REDUCTION TO DIMENSION THREE OF LOCAL SPECTRA OF REAL RANK ZERO C*-ALGEBRAS

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In this paper we deal with C*-algebras of real rank zero that can be represented as inductive limits $A = \varinjlim(A_n, \nu_{n+1,n})$ of direct sums of homogeneous C*-algebras of the form $A_n = \oplus_{i=1}^{s(n)} P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i}$, where $X_{n,i}$ are finite CW complexes and $P_{n,i}$ are selfadjoint projections. Following [3] we called these C*-algebras approximately homogeneous.

Our main result asserts that any approximately homogeneous C*-algebra of real rank zero with $\sup_{n,i}(dim(X_{n,i})) < \infty$ or with slow dimension growth is isomorphic to an inductive limit of direct sums of homogeneous C*-algebras whose spectra have dimension at most three (see Theorem 3.2). By a remarkable recent result of Elliott and Gong, [31], the simple C*-algebras in the latter class are classified up to isomorphism by the ordered, scaled K-theory group $K_* = K_0 \oplus K_1$. Combining the two results one obtains a classification of all simple approximately homogeneous C*-algebras of real rank zero with slow dimension growth (see Theorem 3.4). The conditions on the growth of the dimensions of the local spectra $X_{n,i}$ seem to be necessary in order to avoid pathologies originating in nonstable homotopy theory. The special case of Theorem 3.4 when $K^0(X_{n,i})$ are torsion free was proved in [21]. The case when $K^*(X_{n,i})$ are torsion free is due to Gong [G].

Theorem 3.4 can be regarded as a dynamic hypostasis of Bott periodicity. This can be better understood if we compare Theorem 3.4 with a result from [19] asserting that the asymptotic homotopy type of $C_0(X) \otimes \mathcal{K}$ is determined by the K-theory group $K^*(X)$. Here X is a compact connected metrisable space with base point and $\mathcal{K}$ denotes the compact operators. In particular $C_0(S^1) \otimes \mathcal{K}$ is asymptotically homotopy equivalent (but not homotopy equivalent, [24]) to $C_0(S^3) \otimes \mathcal{K}$ [15]. It should become now visible that the cited result is essentially a reformulation of the Bott periodicity theorem. Its proof is based on the theory of asymptotic morphisms of Connes and Higson [15] and involves a suspension theorem of [22]. In the proof of Theorem 3.2 we use the above version of Bott periodicity to replace the spaces $X_{n,i}$ by lower dimensional spaces with the same K-theory groups. Due to dynamical properties of the real rank zero C*-algebras one can do these changes without changing the isomorphism class of the inductive limit C*-algebra. The present paper should be regarded as a continuation of [21]. In particular the key Lemma 1.7 which relates homotopy of approximate morphisms to approximate unitary equivalence of morphisms was proved in [21].

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One may conjecture as in [31] that any separable nuclear, simple C*-algebras of real rank zero and stable rankone is isomorphic to an approximately homogeneous C*-algebra. Positive evidence for this conjecture is offered by results in [2], [36], [44], [30], [7], [8], [9]. However if one drops the simplicity condition, then the corresponding conjecture is known to be false [11].

The problem of finding suitable invariants for the study of approximately homogeneous C*-algebras was possed by Effros [27]. This problem is now incorporated in the program, formulated by Elliott, and whose ambitious goal is a classification theory of nuclear C*-algebras. The results that are available so far suggest that the nuclear C*-algebras of real rank zero display rigidity properties that give hopes for quite general classification results (see [6], [12], [28], [43], [29], [37], [32], [33], [34], [31], [11], [G], [45], [40]).

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1. Preliminaries

Let A, B, C, D be C*-algebras. Let $\varphi : A \to B$, $\psi : A \to C$ and $\gamma : B \to D$ be $*$-homomorphisms. We denote by $\varphi \oplus \psi : A \to B \oplus C$ (or sometimes by $diag(\varphi, \psi)$) the $*$-homomorphism $a \mapsto \varphi(a) \oplus \psi(a)$ and by $\psi \oplus \gamma : A \oplus B \to C \oplus D$ the $*$-homomorphism $a \oplus b \mapsto \psi(a) \oplus \gamma(b)$. We denote by $[A, B]$ the homotopy classes of $*$-homomorphisms from A to B. The homotopy classes of unital $*$-homomorphisms are denoted by $[A, B]_1$. Let M_k denote the C*-algebra of k by k complex matrices with unit denoted by 1_k. The identity map of M_k is denoted by id_k. If X is a compact Hausdorff space we denote by $C(X)$ the complex valued continuous functions on X and by $M_k(C(X))$ the k by k matrices over $C(X)$. In this paper we reserve the term homogeneous for C*-algebras of the form $QM_N(C(X))Q$ where X is a finite connected CW complex and Q is a selfadjoint projection in $M_N(C(X))$. In classical terminology these correspond to homogeneous C*-algebras with spectrum X and trivial Dixmier-Douady class. Let $B = \bigoplus_j B_j$ and $A = \bigoplus_i A_i$ be finite direct sums of homogeneous C*-algebras where $B_j = P_j M_{k(j)}(C(Y_j))P_j$ and $A_i = Q_i M_{n(i)}(C(X_i))Q_i$. A $*$-homomorphism $\psi : B \to A$ is said to be *m-large* if for all i, j the partial $*$-homomorphisms $\psi^{i,j}$ satisfy the following two conditions.

a) $\psi^{i,j} = 0$ or $rank(\psi^{i,j}(P_j)) \geq m\, rank(P_j)$

b) $rank(Q_i) - \sum_j rank(\psi^{i,j}(P_j))$ is either zero or $\geq m$.

