

## RESIDUALLY AF EMBEDDABLE $C^*$ -ALGEBRAS

MARIUS DADARLAT and VALENTIN DEACONU

*Communicated by Editor*

ABSTRACT. Suppose that a separable exact  $C^*$ -algebra  $A$  is KK-equivalent to a commutative  $C^*$ -algebra and that  $A$  has a separating sequence of unital  $*$ -homomorphisms into simple AF algebras. Under these conditions we show that  $A$  embeds unitaly in a simple AF algebra. We apply this result to a class of amenable groups.

KEYWORDS:  $C^*$ -algebra, residually AF

MSC (2000): Primary 46L05; Secondary 46L55.

### INTRODUCTION

The AF embeddability (or even quasidiagonality) of the  $C^*$ -algebra of a countable discrete amenable group  $\Gamma$  is an important open question. Let us recall that a group is maximally almost periodic if it embeds as a subgroup of the infinite product of the unitary groups  $\prod_{n=1}^{\infty} U(n)$ . It was proved in [5] that if  $\Gamma$  is assumed in addition to be maximally almost periodic, then  $C^*(\Gamma)$  embeds in a simple AF algebra, in fact in the UHF algebra of type  $2^{\infty}$ . The purpose of this note is to generalize this result to countable discrete amenable groups  $\Gamma$  which embed as subgroups of a product of unitary groups  $\prod_{n=1}^{\infty} U(B_n)$  of simple separable unital AF algebras  $B_n$ , see Theorem 2.3. We exhibit examples showing that the class of groups we study here is strictly larger than the class of countable discrete amenable maximally almost periodic groups, see Example 2.5.

The approach of this paper refines an idea that goes back to [4]. Specifically we show that if a separable exact  $C^*$ -algebra  $A$  satisfies the Universal Coefficient Theorem for Kasparov groups, abbreviated as UCT (or equivalently if  $A$  is KK-equivalent to a commutative  $C^*$ -algebra) and if  $A$  embeds in a product  $\prod_{n=1}^{\infty} B_n$  of simple separable unital AF algebras, then  $A$  embeds in an exact tracially AF

algebra satisfying the UCT. The latter algebras are known to be embeddable in simple AF algebras by a result of Lin [10] in the nuclear case, and by [5] in the exact case. In order to apply this result to groups, we first show that  $C^*(\Gamma)$  is residually AF-embeddable for groups  $\Gamma$  as above, and then we invoke a result of Tu [14] according to which the  $C^*$ -algebra of a countable discrete amenable group satisfies the UCT.

## 1. RESIDUAL AF EMBEDDABILITY

We begin by recalling the definition of tracially AF algebras. Since we are interested only in simple algebras, we will work with the following equivalent definition.

DEFINITION 1.1. A simple separable unital  $C^*$ -algebra  $A$  is called *tracially AF* if for any finite set  $\mathcal{F} \subset A$ , any  $\varepsilon > 0$  and any nonzero projection  $p \in A$  there is a nonzero projection  $q \in A$  and a finite dimensional  $C^*$ -subalgebra  $D \subset qAq$  with  $1_D = q$  such that

- 1)  $\|qa - aq\| < \varepsilon$  for all  $a \in \mathcal{F}$ ,
- 2)  $\text{dist}(qaq, D) < \varepsilon$  for all  $a \in \mathcal{F}$ ,
- 3)  $q^\perp = 1 - q$  is unitarily equivalent to a subprojection of  $p$ .

Note that in view of Lemma 2.12 and Proposition 3.8 of [9], the previous definition is equivalent with the original definition from [9] for the class of simple  $C^*$ -algebras. It is easy to verify that it suffices to require that  $D$  is a unital AF algebra rather than a finite dimensional  $C^*$ -algebra.

DEFINITION 1.2. A  $C^*$ -algebra  $A$  is called *residually embeddable in simple AF algebras* if there is a sequence of unital  $*$ -homomorphisms  $\pi_n : A \rightarrow D_n$ , where  $D_n$  are separable simple unital AF algebras, such that for any nonzero  $a \in A$  there is  $n \geq 1$  with  $\pi_n(a) \neq 0$ . This is equivalent to the requirement that  $A$  embeds as a  $C^*$ -subalgebra of  $\prod_{n=1}^{\infty} D_n$ .

