ON THE KK-THEORY OF STRONGLY SELF-ABSORBING C^*-ALGEBRAS

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Abstract. Let $\mathcal{D}$ and A be unital and separable C^*-algebras; let $\mathcal{D}$ be strongly self-absorbing. It is known that any two unital *-homomorphisms from $\mathcal{D}$ to $A \otimes \mathcal{D}$ are approximately unitarily equivalent. We show that, if $\mathcal{D}$ is also K_1-injective, they are even asymptotically unitarily equivalent. This in particular implies that any unital endomorphism of $\mathcal{D}$ is asymptotically inner. Moreover, the space of automorphisms of $\mathcal{D}$ is compactly-contractible (in the point-norm topology) in the sense that for any compact Hausdorff space X, the set of homotopy classes $[X, \mathrm{Aut}\,(\mathcal{D})]$ reduces to a point. The respective statement holds for the space of unital endomorphisms of $\mathcal{D}$. As an application, we give a description of the Kasparov group $KK(\mathcal{D}, A \otimes \mathcal{D})$ in terms of *-homomorphisms and asymptotic unitary equivalence. Along the way, we show that the Kasparov group $KK(\mathcal{D}, A \otimes \mathcal{D})$ is isomorphic to $K_0(A \otimes \mathcal{D})$.

0. Introduction

A unital and separable C^*-algebra $\mathcal{D} \neq \mathbb{C}$ is strongly self-absorbing if there is an isomorphism $\mathcal{D} \xrightarrow{\sim} \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the inclusion map $\mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, $d \mapsto d \otimes \mathbf{1}_{\mathcal{D}}$ ([14]). Strongly self-absorbing C^*-algebras are known to be simple and nuclear; moreover, they are either purely infinite or stably finite. The only known examples of strongly self-absorbing C^*-algebras are the UHF algebras of infinite type (i.e., every prime number that occurs in the respective supernatural number occurs with infinite multiplicity), the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, the Jiang–Su algebra $\mathcal{Z}$ and tensor products of $\mathcal{O}_\infty$ with UHF algebras of infinite type, see [14]. All these examples are K_1-injective, i.e., the canonical map $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D}) \to K_1(\mathcal{D})$ is injective.

It was observed in [14] that any two unital *-homomorphisms $\sigma, \gamma : \mathcal{D} \to A \otimes \mathcal{D}$ are approximately unitarily equivalent, were A is another unital and separable C^*-algebra. If $\mathcal{D}$ is K_1-injective, the unitaries implementing the equivalence may even be chosen to

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be homotopic to the unit. When $\mathcal{D}$ is $\mathcal{O}_2$, $\mathcal{O}_\infty$, it was known that σ and γ are even asymptotically unitarily equivalent – i.e., they can be intertwined by a continuous path of unitaries, parametrized by a half-open interval. Up to this point, it was not clear whether the respective statement holds for the Jiang–Su algebra $\mathcal{Z}$. Theorem 2.2 below provides an affirmative answer to this problem. Even more, we show that the path intertwining σ and γ may be chosen in the component of the unit.

We believe this result, albeit technical, is interesting in its own right, and that it will be a useful ingredient for the systematic further use of strongly self-absorbing C^*-algebras in Elliott's program to classify nuclear C^*-algebras by K-theory data. In fact, this point of view is our main motivation for the study of strongly self-absorbing C^*-algebras; see [8], [10], [16], [17], [18] and [15] for already existing results in this direction.

For the time being, we use Theorem 2.2 to derive some consequences for the Kasparov groups of the form $KK(\mathcal{D}, A \otimes \mathcal{D})$. More precisely, we show that all the elements of the Kasparov group $KK(\mathcal{D}, A \otimes \mathcal{D})$ are of the form $[\varphi] - n[\iota]$ where $\varphi : \mathcal{D} \to \mathcal{K} \otimes A \otimes \mathcal{D}$ is a *-homomorphism and $\iota : \mathcal{D} \to A \otimes \mathcal{D}$ is the inclusion $\iota(d) = \mathbf{1}_A \otimes d$ and $n \in \mathbb{N}$. Moreover, two non-zero *-homomorphisms $\varphi, \psi : \mathcal{D} \to \mathcal{K} \otimes A \otimes \mathcal{D}$ with $\varphi(\mathbf{1}_\mathcal{D}) = \psi(\mathbf{1}_\mathcal{D}) = e$ have the same KK-theory class if and only if there is a unitary-valued continuous map $u : [0, 1) \to e(\mathcal{K} \otimes A \otimes \mathcal{D})e$, $t \mapsto u_t$ such that $u_0 = e$ and $\lim_{t\to 1} \|u_t\, \varphi(d)\, u_t^* - \psi(d)\| = 0$ for all $d \in \mathcal{D}$. In addition, we show that $KK_i(\mathcal{D}, \mathcal{D} \otimes A) \cong K_i(\mathcal{D} \otimes A)$, $i = 0, 1$.

One may note the similarity to the descriptions of $KK(\mathcal{O}_\infty, \mathcal{O}_\infty \otimes A)$ ([8],[10]) and $KK(\mathbb{C}, \mathbb{C} \otimes A)$. However, we do not require that $\mathcal{D}$ satisfies the universal coefficient theorem (UCT) in KK-theory. In the same spirit, we characterize $\mathcal{O}_2$ and the universal UHF algebra $\mathcal{Q}$ using K-theoretic conditions, but without involving the UCT.

