_1(\mathcal{K}(H)) \to K_1(J) \to K_1(\pi(J)) \to 0.$$

Thus it suffices to show that $K_1(X) = 0$ for any hereditary subalgebra X of $Q(\oplus_i M_{n_i}(\mathbb{C}))$.

But if $X \subset Q(\oplus_i M_{n_i}(\mathbb{C}))$ is a hereditary subalgebra then we can find a quasidiagonal extension

$$0 \to \oplus_i M_{n_i}(\mathbb{C}) \to Y \to X \to 0,$$

where $Y \subset \Pi_i M_{n_i}(\mathbb{C})$ is a hereditary subalgebra. Applying [BD, Thm. 8] again it suffices to show that every hereditary subalgebra of $\Pi_i M_{n_i}(\mathbb{C})$ has trivial K_1-group.

But, if $Y \subset \Pi_i M_{n_i}(\mathbb{C})$ is a hereditary σ-unital subalgebra then Y has an increasing approximate unit consisting of projections, say $\{e_n\}$ ([BP]). Hence

$$K_1(Y) = \lim K_1(e_n \Pi_i M_{n_i}(\mathbb{C}) e_n),$$

since $Y = \lim e_n \Pi_i M_{n_i}(\mathbb{C}) e_n$ (by heredity). But for each n it is clear that $e_n \Pi_i M_{n_i}(\mathbb{C}) e_n$ is isomorphic to $\Pi_i M_{k_i}(\mathbb{C})$ for some integers $\{k_i\}$ and hence $K_1(e_n \Pi_i M_{n_i}(\mathbb{C}) e_n) = 0$. □

Proposition 3.3 *Let I be a separable QD C^*-algebra. Then there exists an approximately unital embedding $I \hookrightarrow J$, where J is a σ-unital QD C^*-algebra with $K_1(J) = 0$.*

Proof. Let $\rho : I \to B(H)$ be a nondegenerate faithful representation such that $\rho(I) \cap \mathcal{K}(H) = \{0\}$. By [Br3, Prop. 5.2], there exists a decomposition $H = \oplus_i \mathbb{C}^{n_i}$ such that $\rho(I) \subset \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$. Let J be the hereditary subalgebra of $\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ generated by $\rho(I)$. The conclusion follows from the previous lemma. □

For the remainder of this section we will let $\mathcal{U} = \otimes_n M_n(\mathbb{C})$ be the Universal UHF algebra (i.e. the UHF algebra with $K_0(\mathcal{U}) = \mathbb{Q}$). For any Busby invariant $\gamma : B \to Q(J)$ we let $\gamma^{\mathbb{Q}}$ denote the Busby invariant coming from the short exact sequence

$$0 \to J \otimes \mathcal{U} \to E(\gamma) \otimes \mathcal{U} \to B \otimes \mathcal{U} \to 0.$$

Theorem 3.4 *Let $0 \to I \to E \to B \to 0$ be a short exact sequence where E is separable, I is QD and B is nuclear, QD and satisfies the UCT. If the induced map $\partial : K_1(B) \to K_0(I)$ is zero then E is QD.*

Proof. Let γ be the Busby invariant of the exact sequence in question. By the previous proposition we can find an approximately unital embedding $I \hookrightarrow J$, where J is QD with $K_1(J) = 0$. By the remarks following Definition 3.1 we have an inclusion $E \hookrightarrow E(\eta)$ where $\eta : B \to Q(J)$ is the induced Busby invariant. By naturality we then have that both index maps $\partial : K_1(B) \to K_0(J)$ and $\partial : K_0(B) \to K_1(J)$ are zero. Hence the index maps arising from the stabilization $\eta^s : B \otimes \mathcal{K} \to Q(J \otimes \mathcal{K})$ are also zero.