PROPOSITION 1.3. *Let  $A$  be a separable unital exact  $C^*$ -algebra which is residually embeddable into simple AF algebras. Then  $A$  embeds in a simple tracially AF algebra  $E$ . Moreover, if  $A$  satisfies the UCT, then one can arrange that  $E$  also satisfies the UCT.*

*Proof.* Consider  $\pi_n : A \rightarrow D_n$  as above. We replace the sequence  $(\pi_n)_n$  by a sequence in which each  $\pi_n$  repeats itself infinitely many times. Moreover, we embed unitaly each  $D_n$  into  $B = \bigotimes_{n=1}^{\infty} D_n$ . Thus we have obtained a sequence of unital  $*$ -homomorphisms  $\pi_n : A \rightarrow B$  (where  $B$  is a simple AF algebra) with the property that for each  $k \geq 1$  and each nonzero element  $a \in A$  there is  $n \geq k$  such that  $\pi_n(a) \neq 0$ . After replacing  $B$  by  $B \otimes \text{UHF}(2^\infty)$  we may assume that for each

$n \geq 1$  there is a projection  $e_n \in B$  such that  $n \cdot \langle e_n \rangle < \langle 1 \rangle$  in  $V(B)$ , where  $1$  denotes the unit of  $B$  and  $V(B)$  is the Murray-von Neumann semigroup of equivalence classes of projections in matrices over  $B$ . Let us define  $\varphi_n : A \rightarrow A \otimes B \otimes B$  by

$$\varphi_n(a) = a \otimes 1 \otimes e_n + 1_A \otimes \pi_n(a) \otimes e_n^\perp.$$

Let us set  $B_n = B^{\otimes 2n}$ ,  $A_n = A \otimes B_n$  and define  $\psi_n : A_n \rightarrow A_{n+1}$  by  $\psi_n = \varphi_n \otimes \text{id}_{B_n}$ , for  $n \geq 1$ . Let us denote by  $\psi_{n,m}$  the  $*$ -homomorphism  $A_n \rightarrow A_m$  obtained as  $\psi_{n,m} = \psi_{m-1} \circ \cdots \circ \psi_n$ ,  $m > n$ . Let  $E$  be defined as the inductive limit  $C^*$ -algebra of the system  $(A_n, \psi_n)$ . By [13],  $E$  absorbs  $\text{UHF}(2^\infty)$  tensorially. Alternatively, one can replace  $E$  by  $E \otimes \text{UHF}(2^\infty)$ . Since  $E$  is stably finite as both  $A$  and  $B$  are quasidiagonal, it follows that  $E$  has stable rank one and  $K_0(E)$  is weakly unperforated by [11], and respectively [7].

It is clear that the maps  $\psi_n$  are unital and injective and hence that  $A$  embeds unitaly in  $E$ . We shall prove that  $E$  is simple. Seeking a contradiction, suppose that  $E$  has a proper two-sided closed ideal  $J$ . Since  $B$  is a simple AF-algebra, so is each  $B_n$ . Therefore there are proper two-sided closed ideals  $J_n$  of  $A$  (starting with  $n$  sufficiently large) such that  $\psi_n(J_n \otimes B_n) \subseteq J_{n+1} \otimes B_{n+1}$  and  $J = \varinjlim (J_n \otimes B_n, \psi_n)$ . Let  $a \in J_n, a \neq 0$ . Then there is  $m > n$  with  $\pi_{m-1}(a) \neq 0$ . We shall prove that  $\psi_{n,m}(a \otimes 1_{B_n})$  is a full element of  $A_m$  and hence  $J_m \otimes B_m = A \otimes B_m = A_m$  which will contradict the properness of the ideal  $J_m$ . Indeed one proves by induction on  $m \geq n+1$  that, for  $a \in A$ ,  $\psi_{n,m}(a \otimes 1_{B_n})$  is of the form