As another application of Theorem 2.2 (and the results of [7]), we prove in [4] an automatic trivialization result for continuous fields with strongly self-absorbing fibres over finite dimensional spaces.

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1. Strongly self-absorbing C^*-algebras

In this section we recall the notion of strongly self-absorbing C^*-algebras and some facts from [14].

1.1 Definition: *Let A, B be C^*-algebras and $\sigma, \gamma : A \to B$ be *-homomorphisms. Suppose that B is unital.*

(i) *We say that σ and γ are approximately unitarily equivalent, $\sigma \approx_{\mathrm{u}} \gamma$, if there is a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in B such that*

$$\|u_n\sigma(a)u_n^* - \gamma(a)\| \stackrel{n\to\infty}{\longrightarrow} 0$$

for every $a \in A$. If all u_n can be chosen to be in $\mathcal{U}_0(B)$, the connected component of $\mathbf{1}_B$ of the unitary group $\mathcal{U}(B)$, then we say that σ and γ are strongly approximately unitarily equivalent, written $\sigma \approx_{\mathrm{su}} \gamma$.

(ii) *We say that σ and γ are asymptotically unitarily equivalent, $\sigma \approx_{\mathrm{uh}} \gamma$, if there is a norm-continuous path $(u_t)_{t\in[0,\infty)}$ of unitaries in B such that*

$$\|u_t\sigma(a)u_t^* - \gamma(a)\| \stackrel{t\to\infty}{\longrightarrow} 0$$

for every $a \in A$. If one can arrange that $u_0 = \mathbf{1}_B$ and hence ($u_t \in \mathcal{U}_0(B)$ for all t), then we say that σ and γ are strongly asymptotically unitarily equivalent, written $\sigma \approx_{\mathrm{suh}} \gamma$.

1.2 The concept of strongly self-absorbing C^*-algebras was formally introduced in [14, Definition 1.3]:

Definition: *A separable unital C^*-algebra $\mathcal{D}$ is strongly self-absorbing, if $\mathcal{D} \neq \mathbb{C}$ and there is an isomorphism $\varphi : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ such that $\varphi \approx_{\mathrm{u}} \mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}$.*

1.3 Recall [14, Corollary 1.12]:

Proposition: *Let A and $\mathcal{D}$ be unital C^*-algebras, with $\mathcal{D}$ strongly self-absorbing. Then, any two unital *-homomorphisms $\sigma, \gamma : \mathcal{D} \to A \otimes \mathcal{D}$ are approximately unitarily equivalent. In particular, any two unital endomorphisms of $\mathcal{D}$ are approximately unitarily equivalent.*

We note that the assumption that A is separable which appears in the original statement of [14, Corollary 1.12] is not necessary and was not used in the proof.

1.4 Lemma: *Let $\mathcal{D}$ be a strongly self-absorbing C^*-algebra. Then there is a sequence of unitaries $(w_n)_{n\in\mathbb{N}}$ in the commutator subgroup of $\mathcal{U}(\mathcal{D}\otimes\mathcal{D})$ such that for all $d \in \mathcal{D}$ $\|w_n(d\otimes \mathbf{1}_{\mathcal{D}})w_n^* - \mathbf{1}_{\mathcal{D}} \otimes d\| \to 0$ as $n \to \infty$.*

Proof: Let $\mathcal{F} \subset \mathcal{D}$ be a finite normalized set and let $\varepsilon > 0$. By [14, Prop. 1.5] there is a unitary $u \in \mathcal{U}(\mathcal{D}\otimes\mathcal{D})$ such that $\|u(d\otimes\mathbf{1}_{\mathcal{D}})u^* - \mathbf{1}_{\mathcal{D}}\otimes d\| < \varepsilon$ for all $d \in \mathcal{F}$. Let $\theta : \mathcal{D}\otimes\mathcal{D} \to \mathcal{D}$ be a *-isomorphism. Then $\|(\theta(u^*) \otimes \mathbf{1}_{\mathcal{D}})u(d \otimes \mathbf{1}_{\mathcal{D}})u^*(\theta(u) \otimes \mathbf{1}_{\mathcal{D}}) - \mathbf{1}_{\mathcal{D}} \otimes d\| < \varepsilon$ for all $d \in \mathcal{F}$. By Proposition 1.3 $\theta \otimes \mathbf{1}_{\mathcal{D}} \approx_{\mathrm{u}} \mathrm{id}_{\mathcal{D}\otimes\mathcal{D}}$ and so there is a unitary $v \in \mathcal{U}(\mathcal{D} \otimes \mathcal{D})$ such that $\|\theta(u^*) \otimes \mathbf{1}_{\mathcal{D}} - vu^*v^*\| < \varepsilon$ and hence $\|(\theta(u^*) \otimes \mathbf{1}_{\mathcal{D}})u - vu^*v^*u\| < \varepsilon$. Setting $w = vu^*v^*u$ we deduce that $\|w(d \otimes \mathbf{1}_{\mathcal{D}})w^* - \mathbf{1}_{\mathcal{D}} \otimes d\| < 3\varepsilon$ for all $d \in \mathcal{F}$. ∎

1.5 Remark: In the situation of Proposition 1.3, suppose that the commutator subgroup of $\mathcal{U}(\mathcal{D})$ is contained in $\mathcal{U}_0(\mathcal{D})$. This will happen for instance if $\mathcal{D}$ is assumed to be K_1-injective. Then one may choose the unitaries $(u_n)_{n\in\mathbb{N}}$ which implement the approximate unitary equivalence between σ and γ to lie in $\mathcal{U}_0(A\otimes\mathcal{D})$. This follows from [14, (the proof of) Corollary 1.12], since the unitaries $(u_n)_{n\in\mathbb{N}}$ are essentially images of the unitaries $(w_n)_{n\in\mathbb{N}}$ of Lemma 1.4 under suitable unital *-homomorphisms.