Let $A = \varinjlim(A_n, \nu_{n+1,n})$ be an approximately homogeneous C*-algebra. Here $A_n = \oplus_{j=1}^{k_n} P_{n,j} M_{[n,j]}(C(X_{n,j}))P_{n,j}$, where $X_{n,j}$ are finite connected CW complexes and $P_{n,j}$ are selfadjoint projections. We say that A has *slow dimension growth* if for any n

$$\lim_{r\to\infty} \max_{1\leq i\leq k_r} \frac{dim(X_{r,i})}{\min\{rank(\nu_{r,n}^{i,j}(P_{n,j})) : \ \nu_{r,n}^{i,j} \neq 0,\ 1 \leq j \leq k_n\}} = 0$$

For simple C*-algebras this condition was introduced in [5]. For nonsimple C*-algebras a condition of this type was first considered in [51]. If A is simple, infinite dimensional, and the sequence $dim(\,Spectrum(A_n))$ is bounded, then A has slow dimension growth. It is not hard to see that if A has slow dimension growth then for any $n, s, m \geq 1$, $s > n$, there is $r > s$ such that $\nu_{r,n}^i = \bigoplus_j \nu_{r,n}^{i,j}$ is $m\, dim(X_{r,i})$-large for all i with $1 \leq i \leq k_r$.

The homotopy classes of m-large $*$-homomorphisms can be computed in terms of K-theory and connective KK-theory whenever m is big enough. Actually for the purposes of this paper it is enough to have a classification of $*$-homomorphisms up to the equivalence relation generated by homotopy and unitary equivalence. We now describe this classification in several steps. Let X, Y be finite connected CW complexes. The computation of homotopy classes of unital $*$-homomorphisms from $C(Y)$ to $M_n(C(X))$ is done in terms of connective KK-theory [24]. The connective KK-theory group $kk(X,Y)$ is defined as a direct limit

$$kk(X,Y) = \varinjlim [C(Y), M_n(C(X))]_1,$$

where the connecting maps are given by taking direct sum with an evaluation map (see [24]). The kk-groups have good excision properties in both variables. The composition and the tensor product of $*$-homomorphisms induce a rich multiplicative structure on $kk(X,Y)$. The excision properties and the multiplicative structure make this group computable in many instances. For concrete computations this algebraic apparatus is complemented by a stability result asserting that if $dim\,(X) < 2[n/3]$, $n \geq 3$, then $kk(X,Y)$ is isomorphic to $[C(Y), M_n(C(X))]_1$ (see Corollary 6.4.4 in [24]) . Here $[x]$ denotes the integer part of x.

The next step is to deal with unital $*$-homomorphisms from $C(X)$ to $QM_N(C(X))Q$. Since any such $*$-homomorphism can be regarded as a $*$-homomorphism into $M_N(C(X))$, one can obtain the homotopy classification of these $*$-homomorphisms from Theorem 4.2.11 in [24]. A more direct approach is taken in [34]. Using elementary obstruction theory they show that the isomorphism

$$kk(X,Y) \cong [C(Y), QM_N(C(X))Q]_1,$$

$rank(Q) > 3dim(X) + 3$, can be easily derived from the stability result in [24]. Actually a refinement of their arguments shows that it suffices to assume that $rank(Q) > 2dim(X) + 1$. Under this assumption it follows that any unital $*$-homomorphism $\psi : C(Y) \to QM_N(C(X))Q$ is homotopic to a direct sum $\psi' \oplus \psi''$ where $\psi' : C(Y) \to Q'M_N(C(X))Q'$, Q' is a trivial subprojection of Q and $\psi'' : C(Y) \to (Q - Q')M_N(C(X))(Q - Q')$ is a $*$-homomorphism which factors through $\mathbb{C}$ (see Theorem 4.2.11 in [24] or [34] for a more direct proof). By a trivial (sub)projection we mean a projection corresponding to a trivial vector bundle. Next we deal with $*$-homomorphisms

$$\varphi, \psi : \oplus_{j=1}^{s} M_{k(j)}(C(Y_j)) \to QM_N(C(X))Q.$$

As a preliminary step we consider the case when the spaces Y_j are reducing to points. Suppose that $m > dim(X)$. Then it is known that the unitary equivalence classes of m-large $*$-homomorphisms

$$\varphi'', \psi'' : \oplus_{j=1}^{s} M_{k(j)} \to QM_N(C(X))Q$$

are classified by K-theory. In particular φ'' is unitarily equivalent to ψ'' if and only if they induce the same map on K-theory. The analogue of the Bratteli diagram is now a matrix of vector bundles over X (see [16], [26], [24], [31]). An easy consequence of this fact is the following Lemma.

Lemma 1.1. *Let $\mu, \eta : B = \oplus_{j=1}^{s} M_{k(j)} \to A = QM_N(C(X))Q$ be two $*$-homomorphisms. Suppose that μ is m-large, $m > dim(X)$ and $rank(\mu(1_{k(j)})) - rank(\eta(1_{k(j)})) > k(j)dim(X)$ for all j. Then there is a $*$-homomorphism $\eta' : B \to A$ and a unitary $u \in U(A)$, such that the image of η' is orthogonal to the range of η and $\mu = u(\eta \oplus \eta')u^*$.*

Let $\psi : B = \oplus_{j=1}^{s} M_{k(j)}(C(Y_j)) \to A = QM_N(C(X))Q$ be an m-large $*$-homomorphism. If $m > 4dim(X)$, then the following homotopy decomposition is available. The $*$-homomorphism ψ is homotopic to a direct sum $\psi' \oplus \psi''$ between a $*$-homomorphism ψ' and a $*$-homomorphism ψ'' that factors through a finite dimensional C*-algebra. There is a trivial subprojection Q' of Q such that ψ' is of the form

$$\psi' : \oplus_{j=1}^{s} M_{k(j)}(C(Y_j)) \to \oplus_{j=1}^{s} M_{k(j)m(j)}(C(X)) \subset M_{N'}(C(X)) \cong Q'M_N(C(X))Q'$$

$$\psi' = \psi_1^0 \otimes id_{k(1)} \oplus \cdots \oplus \psi_s^0 \otimes id_{k(s)}, \quad N' = \sum_{j=1}^{s} k(j)m(j)$$

where $\psi_j^0 : C(Y_j) \to M_{m(j)}(C(X))$ are unital $*$-homomorphisms with $\psi_j^0(C_0(Y_j)) \subset M_{m(j)}(C_0(X))$ and

$$\psi'' : \oplus_{j=1}^{s} M_{k(j)}(C(Y_j)) \to (Q - Q')M_N(C(X))(Q - Q')$$

has finite dimensional image (see section 4 in [24] or [31]).