$$\psi_{n,m}(a \otimes 1_{B_n}) = (a \otimes 1 \otimes e_{m-1}) \otimes p_{n,m} +$$

$$(1_A \otimes \pi_{m-1}(a) \otimes e_{m-1}^\perp) \otimes p_{n,m} + (1_A \otimes 1 \otimes 1) \otimes \sigma_{n,m}(a),$$

where  $p_{n,m} \in B_{m-1}$  is a nonzero projection and  $\sigma_{n,m} : A \rightarrow B_{m-1}$  is a  $*$ -morphism such that  $\sigma_{n,m}(1_A)$  is orthogonal to  $p_{n,m}$ . One has  $p_{n,m+1} = 1 \otimes e_{m-1} \otimes p_{n,m}$  and  $\sigma_{n,m+1}(a) = \pi_{m-1}(a) \otimes e_{m-1}^\perp \otimes p_{n,m} + 1 \otimes 1 \otimes \sigma_{n,m}(a)$ . Since  $B$  is simple and  $\pi_{m-1}(a) \neq 0, p_{n,m} \neq 0$ , it follows that the element  $1_A \otimes \pi_{m-1}(a) \otimes e_{m-1}^\perp \otimes p_{n,m}$  is full in  $A_m$  and hence so is  $\psi_{n,m}(a \otimes 1_{B_n})$ .

To prove that  $E$  is tracially AF, consider a finite subset  $\mathcal{F} \subset E$ , a fixed  $\varepsilon > 0$  and a nonzero projection  $p \in E$ . Since  $E$  is simple, there is  $n \geq 1$  such that  $\langle 1_E \rangle < n \cdot \langle p \rangle$  in  $V(E)$ . Without loss of generality, after a small perturbation of  $\mathcal{F}$  we may assume that  $\mathcal{F}$  is equal to the image in  $E$  of some finite subset (denoted again by  $\mathcal{F}$ ) of the algebraic tensor product  $A \otimes_{\text{alg}} B_m \subset A_m$ , for some  $m \geq n$ . Set  $q_m = 1_A \otimes 1 \otimes e_m^\perp \otimes 1_{B_m} \in A_{m+1}$  and let  $q$  denote the image of  $q_m$  in  $E$ . Then  $1_{A_{m+1}} - q_m = 1_A \otimes 1 \otimes e_m \otimes 1_{B_m}$ . By construction,  $m \cdot \langle e_m \rangle \leq \langle 1 \rangle$  and hence  $m \cdot \langle q^\perp \rangle < n \cdot \langle p \rangle$  in  $V(E)$ . Since  $E$  has stable rank one and  $K_0(E)$  is weakly unperforated we conclude that  $q^\perp$  is unitarily equivalent to a subprojection of  $p$ . On the other hand  $q_m$  commutes with the image of  $\psi_m$  and the compression of  $\psi_m(\mathcal{F})$  by  $q_m$  is contained in the AF-algebra  $1_A \otimes B \otimes e_m^\perp \otimes B_{m-1}$ .

If  $A$  satisfies the UCT, then  $A \otimes B^{\otimes 2n}$  also does, hence  $E$  satisfies the UCT. ■

Let us recall the following result contained in the proof of Corollary 6.6 of [5]. The corresponding result for nuclear tracially AF algebras is due to Lin [10].

**PROPOSITION 1.4.** *If  $E$  is a separable simple exact unital tracially AF algebra satisfying the UCT, then  $E$  is embeddable in an AF-algebra.*

*Proof.* If  $E$  is as in the statement, then as explained on page 933 of [5], the image of the canonical map  $\rho : K_0(E) \rightarrow \text{AffS}(K_0(E))$  is a dimension group and hence there is a simple separable AF algebra  $E_0$  with

$$(K_0(E_0), [1_{E_0}]) \cong (\rho(K_0(E)), \rho(1_E)).$$

The map  $\rho : K_0(E) \rightarrow K_0(E_0)$  lifts to a unital  $*$ -monomorphism  $E \rightarrow E_0$  by [5, Theorem 6.4]. ■

Combining Propositions 1.3 and 1.4 we obtain the following

**THEOREM 1.5.** *Let  $A$  be a separable exact  $C^*$ -algebra satisfying the UCT. Suppose that  $A$  is residually embeddable into simple AF algebras. Then  $A$  embeds in a simple unital AF algebra.*

## 2. APPLICATIONS TO GROUPS

The following proposition and its proof was inspired by [1, Proposition 1].