2. Asymptotic vs. approximate unitary equivalence

It is the aim of this section to establish a continuous version of Proposition 1.3.

2.1 Lemma: *Let $\mathcal{D}$ be separable unital strongly self-absorbing C^*-algebra. For any finite subset $\mathcal{F}\subset\mathcal{D}$ and $\varepsilon>0$, there are a finite subset $\mathcal{G}\subset\mathcal{D}$ and $\delta>0$ such that the following holds:*

If A is another unital C^*-algebra and $\sigma:\mathcal{D}\to A\otimes\mathcal{D}$ is a unital *-homomorphism, and if $w\in\mathcal{U}_0(A\otimes\mathcal{D})$ is a unitary satisfying

$$\|[w,\sigma(d)]\|<\delta$$

for all $d\in\mathcal{G}$, then there is a continuous path $(w_t)_{t\in[0,1]}$ of unitaries in $\mathcal{U}_0(A\otimes\mathcal{D})$ such that $w_0=w$, $w_1=\mathbf{1}_{A\otimes\mathcal{D}}$ and

$$\|[w_t,\sigma(d)]\|<\varepsilon$$

for all $d\in\mathcal{F}$, $t\in[0,1]$.

Proof: We may clearly assume that the elements of $\mathcal{F}$ are normalized and that $\varepsilon<1$. Let $u\in\mathcal{D}\otimes\mathcal{D}$ be a unitary satisfying

$$\|u(d\otimes\mathbf{1}_{\mathcal{D}})u^*-\mathbf{1}_{\mathcal{D}}\otimes d\|<\frac{\varepsilon}{20} \tag{1}$$

for all $d\in\mathcal{F}$. There exist $k\in\mathbb{N}$ and elements $s_1,\ldots,s_k,t_1,\ldots,t_k\in\mathcal{D}$ of norm at most one such that

$$\|u-\sum_{i=1}^{k}s_i\otimes t_i\|<\frac{\varepsilon}{20}. \tag{2}$$

Set

$$\delta:=\frac{\varepsilon}{k\cdot 10} \tag{3}$$

and

$$\mathcal{G}:=\{s_1,\ldots,s_k\}\subset\mathcal{D}. \tag{4}$$

Now let $w\in\mathcal{U}_0(A\otimes\mathcal{D})$ be a unitary as in the assertion of the lemma, i.e., w satisfies

$$\|[w,\sigma(s_i)]\|<\delta \tag{5}$$

for all $i=1,\ldots,k$. We proceed to construct the path $(w_t)_{t\in[0,1]}$.

By [14, Remark 2.7] there is a unital *-homomorphism

$$\varphi:A\otimes\mathcal{D}\otimes\mathcal{D}\to A\otimes\mathcal{D}$$

such that

$$\|\varphi(a \otimes \mathbf{1}_{\mathcal{D}}) - a\| < \frac{\varepsilon}{20} \tag{6}$$

for all $a \in \sigma(\mathcal{F}) \cup \{w\}$.

Since $w \in \mathcal{U}_0(A \otimes \mathcal{D})$, there is a path $(\bar{w}_t)_{t\in[\frac{1}{2},1]}$ of unitaries in $A \otimes \mathcal{D}$ such that

$$\bar{w}_{\frac{1}{2}} = w \text{ and } \bar{w}_1 = \mathbf{1}_{A\otimes\mathcal{D}}. \tag{7}$$

For $t \in [\frac{1}{2}, 1]$ define

$$w_t := \varphi((\sigma \otimes \mathrm{id}_{\mathcal{D}})(u)^*(\bar{w}_t \otimes \mathbf{1}_{\mathcal{D}})(\sigma \otimes \mathrm{id}_{\mathcal{D}})(u)) \in \mathcal{U}(A \otimes \mathcal{D}); \tag{8}$$

then $(w_t)_{t\in[\frac{1}{2},1]}$ is a continuous path of unitaries in $A \otimes \mathcal{D}$. For $t \in [\frac{1}{2}, 1]$ and $d \in \mathcal{F}$ we have