Now, if it happens that $K_0(J)$ is a divisible group then the Universal Coefficient Theorem would imply that $[\eta^s] = 0 \in Ext(B \otimes \mathcal{K}, J \otimes \mathcal{K})$ and so by Proposition 2.5 we would be done. Of course this will not be true in general and so may have to replace η^s with $(\eta^s)^{\mathbb{Q}}$. But applying naturality one more time, both boundary maps on K-theory arising from $(\eta^s)^{\mathbb{Q}}$ will also vanish. Hence the theorem follows from the inclusions $E \hookrightarrow E(\eta) \hookrightarrow E(\eta^s) \hookrightarrow E((\eta^s)^{\mathbb{Q}})$ together with Proposition 2.5 applied to $(\eta^s)^{\mathbb{Q}}$. □

In the case that the ideal is nuclear and the quotient is an AF algebra, the next result was obtained by Eilers, Loring and Pedersen ([ELP, Cor. 4.6]).

Corollary 3.5 *Assume that B is a separable nuclear QD C^*-algebra satisfying the UCT and with $K_1(B) = 0$. For any separable QD C^*-algebra I and Busby invariant $\gamma : B \to Q(I)$ we have that $E(\gamma)$ is QD.*

This corollary actually extends to the case where $K_1(B)$ is a torsion group since we can tensor any short exact sequence with $\mathcal{U}$ and $K_1(B \otimes \mathcal{U}) = 0$ in this case. For example, this would cover the case that $B = C_0(\mathbb{R}) \otimes \mathcal{O}_n$, $(2 \leq n \leq \infty)$, where $\mathcal{O}_n$ denotes the Cuntz algebra on n generators. Similarly, it is clear that Theorem 3.4 is valid under the weaker hypothesis that $\partial(K_1(B))$ is contained in the torsion subgroup of $K_0(I)$.

Definition 3.6 For any two QD C^*-algebras I, B let $Ext_{QD}(B, \mathcal{K} \otimes I) \subset Ext(B, \mathcal{K} \otimes I)$ denote the set of classes of Busby invariants γ such that $E(\gamma)$ is QD.

It is easy to check that if $[\gamma] = [\tilde{\gamma}] \in Ext(B, \mathcal{K} \otimes I)$ then $E(\gamma)$ is QD if and only if $E(\tilde{\gamma})$ is QD and hence $Ext_{QD}(B, \mathcal{K} \otimes I)$ is well defined. It is also easy to see that $Ext_{QD}(B, \mathcal{K} \otimes I)$ is a sub-semigroup of $Ext(B, \mathcal{K} \otimes I)$. Finally, we remark that in the case $I = \mathbb{C}$ we do *not* get the semigroup $Ext_{qd}(B, \mathcal{K})$ defined by Salinas; it follows from Corollary 3.7 below, however, that we do get what he called $Ext_{bqt}(B, \mathcal{K})$ in this case (see [Sa1, Definitions 2.7, 2.12 and Thm. 2.14]). One has $Ext_{qd}(B, \mathcal{K}) \subset Ext_{QD}(B, \mathcal{K})$. The elements of $Ext_{QD}(B, \mathcal{K})$ corresponds to C*-algebras $E(\gamma)$ that are QD whereas $[\gamma] \in Ext_{qd}(B, \mathcal{K})$ if the only if the extension $0 \to \mathcal{K} \to E(\gamma) \to B \to 0$ is QD i.e. the concrete set $E(\gamma) \subset M(\mathcal{K})$ is QD.

Recall that there is a natural group homomorphism $\Phi : Ext(B, \mathcal{K} \otimes I) \to Hom(K_1(B), K_0(I))$ taking a Busby invariant to the corresponding boundary map on K-theory. From Theorem 3.4 it follows that we always have an inclusion $Ker(\Phi) \subset Ext_{QD}(B, \mathcal{K} \otimes I)$, when B is nuclear, QD and satisfies the UCT. In general this inclusion will be proper, but we now describe a class of algebras for which we have equality.

There is a natural semigroup $K_0^+(I) \subset K_0(I)$, called the *positive cone*, given by

$$K_0^+(I) = \bigcup_{n \in \mathbb{N}} \{x \in K_0(I) : x = [p], \text{ for some projection } p \in M_n(I)\}.$$

When I is unital this semigroup generates $K_0(I)$ but can also be trivial in general (e.g. if I is stably projectionless). The natural isomorphism $K_0(I) \cong K_0(\mathcal{K} \otimes I)$ induced by an embedding $I = e_{11} \otimes I \subset \mathcal{K} \otimes I$, where e_{11} is a minimal projection in $\mathcal{K}$, preserves the positive cones. We say that $K_0(I)$ is *totally ordered* if for every $x \in K_0(I)$ either x or $-x$ is an element of $K_0^+(I)$.