**PROPOSITION 2.1.** *Let  $\Gamma$  be a discrete countable amenable group. Let  $(B_n)_{n=1}^\infty$  be a sequence of unital  $C^*$ -algebras and let  $(\omega_n)_{n=1}^\infty$  be an infinite-multiplicity sequence of group homomorphisms  $\omega_n : \Gamma \rightarrow U(B_n)$  separating the points of  $\Gamma$ . Then  $C^*(\Gamma)$  embeds unitaly in  $\prod_{n=1}^\infty C_n$ , where  $C_n = \bigotimes_{k=1}^n M_2 \otimes B_k \otimes B_k$ .*

*Proof.* By assumption, the infinite-multiplicity sequence  $(\omega_n)$  separates the points of  $\Gamma$ . Therefore there is an injective map  $s \mapsto n(s)$  from  $\Gamma$  to  $\mathbb{N}$  such that  $\omega_{n(s)}(s) \neq 1$  for  $s \in \Gamma \setminus \{e\}$ . Define  $\mu_n : \Gamma \rightarrow U(M_2 \otimes B_n \otimes B_n)$  by

$$\mu_n(s) = \begin{pmatrix} \omega_n(s) \otimes 1 & 0 \\ 0 & \omega_n(s) \otimes \omega_n(s) \end{pmatrix}.$$

We regard  $M_2 \otimes B_n \otimes B_n$  as acting on a Hilbert space  $\mathcal{H}_n$  and denote by  $\pi_n : C_n \rightarrow \mathcal{L}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$  the corresponding tensor product representation. The spectrum of  $\mu_{n(s)}(s)$  contains at least two points since  $\omega_{n(s)}(s) \neq 1$ . Using the spectral theorem, we find a sequence  $\xi_n \in \mathcal{H}_n$ ,  $\|\xi_n\| = 1$ , so that if  $\varphi_n(s) = \langle \mu_n(s)\xi_n, \xi_n \rangle$ , then  $|\varphi_n(s)| < 1$ , for all  $s \in \Gamma$ ,  $s \in \Gamma \setminus \{e\}$ . Define  $\chi_n : \Gamma \rightarrow U(C_n)$  by  $\chi_n(s) = \mu_1(s) \otimes \cdots \otimes \mu_n(s)$ . Then

$$\varphi_n(s) = \varphi_1(s) \cdots \varphi_n(s) = \langle \pi_n \chi_n(s) \xi_1 \otimes \cdots \otimes \xi_n, \xi_1 \otimes \cdots \otimes \xi_n \rangle$$

is a positive definite function associated with the representation  $\pi_n \chi_n : \Gamma \rightarrow \mathcal{L}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ . Let  $\delta_e$  be the Dirac function at the unit of  $\Gamma$ . Then  $\lim_{n \rightarrow \infty} \phi_n(s) = \delta_e(s)$  for all  $s \in \Gamma$ , since  $\phi_n(e) = 1$  and  $|\phi_n(s)| < 1$  for  $s \neq e$ , and the sequence  $(\phi_n)$  has infinite multiplicity. Since the positive definite function  $\delta_e$  corresponds to a cyclic vector of the left regular representation  $\lambda_\Gamma : \Gamma \rightarrow \mathcal{L}(\ell^2(\Gamma))$ , it follows by [3, 18.1.4] that  $\lambda_\Gamma$  is weakly contained in  $\{\pi_n \chi_n : n \in \mathbb{N}\}$ . Thus if  $\widehat{\lambda}_\Gamma : C^*(\Gamma) \rightarrow \mathcal{L}(\ell^2(\Gamma))$  and  $\widehat{\chi}_n : C^*(\Gamma) \rightarrow C_n$  denote the extensions of  $\lambda_\Gamma$  and  $\chi_n$  to  $C^*(\Gamma)$ , then by [3, 3.4.4]

$$\ker \widehat{\lambda}_\Gamma \supset \bigcap_{n=1}^{\infty} \ker \pi_n \widehat{\chi}_n = \bigcap_{n=1}^{\infty} \ker \widehat{\chi}_n.$$