$$\begin{aligned}
&\|[w_t, \sigma(d)]\| \\
&\quad = \|w_t\sigma(d)w_t^* - \sigma(d)\| \\
&\quad \overset{(6)}{<} \|w_t\varphi(\sigma(d) \otimes \mathbf{1}_{\mathcal{D}})w_t^* - \varphi(\sigma(d) \otimes \mathbf{1}_{\mathcal{D}})\| + 2 \cdot \frac{\varepsilon}{20} \\
&\quad \overset{(8)}{\leq} \|((\sigma \otimes \mathrm{id}_{\mathcal{D}})(u))^*(\bar{w}_t \otimes \mathbf{1}_{\mathcal{D}})((\sigma \otimes \mathrm{id}_{\mathcal{D}})(u(d \otimes \mathbf{1}_{\mathcal{D}})u^*))(\bar{w}_t^* \otimes \mathbf{1}_{\mathcal{D}}) \\
&\qquad \cdot((\sigma \otimes \mathrm{id}_{\mathcal{D}})(u)) - ((\sigma \otimes \mathrm{id}_{\mathcal{D}})(d \otimes \mathbf{1}_{\mathcal{D}}))\| + \frac{\varepsilon}{10} \\
&\quad \overset{(1)}{<} \|((\sigma \otimes \mathrm{id}_{\mathcal{D}})(u))^*(\bar{w}_t \otimes \mathbf{1}_{\mathcal{D}})((\sigma \otimes \mathrm{id}_{\mathcal{D}})(\mathbf{1}_{\mathcal{D}} \otimes d))(\bar{w}_t^* \otimes \mathbf{1}_{\mathcal{D}}) \\
&\qquad \cdot((\sigma \otimes \mathrm{id}_{\mathcal{D}})(u)) - ((\sigma \otimes \mathrm{id}_{\mathcal{D}})(d \otimes \mathbf{1}_{\mathcal{D}}))\| + \frac{\varepsilon}{10} + \frac{\varepsilon}{20} \\
&\quad = \|(\sigma \otimes \mathrm{id}_{\mathcal{D}})(u^*(\mathbf{1}_{\mathcal{D}} \otimes d)u - d \otimes \mathbf{1}_{\mathcal{D}})\| + \frac{\varepsilon}{10} + \frac{\varepsilon}{20} \\
&\quad < \frac{\varepsilon}{20} + \frac{\varepsilon}{10} + \frac{\varepsilon}{20} \\
&\quad < \frac{\varepsilon}{3},
\end{aligned} \tag{9}$$

where for the last equality we have used that the $\bar{w}_t$ are unitaries and that σ is a unital *-homomorphism. Furthermore, we have

$$\begin{aligned}
&\|w_{\frac{1}{2}} - w\| \\
&\stackrel{(7),(8)}{=} \quad \|\varphi(((\sigma \otimes \mathrm{id}_{\mathcal{D}})(u))^*(w \otimes \mathbf{1}_{\mathcal{D}})((\sigma \otimes \mathrm{id}_{\mathcal{D}})(u))) - w\| \\
&\stackrel{(2)}{<} \quad \|\varphi(((\sigma \otimes \mathrm{id}_{\mathcal{D}})(u))^*(w \otimes \mathbf{1}_{\mathcal{D}})(\sum_{i=1}^{k} \sigma(s_i) \otimes t_i)) - w\| + \frac{\varepsilon}{20} \\
&\leq \quad \|\varphi(((\sigma \otimes \mathrm{id}_{\mathcal{D}})(u))^*(\sum_{i=1}^{k} \sigma(s_i) \otimes t_i)(w \otimes \mathbf{1}_{\mathcal{D}})) - w\| \\
&\qquad + \sum_{i=1}^{k} \|[w, \sigma(s_i)]\| \cdot \|t_i\| + \frac{\varepsilon}{20} \\
&\stackrel{(5),(4),(2)}{<} \quad \|\varphi(w \otimes \mathbf{1}_{\mathcal{D}}) - w\| + k \cdot \delta + 2 \cdot \frac{\varepsilon}{20} \\
&\stackrel{(6),(3)}{<} \quad \frac{\varepsilon}{20} + \frac{\varepsilon}{10} + 2 \cdot \frac{\varepsilon}{20} \\
&< \quad \frac{\varepsilon}{3}.
\end{aligned}$$

The above estimate allows us to extend the path $(w_t)_{t \in [\frac{1}{2},1]}$ to the whole interval $[0,1]$ in the desired way: We have $\|w_{\frac{1}{2}} w^* - \mathbf{1}_{\mathcal{D}}\| < \frac{\varepsilon}{3} < 2$, whence -1 is not in the spectrum of $w_{\frac{1}{2}} w^*$. By functional calculus, there is $a = a^* \in A \otimes \mathcal{D}$ with $\|a\| < 1$ such that $w_{\frac{1}{2}} w^* = \exp(\pi i a)$. For $t \in [0, \frac{1}{2})$ we may therefore define a continuous path of unitaries

$$w_t := (\exp(2\pi i t a))w \in \mathcal{U}(A \otimes \mathcal{D}).$$

It is clear that $w_0 = w$ and $w_t \to w_{\frac{1}{2}}$ as $t \to (\frac{1}{2})_-$, whence $(w_t)_{t \in [0,1]}$ is a continuous path of unitaries in A satisfying $w_0 = w$ and $w_1 = \mathbf{1}_A \otimes \mathcal{D}$. Moreover, it is easy to see that

$$\|w_t - w\| \leq \|w_{\frac{1}{2}} - w\| < \frac{\varepsilon}{3}$$

for all $t \in [0, \frac{1}{2})$, whence

$$\|[w_t, \sigma(d)]\| < \|[w_{\frac{1}{2}}, \sigma(d)]\| + \frac{2}{3}\,\varepsilon \stackrel{(9)}{<} \varepsilon$$

for $t \in [0, \frac{1}{2})$, $d \in \mathcal{F}$.