The classification within homotopy and unitary equivalence of the m-large $*$-homomorphisms is particularly nice for big values of m. Let $\psi, \varphi : B \to A$ be two m-large $*$-homomorphisms. Suppose that $m > 4\, dim(X)$. Let $r(B) = \bigoplus_j M_{k(j)}$. Then there is a unitary $u \in A$ such that $u\psi u^*$ is homotopic to φ if and only if $K_0(\psi|r(B)) = K_0(\varphi|r(B))$ and $[\psi_j^0] = [\varphi_j^0] \in kk(X, Y_j)$. In this classification, the invariants associated with a $*$-homomorphism are a matrix of vector bundles and a matrix of kk-elements. More generally, the $*$-homomorphisms

$$\psi : \bigoplus_{j=1}^{s} P_j M_{k(j)}(C(Y_j))P_j \to QM_N(C(X))Q$$

are classified in a similar manner by dilating ψ to a $*$-homomorphism

$$\widehat{\psi} : \oplus_{j=1}^{s} M_{k(j)}(C(Y_j)) \to Q_1 M_N(C(X))Q_1.$$

With the aid of Lemma 2.13 in [31] one shows that the correspondence $\psi \to \widehat{\psi}$ preserves the equivalence relation generated by homotopy and unitary equivalence. Thus ψ is homotopic to a $*$-homomorphism unitarily equivalent to φ if and only if $\widehat{\psi}$ is homotopic to a $*$- homomorphism unitarily equivalent to $\widehat{\varphi}$.

In particular, up to unitary equivalence and homotopy, any m-large $*$-homomorphism $\varphi : QM_N(C(X))Q \to PM_S(C(Y))P$, $m > 4dim(Y)$, can be obtained as the restriction of $\varphi_0 \otimes id_N$ to $QM_N(C(X))Q$, for some unital $*$-homomorphism $\varphi_0 : C(X) \to RM_S(C(Y))R$. The invariants associated with φ are the kk-class of φ_0 denoted by $\alpha \in kk(Y, X)$ and the K-theory class of R denoted by

$[R]=(k,z)\in\mathbb{Z}\oplus\widetilde{K}^0(Y)$. Here k corresponds to the rank of R and z is the reduced K-theory class of R. It is convenient to write these invariants in a matricial form

$$[\varphi]=\begin{pmatrix}\alpha & z\\ 0 & k\end{pmatrix}$$

for the composition of morphisms will correspond to formal multiplication of matrices. This formal multiplication makes sense since a kk-element induces a map on the reduced K-theory groups [24].

Let $A=QM_N(C(X))Q$, $A'=Q'M_N(C(X))Q'$ be homogeneous C*-algebras. A $*$-monomorphism $\gamma:A\to A'$ is called a *simple embedding* if the kk-theory class of γ is equal to the class of the $id_{C(X)}$. The following lemma is a consequence of the above discussion.

Lemma 1.2. *Let X, Y be finite connected CW complexes. Let $A=QM_N(C(X))Q$, $A'=Q'M_N(C(X))Q'$ and $B=PM_S(C(Y))P$ be homogeneous C*-algebras. Let $q=rank(Q)$, $q'=rank(Q')$ and $p=rank(P)$. Suppose that $\gamma:A\to A'$ is a unital simple embedding. Let $\nu:A\to B$ be a unital $*$-homomorphism. Let $\lambda:A\to M_q$ be an evaluation map. Suppose that $q'>4dim(X)q$ and $p>5dim(Y)q'$. Then there exists a unital $*$-homomorphism $\sigma:A'\oplus M_q\to B$ such that $\sigma\,(\gamma\oplus\lambda)$ is homotopic to ν.*

Proof. (Sketch) The invariants associated with ν,γ,λ are of the form

$$[\nu]=\begin{pmatrix}\alpha & z\\ 0 & k\end{pmatrix},\quad [\gamma]=\begin{pmatrix}1 & w\\ 0 & s\end{pmatrix},\quad [\lambda]=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

Using the hypotheses we see that $s>4dim(X)$. Also we can find integers t_1,t_2 such that $k=t_1s+t_2$ and $t_1>4dim(Y)$, $t_2>dim(Y)$. Let $\sigma_1:A'\to B$ and $\sigma_2:M_q\to B$ be $*$-homomorphisms with orthogonal ranges and with invariants

$$[\sigma_1]=\begin{pmatrix}\alpha & 0\\ 0 & t_1\end{pmatrix},\quad [\sigma_2]=\begin{pmatrix}0 & z-\alpha(w)\\ 0 & t_2\end{pmatrix}$$

Define $\sigma=\sigma_1\oplus\sigma_2:A\oplus M_q\to B$. Then $\sigma(\gamma\oplus\lambda)$ and ν have the same invariants. Hence they coincide modulo homotopy and unitary equivalence. □

Definition 1.3. *Let X be a compact connected space and let Q be a projection of rank n in $M_N(C(X))$. The weak variation of a finite set $F\subset QM_N(C(X))Q$ is defined by*

$$w(F)=\sup_{\pi,\sigma}\inf_{u\in U(n)}\max_{a\in F}\|u\pi(a)u^*-\sigma(a)\|$$

where π,σ run though the set of irreducible representations of $QM_N(C(X))Q$ into M_n.

The definition extends to finite subsets $F\subset\bigoplus_i Q_iM_{N(i)}(C(X_i))Q_i$. Thus $w(F)$ is taken to be the maximum of the weak variations in each direct sumand. It is not hard to see that if ψ is a $*$-homomorphism of homogeneous C*-algebras, then $w(\psi(F))\le w(F)$. The above definition is inspired by Definition 1.4.11 in [31]. Following [31] we say that F is weakly approximately constant to within ϵ if $w(F)<\epsilon$. We need the following easy Lemma which is implicit in [31].