Since  $\Gamma$  is amenable,  $\ker \widehat{\lambda}_\Gamma = \{0\}$ , hence the unital  $*$ -homomorphism

$$\prod_{n=1}^{\infty} \widehat{\chi}_n : C^*(\Gamma) \rightarrow \prod_{n=1}^{\infty} C_n$$

is injective. ■

**COROLLARY 2.2.** *If a discrete countable amenable group  $\Gamma$  embeds in a product of unitary groups of separable unital simple AF algebras, then its  $C^*$ -algebra  $C^*(\Gamma)$  is residually embeddable into separable unital simple AF algebras.*

*Proof.* By assumption  $\Gamma$  embeds in a product of unitary groups of simple separable and unital AF algebras  $B_n$ . Therefore we can apply Proposition 2.1 to obtain that  $C^*(\Gamma)$  is residually embeddable into simple AF algebras since each  $C_n$  is a simple separable AF algebra. ■

**THEOREM 2.3.** *If a discrete countable amenable group  $\Gamma$  embeds in a product of unitary groups of separable unital simple AF algebras, then its  $C^*$ -algebra  $C^*(\Gamma)$  embeds in a unital separable simple AF algebra.*

*Proof.* Since  $\Gamma$  is amenable,  $C^*(\Gamma)$  satisfies the UCT by [14, Proposition 10.7] and is residually embeddable into simple AF algebras by Corollary 2.2. The result follows now by applying Theorem 1.5. ■

**COROLLARY 2.4.** *Let  $\Gamma$  be a discrete countable amenable group. The following are equivalent.*

(1) *There is a sequence  $(B_n)$  of simple unital separable AF algebras with  $\Gamma \subset \prod_{n=1}^{\infty} U(B_n)$ .*

(2) *There is a simple unital separable AF algebra  $B$  with  $\Gamma \subset U(B)$ .*

(3) *There is a simple unital separable AF algebra  $B$  such that  $C^*(\Gamma) \subset B$ .*

Let us note that the proof of the equivalence of (1) and (2) is based on their equivalence to (3). We see no alternative way for proving it.

EXAMPLE 2.5. The class of discrete amenable maximally almost periodic groups is obviously contained in the class of discrete amenable groups which embed in a product of unitary groups of unital simple AF algebras. We shall show here that the latter class is strictly larger. By [8, Thm 3.1] if  $G$  and  $H$  are residually finite groups, then the wreath product of  $G$  by  $H$ , denoted by  $G \wr H$ , is residually finite if and only if either  $G$  is abelian or  $H$  is finite. The discrete Heisenberg group  $\mathbb{H}_n$  is residually finite and nonabelian if  $n \geq 2$ , and by [6] its  $C^*$ -algebra embeds in the UHF algebra of type  $2^\infty$ , denoted here by  $D$ . The wreath product  $\mathbb{H}_n \wr \mathbb{Z}^k \cong (\bigoplus_{\mathbb{Z}^k} \mathbb{H}_n) \rtimes \mathbb{Z}^k$  is finitely generated but not residually finite (if  $n \geq 2$ ) and hence it is not maximally almost periodic [15]. On the other hand,

$$C^*(\mathbb{H}_n \wr \mathbb{Z}^k) \cong \left( \bigotimes_{\mathbb{Z}^k} C^*(\mathbb{H}_n) \right) \rtimes \mathbb{Z}^k \subset \left( \bigotimes_{\mathbb{Z}^k} D \right) \rtimes \mathbb{Z}^k$$

and  $(\bigotimes_{\mathbb{Z}^k} D) \rtimes \mathbb{Z}^k \cong D \rtimes \mathbb{Z}^k$  embeds in a simple unital AF algebra by a result of Nate Brown [2]. In conclusion,  $\mathbb{H}_n \wr \mathbb{Z}^k$  is not maximally almost periodic if  $n \geq 2$  and embeds in the unitary group of a separable simple unital AF algebra. The same conclusion holds if  $\mathbb{H}_n$  is replaced by a countable discrete amenable nonabelian maximally almost periodic group  $G$  since  $C^*(G)$  embeds in  $D$  by [6].