Corollary 3.7 *Assume I is separable, QD and $K_0(I)$ is totally ordered. For any separable, nuclear, QD algebra B which satisfies the UCT we have that $Ext_{QD}(B, \mathcal{K} \otimes I) = Ker(\Phi)$.*

Proof. We only have to show $Ext_{QD}(B, \mathcal{K} \otimes I) \subset Ker(\Phi)$. So let $[\gamma] \in Ext(B, \mathcal{K} \otimes I)$. If $E(\gamma)$ is a stably finite C^*-algebra then a result of Spielberg (see Proposition 4.1 of the next section), together with the assumption that $K_0(I)$ is totally ordered, implies that $[\gamma] \in Ker(\Phi)$. But since QD implies stably finite ([Br3, Prop. 3.19]) we have that if $[\gamma] \in Ext_{QD}(B, \mathcal{K} \otimes I)$ then $[\gamma] \in Ker(\Phi)$. □

The classic example for which $K_0(I)$ is totally ordered is the case when $I = \mathcal{K}$. In this setting the corollary above is very similar to a result of Salinas' which describes the closure of $0 \in Ext(B, \mathcal{K})$ in terms of quasidiagonality

([Sa1, Thm. 2.9]). See also [Sa1, Thm. 2.14] for another characterization of $Ext_{QD}(B, \mathcal{K})$ in terms of bi-quasitriangular operators. For a K-theoretical characterization of $Ext_{qd}(B, \mathcal{K})$ see [Sch, Theorem 8.3].

The class of NF algebras introduced in [BK] coincides with the class of separable QD nuclear C*-algebras. It was conjectured in [BK, Conj. 7.1.6] that an asymptotically split extension of NF algebras is NF. We can verify the conjecture under an additional asumption.

Corollary 3.8 *Let $0 \to I \to E \to B \to 0$ be an asymptotically split extension with I and B NF algebras. If B satisfies the UCT, then E is NF.*

Proof. Both index maps are vanishing since the extension is asymptotically split. The conclusion follows from Theorem 3.4. □

4 Extensions and K-theory

In this section we show that the general extension problem for nuclear QD C^*-algebras is equivalent to some natural K-theoretic questions.

We begin by recalling a result of Spielberg which solves the extension problem for stably finite C^*-algebras and shows that it is completely governed by K-theory.

Proposition 4.1 *[Sp, Lemma 1.5] Let $0 \to I \to E \to B \to 0$ be short exact where both I and B are stably finite. Then E is stably finite if and only if $\partial(K_1(B)) \cap K_0^+(I) = \{0\}$, where $\partial : K_1(B) \to K_0(I)$ is the boundary map of the sequence.*

In [BK, Question 7.3.1], it is asked whether every nuclear stably finite C^*-algebra is QD. Support for an affirmative answer to this question is provided by a number of nontrivial examples ([Pi], [Sp], [Br1], [Br2]). In fact, Corollary 3.7 above also provides examples since the proof shows the equivalence of quasidiagonality and stable finiteness (in fact we did not even assume nuclearity of E in that corollary). Hence it is natural to wonder if Spielberg's criterion completely determines quasidiagonality in extensions as well. The following result gives some more evidence for an affirmative answer. If I is a C*-algebra, let $SI = C_0(\mathbb{R}) \otimes I$ denote the suspension of I. Note that $K_0(SI)^+ = \{0\}$ since $SI \otimes \mathcal{K}$ contains no nonzero projections.