Lemma 1.4. *Let X be a compact metric space. Let $A = QM_N(C(X))Q$ be a homogeneous C*-algebra. Let $F \subset A$ be a finite set and let $\epsilon > 0$. Then there exist $k \in \mathbb{N}$ and a unital $*$-homomorphism $\mu : A \to M_k(A)$ of the form $\mu(a) = diag(a(x_1), \dots, a(x_k))$, with $x_1, \dots, x_k \in X$ such that the weak variation of $\{(a, \mu(a)) :\ a \in F\}$ is less than ϵ.*

Definition 1.5. *A C*-algebra A is said to have property (H), if for any finite subset $F \subset A$ and any $\epsilon > 0$, there exist $r \in \mathbb{N}$, a $*$-homomorphism $\tau : A \to M_{r-1}(A)$ and a $*$-homomorphism $\mu : A \to M_r(A)$ with finite dimensional image such that*

$$\|a \oplus \tau(a) - \mu(a)\| < \epsilon$$

for all $a \in F$.

It is clear that any finite dimensional C*-algebra has property (H). Theorem 1.2 in [21] shows that if X is a finite CW complex, then $C(X)$ has property (H). This result is implicit in [31] and is based on techniques developed in [37] and [33]. It is not hard to see that the class of nuclear C*-algebras with property (H) is closed under direct sums and tensor products.

Lemma 1.6. *Let X be a finite connected CW-complex and let $A = QM_N(C(X))Q$ be a homogeneous C*-algebra. Then A has property (H).*

Proof. Let $D = M_N(C(X))$ and fix $\epsilon > 0$, $F \subset A$ finite. Since D has property (H), there exist $r \in \mathbb{N}$, a $*$-homomorphism $\tau : D \to M_{r-1}(D)$ and a $*$-homomorphism $\mu : D \to M_r(D)$ with finite dimensional image such that $\|a \oplus \tau(a) - \mu(a)\| < \epsilon$ for all $a \in F$. Using stability properties of vector bundles we can find integers k, K and partial isometries $v \in M_k(D)$ and $V \in M_K(D)$ such that $v^*v = 1_N - Q$, $vv^* \leq 0_N \oplus Q \otimes 1_{k-1}$ and $V^*V = 1_N \otimes 1_{r-1}$, $VV^* \leq Q \otimes 1_K$. Then $w = (v + Q) \oplus V$ is a partial isometry and $waw^* = a$ for all $a \in A$. Moreover $VM_{r-1}(D)V^* \subset M_K(A)$, and $wM_r(D)w^* \subset M_{k+K}(A)$. Define $\tau_0 : A \to M_{k+K}(A)$ by $\tau_0(a) = V\tau(a)V^*$. Define $\mu_0 : A \to M_{k+K}(A)$ by $\mu_0(a) = w\mu(a)w^*$. Then $\|a \oplus \tau_0(a) - \mu_0(a)\| < \epsilon$ for all $a \in F$. □

Let A, B be C*-algebras. We denote by $Map(A, B)$ the set of all linear, contractive, completely positive maps from A to B. Let $F \subset A$ be a finite set and let $\delta > 0$. We say that a map $\varphi \in Map(A, B)$ is δ-multiplicative on F if $\|\varphi(ab) - \varphi(a)\varphi(b)\| < \delta$ for all $a, b \in F$.

Lemma 1.7. *Let A be a C*-algebra with property (H). Let $\epsilon > 0$ and let $F_1 \subset A$ be a finite set. There are $\delta_H > 0$ and a finite set $F \subset A$ such that if B is any unital C*-algebra and $\Phi \in Map(A, B[0,1])$ is δ_H-multiplicative on F, then there exist $k \in \mathbb{N}$, a $*$-homomorphism $\eta : A \to M_k(B)$ with finite dimensional image, and a unitary $u \in U_{k+1}(B)$ such that*

$$\|u(\Phi^{(0)}(a) \oplus \eta(a))u^* - \Phi^{(1)}(a) \oplus \eta(a)\| < \epsilon$$

for all $a \in F_1$.

Proof. This is a straightforward consequence of Lemma 1.4 in [21]. It is clear that we may arrange that all the elements of F have norm at most one.

The next two elementary results were inspired by [31].

Lemma 1.8. *Let $B = \bigoplus_{j=1}^{s} P_j M_{k(j)}(C(Y_j))P_j$ where Y_j are finite connected CW complexes and P_j are projections with $rank(P_j) = m(j)$. Fix a point y_j in each space Y_j. Let $\xi_0 : B \to r(B) \cong \oplus_j M_{m(j)}$ be the evaluation map at $(y_1, \dots, y_s)$. Let A be a C^*-algebra and suppose that $\eta : B \to A$ is a $*$-homomorphism with finite dimensional image. Let $\epsilon > 0$ and let $F \subset B$ be a finite set. Then there is a $*$-homomorphism $\eta_0 : B \to A$ which factors through ξ_0 and $\|\eta(b) - \eta_0(b)\| < \epsilon + w(F)$ for all $b \in B$.*

Proof. It suffices to give the proof in the case when $s = 1$ and $\eta \neq 0$. Note that in this case $\xi_0(b) = b(y_1)$ for all $b \in B$. Since η has finite dimensional image, its kernel has finite codimension in B. Hence η factors through a finite direct sum of evaluation maps. It follows that there are points $y^1, \dots, y^n$ with $y^i \in Y_1$, and $*$-homomorphisms $\xi = \oplus_i \xi^i : B \to \bigoplus_i M_{m(1)}$, $\xi^i(b) = b(y^i)$ and $\nu : \bigoplus_i M_{m(1)} \to A$ such that $\eta = \xi\nu$. Define $\widehat{\xi_0} : B \to \bigoplus_i M_{m(1)}$ by $\widehat{\xi_0} = \oplus_i \xi_0$. Using the definition of the weak variation we find a unitary u in $\bigoplus_i M_{m(1)}$ such that $\|\xi(b) - u\widehat{\xi_0}(b)u^*\| < \epsilon + w(F)$ for all $b \in F$. Let $\eta_0 = \nu(u)\nu\widehat{\xi_0}\nu(u)^*$. Then $\|\eta(b) - \eta_0(b)\| < \epsilon + w(F)$ and η_0 factors through ξ_0 since $\widehat{\xi_0}$ factors through ξ_0. $\square$