We have now constructed a path $(w_t)_{t \in [0,1]} \subset \mathcal{U}(A)$ with the desired properties. ∎

2.2 Theorem: *Let A and $\mathcal{D}$ be unital C^*-algebras, with $\mathcal{D}$ separable, strongly self-absorbing and K_1-injective. Then, any two unital *-homomorphisms $\sigma, \gamma : \mathcal{D} \to A \otimes \mathcal{D}$ are strongly asymptotically unitarily equivalent. In particular, any two unital endomorphisms of $\mathcal{D}$ are strongly asymptotically unitarily equivalent.*

PROOF: Note that the second statement follows from the first one with $A = \mathcal{D}$, since $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$ by assumption.

Let A be a unital C^*-algebra such that $A \cong A \otimes \mathcal{D}$ and let $\sigma, \gamma : \mathcal{D} \to A$ be unital *-homomorphisms. We shall prove that σ and γ are strongly asymptotically unitarily equivalent. Choose an increasing sequence

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$$

of finite subsets of $\mathcal{D}$ such that $\bigcup \mathcal{F}_n$ is a dense subset of $\mathcal{D}$. Let $1 > \varepsilon_0 > \varepsilon_1 > \ldots$ be a decreasing sequence of strictly positive numbers converging to 0.

For each $n \in \mathbb{N}$, employ Lemma 2.1 (with $\mathcal{F}_n$ and ε_n in place of $\mathcal{F}$ and ε) to obtain a finite subset $\mathcal{G}_n \subset \mathcal{D}$ and $\delta_n > 0$. We may clearly assume that

$$\mathcal{F}_n \subset \mathcal{G}_n \subset \mathcal{G}_{n+1} \text{ and that } \delta_{n+1} < \delta_n < \varepsilon_n \tag{10}$$

for all $n \in \mathbb{N}$.

Since σ and γ are strongly approximately unitarily equivalent by Proposition 1.3 and Remark 1.5, there is a sequence of unitaries $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}_0(A)$ such that

$$\|u_n \sigma(d) u_n^* - \gamma(d)\| < \frac{\delta_n}{2} \tag{11}$$

for all $d \in \mathcal{G}_n$ and $n \in \mathbb{N}$. Let us set

$$w_n := u_{n+1}^* u_n,\ n \in \mathbb{N}.$$

Then $w_n \in \mathcal{U}_0(A)$ and

$$\begin{aligned}
&\|[w_n, \sigma(d)]\| \\
&= \|w_n \sigma(d) w_n^* - \sigma(d)\| \\
&\leq \|u_{n+1}^* u_n \sigma(d) u_n^* u_{n+1} - u_{n+1}^* \gamma(d) u_{n+1}\| \\
&\quad + \|u_{n+1}^* \gamma(d) u_{n+1} - \sigma(d)\| \\
&< \frac{\delta_n}{2} + \frac{\delta_{n+1}}{2} \\
&< \delta_n
\end{aligned}$$

for $d \in \mathcal{G}_n$, $n \in \mathbb{N}$. Now by Lemma 2.1 (and the choice of the $\mathcal{G}_n$ and δ_n), for each n there is a continuous path $(w_{n,t})_{t\in[0,1]}$ of unitaries in $\mathcal{U}_0(A)$ such that $w_{n,0} = w_n$, $w_{n,1} = \mathbf{1}_A$ and

$$\|[w_{n,t}, \sigma(d)]\| < \varepsilon_n \tag{12}$$

for all $d \in \mathcal{F}_n$, $t \in [0, 1]$.

Next, define a path $(\bar{u}_t)_{t\in[0,\infty)}$ of unitaries in $\mathcal{U}_0(A)$ by

$$\bar{u}_t := u_{n+1} w_{n,t-n} \text{ if } t \in [n, n+1).$$

We have that

$$\bar{u}_n = u_{n+1} w_n = u_n \tag{13}$$

and that

$$\bar{u}_t \to u_{n+1}$$

as $t \to n+1$ from below, which implies that the path $(\bar{u}_t)_{t\in[0,\infty)}$ is continuous in $\mathcal{U}_0(A)$. Furthermore, for $t \in [n, n+1)$ and $d \in \mathcal{F}_n$ we obtain

$$\begin{aligned}
&\|\bar{u}_t \sigma(d) \bar{u}_t^* - \gamma(d)\| \\
&\quad = \quad \|u_{n+1} w_{n,t-n} \sigma(d) w_{n,t-n}^* u_{n+1}^* - \gamma(d)\| \\
&\quad \stackrel{(12)}{<} \quad \|u_{n+1} \sigma(d) u_{n+1}^* - \gamma(d)\| + \varepsilon_n \\
&\quad \stackrel{(11),(10)}{<} \quad \frac{\delta_{n+1}}{2} + \varepsilon_n \\
&\quad \stackrel{(10)}{<} \quad 2\varepsilon_n.
\end{aligned}$$

Since the $\mathcal{F}_n$ are nested and the ε_n converge to 0, we have

$$\|\bar{u}_t \sigma(d) \bar{u}_t^* - \gamma(d)\| \stackrel{t\to\infty}{\longrightarrow} 0 \tag{14}$$

for all $d \in \bigcup_{n=0}^{\infty} \mathcal{F}_n$; by continuity and since $\bigcup_{n=0}^{\infty} \mathcal{F}_n$ is dense in $\mathcal{D}$, we have (14) for all $d \in \mathcal{D}$. Since $\bar{u}_0 \in \mathcal{U}_0(A)$ we may arrange that $\bar{u}_0 = \mathbf{1}_A$. ∎

3. The group $KK(\mathcal{D}, A \otimes \mathcal{D})$ and some applications

3.1 For a separable C^*-algebra $\mathcal{D}$ we endow the group of automorphisms $\mathrm{Aut}\,(\mathcal{D})$ with the point-norm topology.