Proposition 4.2 *Let $0 \to SI \to E \to B \to 0$ be exact, where I is σ-unital and B is separable, QD, nuclear. Then E is QD.*

Proof. The suspension SI of I is QD by [Vo1]. We may assume that I is stable. Let $\alpha : SI \hookrightarrow SI$ be a null-homotopic approximately unital embedding and let $\widehat{\alpha} : Q(SI) \hookrightarrow Q(SI)$ be the corresponding $*$-monomorphism. Then for any Busby invariant $\gamma : B \to M(SI)$, $[\widehat{\alpha} \circ \gamma] = 0 \in Ext(B, SI)$ by the homotopy invariance of $Ext(B, SI)$ in the second variable [Kas]. It follows that $E(\gamma) \hookrightarrow E(\widehat{\alpha} \circ \gamma)$ is QD by Proposition 2.5. □

Definition 4.3 Say that a QD C^*-algebra A has the *QD extension property* if for every separable, nuclear, QD algebra B which satisfies the UCT and Busby invariant $\gamma : B \to Q(\mathcal{K} \otimes A)$ we have that $E(\gamma)$ is QD if and only if $E(\gamma)$ is stably finite (which is if and only if $\partial(K_1(B)) \cap K_0^+(\mathcal{K} \otimes A) = \{0\}$, by Proposition 4.1).

The QD extension property is closely related to a certain embedding property for the K-theory of A which we now describe. The interest in controlling the K-theory of embeddings of C^*-algebras goes back to the seminal work of Pimsner and Voiculescu on AF embeddings of irrational rotation algebras ([PV]). Since then other authors have studied the K-theory of (AF) embeddings ([Lo], [EL], [DL], [Br1], [Br1]).

Definition 4.4 Say that a QD C^*-algebra A has the *K_0-embedding property* if for every subgroup $G \subset K_0(A)$ such that $G \cap K_0^+(A) = \{0\}$ there exists an embedding $\rho : A \hookrightarrow C$, where C is also QD, such that $\rho_*(G) = 0$.

It is not hard to see that if C is a stably finite C^*-algebra and $p \in C$ is a nonzero projection then $[p]$ must be a nonzero element of $K_0(C)$. From this remark it follows that the condition $G \cap K_0^+(A) = \{0\}$ is necessary. Hence the K_0-embedding property states that this condition is also sufficient.

A number of QD C^*-algebras have the K_0-embedding property. For example, commutative C*-algebras, AF algebras ([Sp, Lem. 1.14]), crossed products of AF algebras by $\mathbb{Z}$ ([Br1, Thm. 5.5]) and simple nuclear unital C*-algebras with unique trace.

Our next goal is to connect the QD extension and K_0-embedding properties. But we first need a simple lemma.

Lemma 4.5 *Let C be a hereditary subalgebra of a unital C^*-algebra D. If C has an approximate unit consisting of projections and $K_0(D)$ has cancellation then the inclusion $C \hookrightarrow D$ induces an injective map $K_0(C) \hookrightarrow K_0(D)$.*

Proof. By *cancellation* we mean that if $p, q \in M_n(D)$ are projections with $[p] = [q]$ in $K_0(D)$ then there exists a partial isometry $v \in M_n(D)$ such that $vv^* = p$ and $v^*v = q$.

Let $x = [p] - [q] \in K_0(C)$ be an element such that $x = 0 \in K_0(D)$. Since C has an approximate unit of projections, say $\{e_\lambda\}$, we may assume that p and q are projections in $(e_\lambda \otimes 1)C \otimes M_n(\mathbb{C})(e_\lambda \otimes 1)$ for sufficiently large n and λ. Since $[p] = [q]$ in $K_0(D)$ and this group has cancellation we can find a partial isometry $v \in M_n(D)$ such that $vv^* = p$ and $v^*v = q$.