Corollary 1.9. *Let X, $Y_1, \dots Y_s$ be finite connected CW complexes. Let $B = \bigoplus_{j=1}^{s} P_j M_{k(j)}(C(Y_j))P_j$, $A = QM_N(C(X))Q$ be homogeneous C^*-algebras. Let $\varphi, \psi \in Map(B, A)$, $\epsilon > 0$ and let $F \subset B$ be a finite set. Suppose that there are $k \in \mathbb{N}$, a $*$-homomorphism $\eta : B \to M_k(A)$ with finite dimensional image and a unitary $u \in U_{k+1}(A)$ such that*

$$\|u(\varphi(b) \oplus \eta(b))u^* - \psi(b) \oplus \eta(b)\| < \epsilon$$

for all $b \in F$. There is an integer m such that if $\mu : B \to M_R(A)$ is any m-large $*$-homomorphism with finite dimensional image and $\mu(P_j) \neq 0$ for $j = 1, \dots, s$, then there is a unitary $v \in U_{R+1}(A)$ such that

$$\|v(\varphi(b) \oplus \mu(b))v^* - \psi(b) \oplus \mu(b)\| < \epsilon + 4w(F)$$

for all $b \in F$.

Proof. (Sketch) With the aid of Lemma 1.8, one replaces η and μ by $*$-homomorphisms that factors through $r(B)$. Then one uses Lemma 1.1 to absorb η as a direct summand in μ up to a unitary equivalence.

2. Approximate factorizations of morphisms

The purpose of this section is to establish Lemma 2.4 which can be regarded as an approximate factorization result for morphisms of homogeneous C*-algebras. First we use connective kk-theory and E-theory to produce factorizations of morphisms at the level of homotopy. Then we use Lemma 1.7 to pass from homotopy factorizations to approximate factorizations.

The notion of asymptotic morphism due to Connes and Higson led to a geometric realization of E-theory [15], [14]. Let A, B be separable C*-algebras. Roughly speaking, an asymptotic morphism from A to B is a continuous family of maps

$\varphi_t : A \to B$, $t \in T = [1, \infty)$, which satisfies asymptotically the axioms for $*$-homomorphisms. A homotopy of asymptotic morphisms $\varphi_t, \psi_t : A \to B$ is given by an asymptotic morphism $\Phi_t : A \to B[0,1]$ such that $\Phi_t^{(0)} = \varphi_t$ and $\Phi_t^{(1)} = \psi_t$. Here $B[0,1]$ denotes the C*-algebra of continuous functions from the unit interval to B. The homotopy classes of asymptotic morphisms from A to B are denoted by $[[A, B]]$ and the class of φ_t by $[[\varphi_t]]$. Two asymptotic morphisms φ_t, ψ_t are said equivalent if $\varphi_t(a) - \psi_t(a) \to 0$, as $t \to \infty$ for all $a \in A$. Equivalent asymptotic morphisms are homotopic. In this paper we deal exclusively with asymptotic morphisms from nuclear C*-algebras. It was observed in [16] that if A is nuclear then any asymptotic morphism from A to B is equivalent to a completely positive linear contractive asymptotic morphism . That is an asymptotic morphism with each individual map φ_t being a completely positive linear contraction. This is a consequence of the Choi-Effros theorem [13], and it holds for homotopies of asymptotic morphisms as well. *Henceforth, throughout the paper, by an asymptotic morphism we will mean a completely positive linear contractive asymptotic morphism unless stated otherwise.* Let M_∞ denote the dense $*$-subalgebra of the compact operators $\mathcal{K}$ obtained as the union of the C*-algebras M_n. Using approximate units it is not hard to see that any asymptotic morphism from A to $B \otimes \mathcal{K}$ is equivalent to an asymptotic morphism $\varphi_t : A \to B \otimes M_\infty$ for which there is a function $\alpha : T \to \mathbb{N}$ such that $\varphi_t(A) \subset B \otimes M_{\alpha(t)}$. The map α is called a dominating function for φ_t. This applies also to homotopies and yields a bijection

$$[[A, B \otimes M_\infty]] \to [[A, B \otimes \mathcal{K}]].$$

We consider here only asymptotic morphisms that are dominated by functions α as above. Recall that if A is nuclear, then the Kasparov group $KK(A, B)$ is isomorphic to $[[SA, SB \otimes \mathcal{K}]]$ (see [15]).

Let X be a finite connected CW complex with base point x_0. We let $C_0(X)$ denote the C*-algebra of continuous functions on X vanishing at the base point x_0. Then by a suspension theorem of [22], $[[C_0(X), B \otimes M_\infty]] \cong KK(C_0(X), B)$. Let $\varphi_t : C_0(X) \to B \otimes M_\infty$ be an asymptotic morphism and let α be a dominating function for φ_t. For each $t \in T$ we let $\varphi_t^\alpha : C(X) \to B \otimes M_{\alpha(t)}$ denote the unital extension of $\varphi_t : C_0(X) \to B \otimes M_{\alpha(t)}$ with $\varphi_t^\alpha(1) = 1_B \otimes 1_{\alpha(t)}$. Note that if $\alpha(t) \le \beta(t)$ then $\varphi_t^\alpha = 1_B \otimes 1_{\alpha(t)}\ \varphi_t^\beta\ 1_B \otimes 1_{\alpha(t)}$.