Corollary: *Let $\mathcal{D}$ be a separable, unital, strongly self-absorbing and K_1-injective C^*-algebra. Then $[X, \mathrm{Aut}(\mathcal{D})]$ reduces to a point for any compact Hausdorff space X.*

Proof: Let $\varphi, \psi : X \to \mathrm{Aut}\,(\mathcal{D})$ be continuous maps. We identify φ and ψ with unital *-homomorphisms $\varphi, \psi : \mathcal{D} \to \mathcal{C}(X) \otimes \mathcal{D}$. By Theorem 2.2, φ is strongly asymptotically unitarily equivalent to ψ. This gives a homotopy between the two maps $\varphi, \psi : X \to \mathrm{Aut}\,(\mathcal{D})$. ∎

3.2 Remark: The conclusion of Corollary 3.1 was known before for $\mathcal{D}$ a UHF algebra of infinite type and X a CW complex by [13], for $\mathcal{D} = \mathcal{O}_2$ by [8] and [10], and for $\mathcal{D} = \mathcal{O}_\infty$ by [2]. It is new for the Jiang–Su algebra.

3.3 For unital C^*-algebras $\mathcal{D}$ and B we denote by $[\mathcal{D}, B]$ the set of homotopy classes of unital *-homomorphisms from $\mathcal{D}$ to B. By a similar argument as above we also have the following corollary.

Corollary: *Let $\mathcal{D}$ and A be unital C^*-algebras. If $\mathcal{D}$ is separable, strongly self-absorbing and K_1-injective, then $[\mathcal{D}, A\otimes\mathcal{D}]$ reduces to a singleton.*

3.4 For separable unital C^*-algebras $\mathcal{D}$ and B, let $\chi_i : KK_i(\mathcal{D}, B) \to KK_i(\mathbb{C}, B) \cong K_i(B)$, $i = 0, 1$ be the morphism of groups induced by the unital inclusion $\nu : \mathbb{C} \to \mathcal{D}$.

Theorem: *Let $\mathcal{D}$ be a unital, separable and strongly self-absorbing C^*-algebra. Then for any separable C^*-algebra A, the map $\chi_i : KK_i(\mathcal{D}, A\otimes\mathcal{D}) \to K_i(A\otimes\mathcal{D})$ is bijective, for $i = 0, 1$. In particular both groups $KK_i(\mathcal{D}, A\otimes D)$ are countable and discrete with respect to their natural topology.*

Proof: Since $\mathcal{D}$ is KK-equivalent to $\mathcal{D}\otimes\mathcal{O}_\infty$, we may assume that $\mathcal{D}$ is purely infinite and in particular K_1-injective by [11, Prop. 4.1.4]. Let $C_\nu\mathcal{D}$ denote the mapping cone C^*-algebra of ν. By [3, Cor. 3.10], there is a bijection $[\mathcal{D}, A\otimes\mathcal{D}] \to KK(C_\nu\mathcal{D}, SA\otimes\mathcal{D})$ and hence $KK(C_\nu\mathcal{D}, SA\otimes\mathcal{D}) = 0$ for all separable and unital C^*-algebras A as a consequence of Corollary 3.3. Since $KK(C_\nu\mathcal{D}, A\otimes\mathcal{D})$ is isomorphic to $KK(C_\nu\mathcal{D}, S^2A\otimes\mathcal{D})$ by Bott periodicity and the latter group injects in $KK(C_\nu\mathcal{D}, SC(\mathbb{T})\otimes A\otimes\mathcal{D}) = 0$, we have that $KK_i(C_\nu\mathcal{D}, \mathcal{D}\otimes A) = 0$ for all unital and separable C^*-algebras A and $i = 0, 1$. Since $KK_i(C_\nu\mathcal{D}, \mathcal{D}\otimes A)$ is a subgroup of $KK_i(C_\nu\mathcal{D}, \mathcal{D}\otimes\widetilde{A}) = 0$ (where $\widetilde{A}$ is the unitization of A) we see that $KK_i(C_\nu\mathcal{D}, \mathcal{D}\otimes A) = 0$ for all separable C^*-algebras A. Using the Puppe exact sequence, where $\chi_i = \nu^*$,

$$KK_{i+1}(C_\nu\mathcal{D}, A\otimes\mathcal{D}) \longrightarrow KK_i(\mathcal{D}, A\otimes\mathcal{D}) \xrightarrow{\chi_i} KK_i(\mathbb{C}, A\otimes\mathcal{D}) \longrightarrow KK_i(C_\nu\mathcal{D}, A\otimes\mathcal{D})$$

we conclude that χ_i is an isomorphism, $i = 0, 1$. The map $\chi_i = \nu^*$ is continuous since it is given by the Kasparov product with a fixed element (we refer the reader to [12], [9] or [1] for a background on the topology of the Kasparov groups). Since the topology of K_i is discrete and χ_i is injective, it follows that the topology of $KK_i(\mathcal{D}, A\otimes D)$ is also discrete. The countability of $KK_i(\mathcal{D}, A\otimes D)$ follows from that of $K_i(A\otimes D)$, as $A\otimes\mathcal{D}$ is separable. ∎

3.5 Remark: In contrast to Theorem 3.4, if $\mathcal{D}$ is the universal UHF algebra, then $KK(\mathcal{D}, \mathbb{C}) \cong \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{Q}^{\mathbb{N}}$ has the power of the continuum [6, p. 221].