We claim that actually $v \in M_n(C)$ (which will evidently prove the lemma). To see this we first note that $v = vv^*(v)v^*v = pvq$ and hence

$$v = pvq = (e_\lambda \otimes 1)pvq(e_\lambda \otimes 1) = (e_\lambda \otimes 1)v(e_\lambda \otimes 1).$$

Hence $v \in (e_\lambda \otimes 1)D \otimes M_n(\mathbb{C})(e_\lambda \otimes 1)$. But since C is hereditary in D, $C \otimes M_n(\mathbb{C})$ is hereditary in $D \otimes M_n(\mathbb{C})$ and thus

$$v \in (e_\lambda \otimes 1)D \otimes M_n(\mathbb{C})(e_\lambda \otimes 1) \subset C \otimes M_n(\mathbb{C}). \qquad \square$$

Proposition 4.6 *Let A be a separable QD C^*-algebra. Then A satisfies the QD extension property if and only if A satisfies the K_0-embedding property.*

Proof. We begin with the easy direction. Assume that A has the QD extension property and let $G \subset K_0(A)$ be a subgroup such that $G \cap K_0^+(A) = \{0\}$. Since abelian C^*-algebras satisfy the UCT we can construct an extension

$$0 \to \mathcal{K} \otimes A \to E \to \oplus_{\mathbb{N}} C(\mathbb{T}) \to 0,$$

such that $\partial(K_1(\oplus_{\mathbb{N}} C(\mathbb{T}))) = \partial(\oplus_{\mathbb{N}} \mathbb{Z}) = G$. But since A has the QD extension property E must be a QD C^*-algebra. Thus the six-term K-theory exact sequence implies that A has the K_0-embedding property (i.e. the embedding into E will work).

Conversely, assume that A has the K_0-embedding property and let

$$0 \to \mathcal{K} \otimes A \to E \to B \to 0$$

be a short exact sequence where B is separable, nuclear, QD, satisfies the UCT and E is stably finite.

Let $G = \partial(K_1(B)) \subset K(\mathcal{K} \otimes A) \cong K_0(A)$. Since E is stably finite, $G \cap K_0^+(A) = \{0\}$. By the K_0-embedding property we can find a QD C^*-algebra C and an embedding $\rho : A \hookrightarrow C$ such that $\rho_*(G) = 0$. Since A is separable we may assume that C is also separable. Indeed $K_0(A)$ (and hence G) is countable. Thus it only takes a countable number of projections and partial isometries in matrices over C to kill off $\rho_*(G)$. From this observation it is easy to see that we may assume that C is also separable.

Let $\pi : C \hookrightarrow \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ be an embedding (the existence of which is ensured by the separability of C) as in the proof of Proposition 3.3. Let $J \subset \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ be the hereditary subalgebra generated by $\pi \circ \rho(A)$. Since $\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ has real rank zero and stable rank one it follows from Lemma 4.5 that the inclusion $J \hookrightarrow \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ induces an injective map $K_0(J) \hookrightarrow K_0(\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H))$. Since G is in the kernel of the K-theory map induced by the embedding $\pi \circ \rho : A \to \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ it follows that G is also in the kernel of the K-theory map induced by the embedding $\pi \circ \rho : A \to J$. But the embedding into J is approximately unital by construction and so we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} \otimes A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K} \otimes J & \longrightarrow & E(\eta) & \longrightarrow & B & \longrightarrow & 0, \end{array}$$

where η is the induced Busby invariant and the two vertical maps on the left are injective.

Now we are done since naturality of the boundary map implies that the homomorphism $\partial : K_1(B) \to K_0(\mathcal{K} \otimes J)$ is zero and hence $E(\eta)$ is QD by Theorem 3.4. □

We now wish to point out a connection between extensions of QD C^*-algebras and another very natural K-theoretic question. For brevity, we say a linear map $\varphi : A \to B$ is *ccp* if it is contractive and completely positive ([Pa]). We recall a theorem of Voiculescu.

Theorem 4.7 *[Vo1, Thm. 1] Let A be a separable C^*-algebra. Then A is QD if and only if there exists an asymptotically multiplicative, asymptotically isometric sequence of ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ for some sequence of natural numbers k_n (i.e. $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0$ and $\|\varphi_n(a)\| \to \|a\|$ for all $a, b \in A$).*

Given this abstract characterization of QD C^*-algebras it is natural to ask how well these approximating maps capture the relevant K-theoretic data.