Lemma 2.1. *Let X, $Y = Y_0, Y_1, \dots Y_s$ be finite connected CW complexes. Let $B_1 = \oplus_{j=1}^s M_{k(j)}(C(Y_j))$, $A_1 = M_N(C(X))$ and let $\psi_1 : B_1 \to A_1$ be a $*$-homomorphism which is homotopic to a direct sum between an m-large $*$-homomorphism, $m > 4dim(X)$, and a $*$-homomorphism with finite dimensional image. Suppose that $dim(Y_j) \le 3$ and $H^3(Y_j)$ is finite for $j = 0, \dots, s$. Suppose that $K^*(X) \cong K^*(Y)$. Let $G \subset B_1$, $F \subset A_1$ be finite sets. Then for any $\delta_H > 0$ there exists a diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\ \gamma\ } & A_1' \\ {\scriptstyle\psi_1}\big\uparrow & & \big\uparrow{\scriptstyle\psi_2} \\ B_1 & \xrightarrow[\ \xi\ ]{} & B_2 \end{array}$$

where

$A_1' = M_L(C(X))$, $B_2 = M_S(C(Y))$;

ξ is a $*$-homomorphism, ψ_2 is a unital $*$-homomorphism, and γ is a unital simple embedding;

$\varphi \in Map(A_1, B_2)$ is δ_H-multiplicative on F;

Moreover there exist homotopies $\Phi \in Map(B_1, B_2[0,1])$ and $\Psi \in Map(A_1, A_1'[0,1])$ such that Φ is δ_H-multiplicative on G, Ψ is δ_H-multiplicative on F and

$\Phi^{(1)} = \xi$, $\Phi^{(0)} = \varphi\psi_1$, $\Psi^{(0)} = \psi_2\varphi$, $\Psi^{(1)} = \gamma$.

Proof. The proof is divided into several steps

a) We may assume that $\psi_1(1_{k(j)}) \neq 0$ for all j. By the results cited in section 1, ψ_1 is homotopic to a direct sum of $*$-homomorphisms $\psi' \oplus \psi''$ where

$$\psi' : \oplus_{j=1}^{s} M_{k(j)}(C(Y_j)) \to \oplus_{j=1}^{s} M_{k(j)m(j)}(C(X)) \subset M_{N'}(C(X))$$

$$\psi' = \psi_1^0 \otimes id_{k(1)} \oplus \cdots \oplus \psi_s^0 \otimes id_{k(s)}, \quad N' = \sum_{j=1}^{s} k(j)m(j),$$

$\psi_j^0 : C(Y_j) \to M_{m(j)}(C(X))$ are unital $*$-homomorphisms with $\psi_j^0(C_0(Y_j)) \subset M_{m(j)}(C_0(X))$ and

$$\psi'' : \oplus_{j=1}^{s} M_{k(j)}(C(Y_j)) \to M_{N''}(C(X))$$

is a $*$-homomorphism with finite dimensional image, $N = N' + N''$. Note that in certain cases one may have $\psi' = 0$ and $N' = 0$.

b) Since X and Y are connected and $K^*(X) \cong K^*(Y)$ there is an asymptotic morphism $\varphi_t : C_0(X) \to C_0(Y) \otimes M_\infty$ inducing a KK-equivalence (see [47], [15] and [22]). By Theorem 3.3 of [31], since $H^3(Y_j)$ is finite and $dim\ (Y_j) \leq 3$, there is a $*$-homomorphism ξ_j such that the diagram

$$\begin{array}{ccc} C_0(Y_j) & \xrightarrow{\psi_j^0} & M_{m(j)}(C_0(X)) \\ & & \Big\downarrow{\scriptstyle \varphi_t \otimes id_{m(j)}} \\ & & M_{m(j)}(C_0(Y) \otimes M_\infty) \end{array}$$

is commutative at the level of KK-theory. By Theorem 4.1 in [22] there is a homotopy of asymptotic morphisms

$$\Phi_{j,t} : C_0(Y_j) \to M_{m(j)}(C_0(Y)[0,1] \otimes M_\infty)$$

with $\Phi_{j,t}^{(0)} = (\varphi_t \otimes id_{m(j)})\psi_j^0$ and $\Phi_{j,t}^{(1)} = \xi_j$. We may assume that the homotopy is dominated by a function $\alpha : T \to \mathbb{N}$.

c) Let φ_t be as above. By a similar argument one obtains a diagram

$$\begin{array}{ccc} C_0(X) & \xrightarrow{\gamma_0} & C_0(X) \otimes M_\ell \otimes M_\infty \\ {\scriptstyle \varphi_t}\Big\downarrow & & \\ C_0(Y) \otimes M_\infty & & \end{array}$$

where $\gamma_0(a) = a \otimes e$, e is a minimal projection, and $\psi : C(Y) \to C(X) \otimes M_\ell$ is a $*$-homomorphism with $\psi(C_0(Y)) \subset C_0(X) \otimes M_\ell$, which induces an inverse of $[[\varphi_t]]$ in KK-theory. Hence there is a homotopy of asymptotic morphisms

$$\Psi_t : C_0(X) \to C_0(X)[0,1] \otimes M_\ell \otimes M_\infty$$

with $\Psi_t^{(0)} = (\psi \otimes id_\infty)\varphi_t$ and $\Psi_t^{(1)} = \gamma_0$.

d) We may assume the maps γ_0, ξ_j, φ_t, Φ_t and Ψ_t are dominated by the same function α. For each $t \in T$ we consider the following unital extensions of those maps:

$$\varphi_t^\alpha : C(X) \to C(Y) \otimes M_{\alpha(t)}$$

$$\xi_j^\alpha : C(Y_j) \to M_{m(j)}(C(Y) \otimes M_{\alpha(t)})$$

$$\Phi_{j,t}^\alpha : C(Y_j) \to M_{m(j)}(C(Y)[0,1] \otimes M_{\alpha(t)})$$

$$\Psi_t^\alpha : C(X) \to C(X)[0,1] \otimes M_\ell \otimes M_{\alpha(t)}$$

$$\gamma_0^\alpha : C(X) \to C(X) \otimes M_\ell \otimes M_{\alpha(t)}.$$

Set

$$\Phi_t^{\prime,\alpha} \overset{\text{def}}{=} \oplus_j \Phi_{j,t}^\alpha \otimes id_{k(j)} : \oplus_{j=1}^s M_{k(j)}(C(Y_j)) \to M_{N'}(C(Y)[0,1] \otimes M_{\alpha(t)})$$

and note that $\Phi_t^{\prime,\alpha}$ is a homotopy from $(\varphi_t^\alpha \otimes id_{N'})\psi'$ to $\xi^{\prime,\alpha} \overset{\text{def}}{=} \oplus_j(\xi_j^\alpha \otimes id_{k(j)})$. Similarly, Ψ_t^α is a homotopy from $(\psi \otimes id_{\alpha(t)})\varphi_t^\alpha$ to γ_0^α.