3.6 Let $\mathcal{D}$ and A be as in Theorem 3.4 and assume in addition that $\mathcal{D}$ is K_1-injective and A is unital. Let $\iota : \mathcal{D} \to A\otimes\mathcal{D}$ be defined by $\iota(d) = \mathbf{1}_A \otimes d$.

Corollary: *If $e \in \mathcal{K}\otimes A\otimes\mathcal{D}$ is a projection, and $\varphi, \psi : \mathcal{D} \to e(\mathcal{K}\otimes A\otimes\mathcal{D})e$ are two unital *-homomorphisms, then $\varphi \approx_{\mathrm{suh}} \psi$ and hence $[\varphi] = [\psi] \in KK(\mathcal{D}, A\otimes\mathcal{D})$. Moreover:*

$$KK(\mathcal{D}, A\otimes\mathcal{D}) = \{[\varphi] - n[\iota] \,|\, \varphi : \mathcal{D} \to \mathcal{K}\otimes A\otimes\mathcal{D} \textit{ is a } ^*\textit{-homomorphism},\, n \in \mathbb{N}\}.$$

Proof: Let φ, ψ and e be as in the first part of the statement. By [14, Cor. 3.1], the unital C^*-algebra $e(\mathcal{K} \otimes A \otimes \mathcal{D})e$ is $\mathcal{D}$-stable, being a hereditary subalgebra of a $\mathcal{D}$-stable C^*-algebra. Therefore $\varphi \approx_{\mathrm{suh}} \psi$ by Theorem 2.2.

Now for the second part of the statement, let $x \in KK(\mathcal{D}, A \otimes \mathcal{D})$ be an arbitrary element. Then $\chi_0(x) = [e] - n[\mathbf{1}_{A\otimes\mathcal{D}}]$ for some projection $e \in \mathcal{K} \otimes A \otimes \mathcal{D}$ and $n \in \mathbb{N}$. Since $e(\mathcal{K} \otimes A \otimes \mathcal{D})e$ is $\mathcal{D}$-stable, there is a unital *-homomorphism $\varphi : \mathcal{D} \to e(\mathcal{K} \otimes A \otimes \mathcal{D})e$. Then

$$\chi_0([\varphi] - n[\iota]) = [\varphi(\mathbf{1}_{\mathcal{D}})] - n[\iota(\mathbf{1}_{\mathcal{D}})] = [e] - n[\mathbf{1}_{A\otimes\mathcal{D}}] = \chi_0(x),$$

and hence $[\varphi] - n[\iota] = x$ since χ_0 is injective by Theorem 3.4. ▮

In the remainder of the paper we give characterizations for the Cuntz algebra $\mathcal{O}_2$ and for the universal UHF-algebra which do not require the UCT. The latter result is a variation of a theorem of Effros and Rosenberg [5].

3.7 Proposition: *Let $\mathcal{D}$ be a separable unital strongly self-absorbing C^*-algebra. If $[\mathbf{1}_{\mathcal{D}}] = 0$ in $K_0(\mathcal{D})$, then $\mathcal{D} \cong \mathcal{O}_2$.*

Proof: Since $\mathcal{D}$ must be nuclear (see [14]), $\mathcal{D}$ embeds unitally in $\mathcal{O}_2$ by Kirchberg's theorem. $\mathcal{D}$ is not stably finite since $[\mathbf{1}_{\mathcal{D}}] = 0$. By the dichotomy of [14, Thm. 1.7] $\mathcal{D}$ must be purely infinite. Since $[\mathbf{1}_{\mathcal{D}}] = 0$ in $K_0(\mathcal{D})$, there is a unital embedding $\mathcal{O}_2 \to \mathcal{D}$, see [11, Prop. 4.2.3]. We conclude that $\mathcal{D}$ is isomorphic to $\mathcal{O}_2$ by [14, Prop. 5.12]. ▮

3.8 Proposition: *Let $\mathcal{D}$, A be separable, unital, strongly self-absorbing C^*-algebras. Suppose that for any finite subset $\mathcal{F}$ of $\mathcal{D}$ and any $\varepsilon > 0$ there is a u.c.p. map $\varphi : \mathcal{D} \to A$ such that $\|\varphi(cd) - \varphi(c)\varphi(d)\| < \varepsilon$ for all $c, d \in \mathcal{F}$. Then $A \cong A \otimes \mathcal{D}$.*