In the last part of the paper we give an application to transformation  $C^*$ -algebras.

LEMMA 2.6. *Let  $\Gamma$  be a discrete group acting by automorphisms on a unital  $C^*$ -algebra  $A$ . Suppose that  $A$  has a sequence of  $\Gamma$ -invariant two-sided closed ideals  $(I_n)$  with  $\bigcap_{n=1}^{\infty} I_n = \{0\}$ . Then  $A \rtimes_r \Gamma$  embeds unitaly in  $\prod_{n=1}^{\infty} (A/I_n \rtimes_r \Gamma)$ .*

*Proof.* This is similar to the proof of [12, Theorem 4.1.10]. The map

$$\ell^1(\Gamma, A) \rightarrow A, \quad (a_s)_{s \in \Gamma} \mapsto a_e$$

extends to a faithful conditional expectation  $E_A : A \rtimes_r \Gamma \rightarrow A$  by [16, 4.12]. Consider the commutative diagram

$$\begin{array}{ccc} A \rtimes_r \Gamma & \xrightarrow{\pi_n} & A/I_n \rtimes_r \Gamma \\ \downarrow E_A & & \downarrow E_{A/I_n} \\ A & \longrightarrow & A/I_n. \end{array}$$

We claim that the map  $\prod \pi_n : A \rtimes_r \Gamma \rightarrow \prod (A/I_n \rtimes_r \Gamma)$  is injective. Indeed, let  $x \in A \rtimes_r \Gamma$ ,  $x \geq 0$ , be such that  $\pi_n(x) = 0$  for all  $n$ . From the commutative diagram, we obtain that  $E_A(x) \in I_n$  for all  $n$ , hence  $E_A(x) = 0$  since  $\bigcap_{n=1}^{\infty} I_n = \{0\}$  by assumption. Therefore  $x = 0$  since  $E_A$  is faithful. ■

**COROLLARY 2.7.** *Let  $\Gamma$  be a discrete countable amenable group which is isomorphic to a subgroup of a countable product of unitary groups of simple unital separable AF algebras. Suppose that  $\Gamma$  acts on a compact metrisable space such that the points with finite orbits are dense in  $X$ . Then  $C(X) \rtimes_r \Gamma$  embeds in a unital simple separable AF algebra.*

*Proof.* Since  $\Gamma$  is amenable, so is any of its subgroups  $H$ , and  $C_r^*(H) \cong C^*(H) \subset C^*(\Gamma) \cong C_r^*(\Gamma)$ . By assumption there is a dense sequence  $(x_n)$  of points of  $X$  such that each isotropy group  $\Gamma_{x_n} = \{s \in \Gamma : s \cdot x_n = x_n\}$  has finite index in  $\Gamma$ ,  $[\Gamma : \Gamma_{x_n}] = m(n) < \infty$ . Let  $I_n$  denote the ideal of  $C(X)$  consisting of all functions vanishing on the orbit  $X_n$  of  $x_n$ . Then

$$A/I_n \rtimes_r \Gamma \cong C(X_n) \rtimes_r \Gamma \cong C_r^*(\Gamma_{x_n}) \otimes M_{m(n)} \subset C_r^*(\Gamma) \otimes M_{m(n)}.$$

We have  $\bigcap_{n=1}^{\infty} I_n = \{0\}$  since  $(x_n)$  is dense in  $X$ . Therefore  $C(X) \rtimes_r \Gamma$  embeds unitaly in  $\prod_{n=1}^{\infty} C_r^*(\Gamma) \otimes M_{m(n)}$  by Lemma 2.6. By Theorem 2.3,  $C_r^*(\Gamma) \otimes M_{m(n)}$  embeds unitaly in a simple unital AF algebra  $B$ , hence  $C(X) \rtimes_r \Gamma \subset \prod_{n=1}^{\infty} B$ . The groupoid  $X \times \Gamma$  is amenable, hence  $C(X) \rtimes_r \Gamma$  satisfies the UCT by [14, Proposition 10.7]. We conclude the proof by applying Theorem 2.3. ■