e) Since $\psi''(B_1)$ is finite dimensional and φ_t is an asymptotic morphism , for each $t \in T$, there is a $*$-homomorphism with finite dimensional image $\xi_t^{\prime\prime,\alpha} : B_1 \to M_{N''}(C(Y) \otimes M_{\alpha(t)})$ such that $\lim_{t\to\infty} \|(\varphi_t^\alpha \otimes id_{N''})\psi''(b) - \xi_t^{\prime\prime,\alpha}(b)\| = 0$ for all $b \in B_1$. The map $\xi_t^{\prime\prime,\alpha}$ is obtained by perturbing the restriction of $\varphi_t^\alpha \otimes id_{N''}$ to $\psi''(B_1)$, to a $*$-homomorphism with finite dimensional image. The various maps we are dealing with are pictured in the diagram

$$\begin{array}{ccccc}
B_1 \xrightarrow{\psi' \oplus \psi''} & M_{N'}(C(X)) \oplus \psi''(B_1) & \longrightarrow & M_N(C(X)) \\
& \Big\downarrow {\scriptstyle \varphi_t^\alpha \otimes id_{N'} \oplus \varphi_t^\alpha \otimes id_{N''}} & & \Big\downarrow {\scriptstyle \varphi_t^\alpha \otimes id_N} \\
& M_{N'}(C(Y) \otimes M_{\alpha(t)}) \oplus M_{N''}(C(Y) \otimes M_{\alpha(t)}) & \longrightarrow & M_N(C(Y) \otimes M_{\alpha(t)})
\end{array}$$

Define $\Phi_t^{\prime\prime(s)} = s(\varphi_t^\alpha \otimes id_{N''})\psi'' + (1-s)\xi_t^{\prime\prime,\alpha}$. Let $\chi : B_1 \to A_1[0,1]$ be a homotopy from ψ_1 to $\psi' \oplus \psi''$. Let Φ_t^α denote the juxtaposition of the homotopies $(\varphi_t^\alpha \otimes id_N)\chi^{(s)}$ and $\Phi_t^{\prime,\alpha} \oplus \Phi_t''$. Then Φ_t^α is a homotopy from $(\varphi_t^\alpha \otimes id_N)\psi_1$ to $\xi_t^\alpha = \xi^{\prime,\alpha} \oplus \xi_t^{\prime\prime,\alpha}$.

$$\begin{array}{ccc}
M_N(C(X)) & \xrightarrow{\gamma_0^\alpha \otimes id_N} & M_N(C(X) \otimes M_{\ell\alpha(t)}) \\
{\scriptstyle \psi_1} \Big\uparrow & & \Big\uparrow {\scriptstyle \psi \otimes id_{\alpha(t)} \otimes id_N} \\
B_1 & \xrightarrow[\xi_t^\alpha]{} & M_N(C(Y) \otimes M_{\alpha(t)})
\end{array}$$

Since the sets G and F are finite, by taking t large enough, we can arrange that $\varphi_t^\alpha \otimes id_N$ and $\Psi_t^\alpha \otimes id_N$ are δ_H-multiplicative on F and Φ_t^α is δ_H-multiplicative on G. $\square$

Lemma 2.2. *Let X, $Y = Y_0, Y_1, \dots Y_s$ be finite connected CW complexes. Let $B_1 = \oplus_{j=1}^{s} M_{k(j)}(C(Y_j))$, $A_1 = M_N(C(X))$ and let $\psi_1 : B_1 \to A_1$ be a $*$-homomorphism which is homotopic to a direct sum between an m-large $*$-homomorphism, $m > 4dim(X)$, and a $*$-homomorphism with finite dimensional image. Suppose that $dim(Y_j) \leq 3$ and $H^3(Y_j)$ is finite for $j = 0, \dots, s$. Suppose that $K^*(X) \cong K^*(Y)$. Let $G_1 \subset B_1$, $F_1 \subset \widehat{F} \subset A_1$ be finite sets . Then for any $\epsilon > 0$ and any $\delta > 0$ there exists a diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma} & A_1' \\ {\scriptstyle \psi_1}\uparrow & & \uparrow{\scriptstyle \psi_2} \\ B_1 & \xrightarrow[\xi]{} & B_2 \end{array}$$

where

$A_1' = M_L(C(X))$, $B_2 = M_S(C(Y))$;
ξ, ψ_2, γ are $*$-homomorphisms ,ψ_2 is unital and γ is a simple unital embedding;
$\varphi \in Map(A_1, B_2)$ is δ-multiplicative on $\widehat{F}$;
$\|\varphi\psi_1(b) - \xi(b)\| < \epsilon + 4w(G_1)$ for all $b \in G_1$.
$\|\psi_2\varphi(a) - \gamma(a)\| < \epsilon + 4\,w(F_1)$ for all $a \in F_1$.