Definition 4.8 Say that a QD C^*-algebra A has the *K_0-Hahn-Banach property* if for each $x \in K_0(A)$ such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$, where $\mathbb{Z}x = \{kx : k \in \mathbb{Z}\}$, there exists a sequence of asymptotically multiplicative, asymptotically isometric ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ such that $(\varphi_n)_*(x) = 0$ for all n large enough.

It is easy to see that if $y \in K_0(A)$ and there exists a nonzero integer k such that $ky \in K_0^+(A)$ then *for every* asymptotically multiplicative, asymptotically isometric sequence of ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ we have $(\varphi_n)_*(y) > 0$ (if $k > 0$) or $(\varphi_n)_*(y) < 0$ (if $k < 0$), for all sufficiently large n. Hence this K_0-Hahn-Banach property states that one can separate elements $x \in K_0(A)$ such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ from (finite subsets of) the positive cone using finite dimensional approximate morphisms.

Another way of thinking about this property is that A has the K_0-Hahn-Banach property if and only if finite dimensional approximate morphisms determine the order on $K_0(A)$ to a large extent. A more precise formulation is contained in the next proposition (not needed for the rest of the paper).

Proposition 4.9 *The K_0-Hahn-Banach property is equivalent to the following property: If $x \in K_0(A)$ and for every sequence of asymptotically multiplicative, asymptotically isometric ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ we have that $(\varphi_n)_*(x) > 0$ for all large n then there exists a positive integer k such that $kx \in K_0^+(A)$.*

Proof. We first show that the (contrapositive of the) second property above follows from the K_0-Hahn-Banach property. So assume we are given an element $x \in K_0(A)$ and assume that there is *no* positive integer k such that $kx \in K_0^+(A)$. We must exhibit a sequence of asymptotically multiplicative, asymptotically isometric ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ such that $(\varphi_n)_*(x) \leq 0$ for all sufficiently large n. There are two cases.

If there exists a negative integer k such that $kx \in K_0^+(A)$ then for every sequence $\varphi_n : A \to M_{k_n}(\mathbb{C})$ we have $(\varphi_n)_*(x) < 0$ for all sufficiently large n (see the discussion following definition 4.7). The second case is if $\mathbb{Z}x \cap K_0^+(A) = \{0\}$. This case is obviously handled by the K_0-Hahn-Banach property.

Now we show how the second property above implies the K_0-Hahn-Banach property. So let $x \in K_0(A)$ be such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$. Since no positive multiple of x is in $K_0^+(A)$ the second property implies that we can find some sequence $\varphi_n : A \to M_{k_n}(\mathbb{C})$ such that $(\varphi_n)_*(x) \leq 0$ for all sufficiently large n. Similarly, since no positive multiple of $-x$ is in $K_0^+(A)$ we can find a sequence $\psi_n : A \to M_{j_n}(\mathbb{C})$ such that $(\psi_n)_*(x) \geq 0$ for all sufficiently large n. If either of $\{\varphi_n\}$ or $\{\psi_n\}$ contains a subsequence with equality at 0 then we are done so we assume that $(\varphi_n)_*(x) = -s_n < 0$ and $(\psi_n)_*(x) = t_n > 0$ for all (sufficiently large) n. It is now clear what to do: we simply add up appropriate numbers of copies of φ_n and ψ_n so that these positive and negative ranks cancel. More precisely we define maps

$$\Phi_n = (\bigoplus_1^{t_n} \varphi_n) \oplus (\bigoplus_1^{s_n} \psi_n)$$

and regard these maps as taking values in the $(t_n k_n + s_n j_n) \times (t_n k_n + s_n j_n)$ matrices. □

Proposition 4.10 *If a separable QD C^*-algebra A has the QD extension property or, equivalently, the K_0-embedding property then A also has the K_0-Hahn-Banach property.*

Proof. Assume that A has the K_0-embedding property and we are given $x \in K_0(A)$ such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$, where $\mathbb{Z}x = \{kx : k \in \mathbb{Z}\}$. By the K_0-embedding property we can find an embedding $\rho : A \hookrightarrow C$, where C is QD and $\rho_*(x) = 0$. As in the proof of Proposition 4.6 we may assume that C is also separable. But then take any asymptotically multiplicative, asymptotically isometric sequence of contractive completely positive maps $\varphi_n : C \to M_{k_n}(\mathbb{C})$ and we get that $(\varphi_n \circ \rho)_*(x) = 0$ for all sufficiently large n. □

We do not know if the converse of the previous proposition holds. However our final result will complete the circle for the class of nuclear C^*-algebras. Moreover, the next theorem also states that in order to prove that every separable, nuclear, QD C^*-algebra has any of the properties we have been studying, it actually suffices to consider very special cases of either the QD extension problem or K_0-embedding problem.