Note that this corollary applies to actions of discrete countable amenable residually finite groups (including  $\mathbb{Z}^n$ ), provided that they have dense sets of points with finite orbits. However this condition is not in general necessary for the AF-embeddability of  $C(X) \rtimes_r \mathbb{Z}^n$ .

*Acknowledgements.* The research of M.D. was supported in part by NSF Grant #DMS-103066. The research of V.D. was supported in part by a UNR Grant.

## REFERENCES

- [1] M. B. BEKKA, On the full  $C^*$ -algebras of arithmetic groups and the congruence subgroup problem, *Forum Math.* **11** (1999), no. 6, 705–715.
- [2] N. BROWN, Crossed products of UHF algebras by some amenable groups, *Hokkaido Math. J.* **29** (2000), no. 1, 201–211.
- [3] J. DIXMIER, *Les  $C^*$ -algèbres et leurs représentations*, (French) Deuxième édition. Cahiers Scientifiques, Fasc. XXIX. Gauthier-Villars Éditeur, Paris 1969 xv+390 pp.
- [4] M. DADARLAT, Residually finite-dimensional  $C^*$ -algebras, *Operator algebras and operator theory (Shanghai, 1997)*, 45–50, *Contemp. Math.*, **228**, Amer. Math. Soc., Providence, RI, 1998.
- [5] M. DADARLAT, Morphisms of simple tracially AF algebras, *Internat. J. Math.* **15** (2004), no. 9, 919–957.

- [6] M. DADARLAT, On the topology of the Kasparov groups and its applications, *J. Funct. Anal.* **228** (2005), no. 2, 394–418.
- [7] G. GONG, X. JIANG AND H. SU, Obstructions to  $\mathcal{Z}$ -stability for unital simple  $C^*$ -algebras, *Canadian Math. Bull.* **43** (2000), 418–426.
- [8] K. W. GRUENBERG, Residual properties of infinite soluble groups, *Proc. London Math. Soc.* **7** (1957) no. 3, 29–62.
- [9] H. LIN, Tracially AF  $C^*$ -algebras, *Trans. Amer. Math. Soc.* **353** (2001), no. 2, 693–722.
- [10] H. LIN, Residually finite dimensional and AF-embeddable  $C^*$ -algebras, *Proc. Amer. Math. Soc.* **129** (2001), no. 6, 1689–1696.
- [11] M. RØRDAM, On the Structure of Simple  $C^*$ -algebras Tensored with a UHF-Algebra, *J. Funct. Anal.* **100** (1991), 1–17.
- [12] J. TOMIYAMA, *Invitation to  $C^*$ -algebras and topological dynamics*, World Scientific Advanced Series in Dynamical Systems, 3. World Scientific Publishing Co., Singapore, 1987. x+167 pp.
- [13] A. S. TOMS AND W. WINTER, Strongly self-absorbing  $C^*$ -algebras, *Trans. Amer. Math. Soc.* **359** (2007), 3999–4029.
- [14] J. L. TU, La conjecture de Baum-Connes pour les feuilletages moyennable, *K-Theory* **17** (1999), no. 3, 215–264.
- [15] S. WASSERMANN, *Exact  $C^*$ -algebras and related topics*, Lecture Notes Series, 19 Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1994.
- [16] G. ZELLER-MEIER, Produits croisés d’une  $C^*$ -algèbre par une groupe d’automorphismes, *J. Math. Pures. Appl.* **47** (1968), 101–239.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE IN 47907,  
USA

*E-mail address*, Marius Dadarlat: mdd@math.purdue.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEVADA, RENO  
NV 89557-0084, USA

*E-mail address*, Valentin Deaconu: vdeaconu@unr.edu

Received Month dd, yyyy; revised Month dd, yyyy.