Proof: By [14, Thm. 2.2] it suffices to show that for any given finite subsets $\mathcal{F}$ of $\mathcal{D}$, $\mathcal{G}$ of A and any $\varepsilon > 0$ there is u.c.p. map $\Phi : \mathcal{D} \to A$ such that (i) $\|\Phi(cd) - \Phi(c)\Phi(d)\| < \varepsilon$ for all $c, d \in \mathcal{F}$ and (ii) $\|[\Phi(d), a]\| < \varepsilon$ for all $d \in \mathcal{F}$ and $a \in \mathcal{G}$. We may assume that $\|d\| \leq 1$ for all $d \in \mathcal{F}$. Since A is strongly self-absorbing, by [14, Prop. 1.10] there is a unital *-homomorphism $\gamma : A \otimes A \to A$ such that $\|\gamma(a \otimes \mathbf{1}_A) - a\| < \varepsilon/2$ for all $a \in \mathcal{G}$. On the other hand, by assumption there is a u.c.p. map $\varphi : \mathcal{D} \to A$ such that $\|\varphi(cd) - \varphi(c)\varphi(d)\| < \varepsilon$ for all $c, d \in \mathcal{F}$. Let us define a u.c.p. map $\Phi : \mathcal{D} \to A$ by $\Phi(d) = \gamma(\mathbf{1}_A \otimes \varphi(d))$. It is clear that Φ satisfies (i) since γ is a *-homomorphism. To conclude the proof we check now that Φ also satisfies (ii). Let $d \in \mathcal{F}$ and $a \in \mathcal{G}$. Then

$$\begin{aligned} &\|[\Phi(d), a]\| \\ &\quad\leq \|[\Phi(d), a - \gamma(a \otimes \mathbf{1}_A)]\| + \|[\Phi(d), \gamma(a \otimes \mathbf{1}_A)]\| \\ &\quad\leq 2\|\Phi(d)\|\|a - \gamma(a \otimes \mathbf{1}_A)\| + \|[\gamma(\mathbf{1}_A \otimes \varphi(d)), \gamma(a \otimes \mathbf{1}_A)]\| \\ &\quad< 2\varepsilon/2 + 0 = \varepsilon. \end{aligned}$$

▮

3.9 PROPOSITION: *Let $\mathcal{D}$ be a separable, unital, strongly self-absorbing C^*-algebra. Suppose that $\mathcal{D}$ is quasidiagonal, it has cancellation of projections and that $[\mathbf{1}_{\mathcal{D}}] \in nK_0(\mathcal{D})^+$ for all $n \geq 1$. Then $\mathcal{D}$ is isomorphic to the universal UHF algebra $\mathcal{Q}$ with $K_0(\mathcal{Q}) \cong \mathbb{Q}$.*

PROOF: Since $\mathcal{D}$ is separable unital and quasidiagonal, there is a unital *-representation $\pi : \mathcal{D} \to B(H)$ on a separable Hilbert space H and a sequence of nonzero projections $p_n \in B(H)$ of finite rank $k(n)$ such that $\lim_{n\to\infty} \|[p_n, \pi(d)]\| = 0$ for all $d \in \mathcal{D}$. Then the sequence of u.c.p. maps $\varphi_n : \mathcal{D} \to p_n B(H) p_n \cong M_{k(n)}(\mathbb{C}) \subset \mathcal{Q}$ is asymptotically multiplicative, i.e $\lim_{n\to\infty} \|\varphi_n(cd) - \varphi_n(c)\varphi_n(d))\| = 0$ for all $c, d \in \mathcal{D}$. Therefore $\mathcal{Q} \cong \mathcal{Q} \otimes \mathcal{D}$ by Proposition 3.8.

In the second part of the proof we show that $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Q}$. Let $E_n : \mathcal{Q} \to M_{n!}(\mathbb{C}) \subset \mathcal{Q}$ be a conditional expectation onto $M_{n!}(\mathbb{C})$. Then $\lim_{n\to\infty} \|E_n(a) - a\| = 0$ for all $a \in \mathcal{Q}$.

By assumption, for each n there is a projection e in $\mathcal{D} \otimes M_m(\mathbb{C})$ (for some m) such that $n![e] = [\mathbf{1}_{\mathcal{D}}]$ in $K_0(\mathcal{D})$. Let $\varphi : M_{n!}(\mathbb{C}) \to M_{n!}(\mathbb{C}) \otimes e(\mathcal{D} \otimes M_m(\mathbb{C}))e$ be defined by $\varphi(b) = b \otimes e$. Since $\mathcal{D}$ has cancellation of projections and since $n![e] = [\mathbf{1}_{\mathcal{D}}]$, there is a partial isometry $v \in M_{n!}(\mathbb{C}) \otimes D \otimes M_m(\mathbb{C})$ such that $v^*v = \mathbf{1}_{M_{n!}(\mathbb{C})} \otimes e$ and $vv^* = e_{11} \otimes \mathbf{1}_{\mathcal{D}} \otimes e_{11}$. Therefore $b \mapsto v\,\varphi(b)\,v^*$ gives a unital embedding of $M_{n!}(\mathbb{C})$ into $\mathcal{D}$. Finally, $\psi_n(a) = v\,(\varphi \circ E_n(a))\,v^*$ defines a sequence of asymptotically multiplicative u.c.p. maps $\mathcal{Q} \to \mathcal{D}$. Therefore $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Q}$ by Proposition 3.8. ▮

3.10 REMARK: Let $\mathcal{D}$ be a separable, unital, strongly self-absorbing and quasidiagonal C^*-algebra. Then $\mathcal{D} \otimes \mathcal{Q} \cong \mathcal{Q}$ by the first part of the proof of Proposition 3.9. In particular $K_1(\mathcal{D}) \otimes \mathbb{Q} = 0$ and $K_0(\mathcal{D}) \otimes \mathbb{Q} \cong \mathbb{Q}$ by the Künneth formula (or by writing $\mathcal{Q}$ as an inductive limit of matrices).

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