Theorem 4.11 *The following statements are equivalent.*

1. *Every separable, nuclear, QD C^*-algebra has the QD extension property.*

2. *Every separable, nuclear, QD C^*-algebra has the K_0-embedding property.*

3. *Every separable, nuclear, QD C^*-algebra has the K_0-Hahn-Banach property.*

4. *If A is any separable, nuclear, QD C^*-algebra and $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists an embedding $\rho : A \hookrightarrow C$, where C is QD (but not necessarily separable or nuclear), such that $\rho_*(x) = 0$.*

5. *If A is any separable, nuclear, QD C^*-algebra and $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists a short exact sequence $0 \to \mathcal{K} \otimes A \to E \to C(\mathbb{T}) \to 0$ where E is QD and $x \in \partial(K_1(C(\mathbb{T}))) = \partial(\mathbb{Z})$.*

Proof. The proof of Proposition 4.6 carries over verbatim to show the equivalence of 1 and 2. That proof also shows the equivalence of 4 and 5. The previous proposition shows that 2 implies 3 and hence we are left to show that 3 implies 5 and 4 implies 2.

We begin with the easier implication $4 \implies 2$. So, let A be any separable, nuclear, QD C^*-algebra and $G \subset K_0(A)$ be a subgroup such that $G \cap K_0^+(A) = \{0\}$. As in the proof of Proposition 4.6 we can construct a short exact sequence

$$0 \to \mathcal{K} \otimes A \to E \to \bigoplus_1^\infty C(\mathbb{T}) \to 0,$$

such that $\partial(K_1(\oplus_{\mathbb{N}} C(\mathbb{T}))) = \partial(\oplus_{\mathbb{N}}\mathbb{Z}) = G$. We will prove that E is QD and, by exactness of $\oplus_{\mathbb{N}}\mathbb{Z} \xrightarrow{\partial} K_0(A) \to K_1(E)$, this will show 2.

For each n there is a short exact sequence

$$0 \to \mathcal{K} \otimes A \to E_n \to \bigoplus_1^n C(\mathbb{T}) \to 0,$$

where each $E_n \subset E$ is an ideal and $E = \overline{\cup_n E_n}$. Note also that each E_n is nuclear since extensions of nuclear algebras are again nuclear. Since a locally QD algebra is actually QD it suffices to show that each E_n is QD. Since E_1 is stably finite (being a subalgebra of E) we have that the boundary map

$\partial : K_1(C(\mathbb{T})) \to K_0(E_1)$ takes no positive values. But then the proof of Proposition 4.6 shows that if we assume 4 then E_1 will be QD. Proceeding by induction we may assume that E_{n-1} is QD. Since E_n is also stably finite, E_{n-1} is an ideal in E_n and $E_n/E_{n-1} = C(\mathbb{T})$, applying the same argument to the exact sequence $0 \to E_{n-1} \to E_n \to C(\mathbb{T}) \to 0$ we see that E_n is also QD.

We now show that $3 \implies 5$, which will complete the proof. So let A be any separable, nuclear, QD C^*-algebra and $x \in K_0(A)$ be such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$. Construct a short exact sequence $0 \to \mathcal{K} \otimes A \to E \to C(\mathbb{T}) \to 0$ such that $\partial(1) = x$. We will show that E must be QD.

We can use the K_0-Hahn-Banach property to construct an embedding $\rho : \mathcal{K} \otimes A \to Q(\oplus_i M_{n_i}(\mathbb{C}))$ such that $\rho_*(x) = 0$. Let $D \subset Q(\oplus_i M_{n_i}(\mathbb{C}))$ be the hereditary subalgebra generated by $\rho(\mathcal{K} \otimes A)$. Let $\pi : C(\mathbb{T}) \to B(H)$ be any faithful unital representation such that $\pi(C(\mathbb{T})) \cap \mathcal{K}(H) = \{0\}$. We first claim that there is an embedding of E into $(\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D}$, where $\tilde{D}$ is the unitization of D. Indeed, since the embedding $\rho : \mathcal{K} \otimes A \to D$ is approximately unital we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} \otimes A & \longrightarrow & E & \longrightarrow & C(\mathbb{T}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & F & \longrightarrow & C(\mathbb{T}) & \longrightarrow & 0, \end{array}$$

for some algebra F and the map $E \to F$ is injective. Since $\rho_*(x) = 0 \in K_0(D)$ (by Lemma 4.5) and $K_1(D) = 0$ (by the proof of Lemma 3.2) it follows that both boundary maps arising from the sequence $0 \to D \to F \to C(\mathbb{T}) \to 0$ are zero. Hence we may appeal to the UCT, add on a trivial absorbing extension and eventually find an embedding of F into $\pi(C(\mathbb{T})) \otimes 1 + \mathcal{K}(H) \otimes D \subset (\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D}$.

Since E is nuclear it now suffices to show that every nuclear subalgebra of $(\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D}$ is QD. Hence, by [Br3, Prop. 8.3] and the Choi-Effros lifting theorem ([CE]) it suffices to show that there exists a short exact sequence

$$0 \to J \to C \to (\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D} \to 0,$$

where C is QD and J contains an approximate unit consisting of projections which is quasicentral in C (i.e. the extension is quasidiagonal). However, this is now trivial since $D \subset Q(\oplus_i M_{n_i}(\mathbb{C}))$ implies that there is a quasidiagonal extension

$$0 \to \oplus_i M_{n_i}(\mathbb{C}) \to R \to \tilde{D} \to 0,$$

where $R \subset \Pi_i M_{n_i}(\mathbb{C})$. But since $X = \pi(C(\mathbb{T})) + \mathcal{K}(H)$ is nuclear the sequence

$$0 \to (\oplus_i M_{n_i}(\mathbb{C})) \otimes X \to R \otimes X \to \tilde{D} \otimes X \to 0$$

is exact and since X is unital the extension is also quasidiagonal. □

Though Theorem 4.11 is stated for the class of nuclear QD C^*-algebras a close inspection of the proof shows that this assumption was only used in the proof of 4 $\implies$ 2. Hence we also have the following result which applies to individual nuclear C*-algebras.

Theorem 4.12 *Let A be a separable nuclear QD C^*-algebra and consider the following statements.*

1. *A has the QD extension property.*
2. *A has the K_0-embedding property.*
3. *A has the K_0-Hahn-Banach property.*
4. *If $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists an embedding $\rho : A \hookrightarrow C$, where C is QD (but not necessarily separable or nuclear), such that $\rho_*(x) = 0$.*
5. *If $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists a short exact sequence $0 \to \mathcal{K} \otimes A \to E \to C(\mathbb{T}) \to 0$ where E is QD and $x \in \partial(K_1(C(\mathbb{T}))) = \partial(\mathbb{Z})$.*

Then 1 $\iff$ 2 $\implies$ 3 $\iff$ 4 $\iff$ 5.

Remark. There is another version of Theorem 4.11 where the class of nuclear C*-algebras is replaced by a class $\mathcal{A}$ of separable C*-algebras with the following closure property. If $0 \to A \otimes \mathcal{K} \to E \to B \to 0$ is exact with $A \in \mathcal{A}$ and B separable abelian, then $E \in \mathcal{A}$. For instance $\mathcal{A}$ can be the class of all separable C*-algebras or the class of all separable exact C*-algebras. Then the statements 1-5 of Theorem 4.11 formulated for the class $\mathcal{A}$ (rather then for the class of nuclear C*-algebras) are related as follows: 1 $\iff$ 2 $\iff$ 4 $\iff$ 5 $\implies$ 3.

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