Proof. The C*-algebras A_1 and B_1 have property (H). We are going to apply Lemma 1.7 for ϵ and $F_1 \subset A_1$, and for ϵ and $G_1 \subset B_1$. Let $\delta_H > 0$, $F \subset A_1$ and $G \subset B_1$ be as provided by Lemma 1.7. We may assume that $\psi_1(1_{k(j)}) \neq 0$ for all j, $\psi_1(G) \subset \widehat{F} \subset F$ and $\delta_H < \delta$. Now we apply Lemma 2.1 for δ_H, F, G. Let B_2, A_1', φ, ξ, ψ_2, γ, Φ, Ψ be as in the conclusion of Lemma 2.1. By Lemma 1.7 there are $k \in \mathbb{N}$, $*$-homomorphisms $\eta_1 : B_1 \to M_k(B_2)$, $\eta_2 : A_1 \to M_k(A_1')$ both with finite dimensional image and unitaries $u \in U_{k+1}(B_2)$, and $v \in U_{k+1}(A_1')$ such that

$$\|u(\xi(b) \oplus \eta_1(b))u^* - \varphi\psi_1(b) \oplus \eta_1(b)\| < \epsilon \tag{1}$$

$$\|v(\psi_2\varphi(a) \oplus \eta_2(a))v^* - \gamma(a) \oplus \eta_2(a)\| < \epsilon \tag{2}$$

for all $b \in G_1$ and $a \in F_1$. Let $\eta : A_1 \to M_m(B_2)$ be a unital $*$-homomorphism that factors through M_N. By taking m large enough, we can apply Corollary 1.9 for (1) with $\mu = \eta\psi_1$, and for (2) with $\mu = (\psi_2 \otimes id_m)\eta$. Hence we find unitaries $U \in U_{m+1}(B_2)$ and $V \in U_{m+1}(A_1')$ such that the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma\oplus(\psi_2\otimes id_m)\eta} & M_{m+1}(A_1') \\ {\scriptstyle \psi_1}\uparrow & & \uparrow{\scriptstyle V(\psi_2\otimes id_{m+1})V^*} \\ B_1 & \xrightarrow[U(\xi\oplus\eta\psi_1)U^*]{} & M_{m+1}(B_2) \end{array}$$

approximately commutes on G_1,(resp. F_1) to within $\epsilon + 4\,w(G_1)$ (resp. $\epsilon + 4\,w(F_1)$). The $*$-homomorphism $\gamma \oplus (\psi_2 \otimes id_m)\eta$ is a simple embedding since γ is a simple embedding and η has finite dimensional image. □

Next we obtain a version of Lemma 2.2 for homogeneous C*-algebras.

Lemma 2.3. *Let X, $Y = Y_0$, $Y_1, \ldots Y_s$ be finite connected CW complexes. Let $B_1 = \bigoplus_{j=1}^{s} P_j M_{k(j)}(C(Y_j))P_j$, $A_1 = QM_N(C(X))Q$ and let $\psi_1 : B_1 \to A_1$ be a $*$-homomorphism which is homotopic to a direct sum between an m-large $*$-homomorphism, $m > 4dim(X)$, and a $*$-homomorphism with finite dimensional image. Suppose that $dim(Y_j) \le 3$ and $H^3(Y_j)$ is finite for $j = 0, \ldots, s$. Suppose that $K^*(X) \cong K^*(Y)$. Let $G_1 \subset B_1$, $F_1 \subset \widehat{F} \subset A_1$ be finite sets with all elements of norm ≤ 1, $w(G_1) < 10^{-1}$, $w(F_1) < 10^{-1}$. Then for any $\epsilon > 0$ and any $\delta > 0$ there exists a diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma} & A_1' \\ {\scriptstyle\psi_1}\uparrow & & \uparrow{\scriptstyle\psi_2} \\ B_1 & \xrightarrow[\xi]{} & B_2 \end{array}$$

where

$A_1' = Q_1 M_L(C(X))Q_1$, $B_2 = PM_S(C(Y))P$;

ξ, ψ_2, γ are $*$-homomorphisms with ψ_2 unital, and γ a unital, simple embedding; If ψ_1 is unital, then one can arrange to have ξ unital too.

$\varphi \in Map(A_1, B_2)$ is δ-multiplicative on $\widehat{F}$;

$\|\varphi\psi_1(b) - \xi(b)\| < \epsilon + 4\,w(G_1) + 16\,w(G_1)^{1/2}$ for all $b \in G_1$

$\|\psi_2\varphi(a) - \gamma(a)\| < \epsilon + 4\,w(F_1) + 16\,w(F_1)^{1/2}$ for all $a \in F_1$.

Proof. We may assume that $\psi_1(P_j) \neq 0$ for all j. By Lemma 2.13 in [31], ψ_1 can be dilated to a $*$-homomorphism $\psi_1' : B' \to A'$ where $B' = \oplus_{j=1}^{s} M_{k(j)}(C(Y_j))$ and $A' = M_N(C(X))$. Hence there is a commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma'} & A' \\ {\scriptstyle\psi_1}\uparrow & & \uparrow{\scriptstyle\psi_1'} \\ B_1 & \xrightarrow[\xi']{} & B' \end{array}$$

where γ' and ξ' are simple embeddings. Moreover ψ_1' is homotopic to a direct sum between an m-large $*$-homomorphism, $m > 4dim(X)$, and a $*$-homomorphism with finite dimensional image. We may assume that $1_{B_1} \in G_1$, $1_{A_1} \in F_1$ and $\psi_1(G_1) \subset \widehat{F}$. Set $F' = \gamma'(F_1)$, $\widehat{F'} = \gamma'(\widehat{F})$ and $G' = \xi'(G_1)$. Then $w(F') \le w(F_1)$ and $w(G') \le w(G_1)$. Applying Lemma 2.2 for ψ_1', F', $\widehat{F'}$, G', $\epsilon < 10^{-3}$, $\delta < 10^{-3}$, we complete the above diagram to an approximately commuting diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{\gamma'} & A' & \xrightarrow{\gamma} & A_1' \\ {\scriptstyle\psi_1}\uparrow & & {\scriptstyle\psi_1'}\uparrow & & \uparrow{\scriptstyle\psi_2} \\ B_1 & \xrightarrow[\xi']{} & B' & \xrightarrow[\xi]{} & B_2 \end{array}$$

where ξ, ψ_2, φ, γ, A_1', B_2 are as in the conclusion of Lemma 2.2. The result is now obtained by a standard approximation argument. Let $Q' = \gamma'(1_{A_1})$, $P' = \xi'(1_{B_1})$, hence $\psi_1'(P') \le Q'$. The element $\varphi(Q')$ is selfadjoint, of norm ≤ 1 and $\|\varphi(Q')^2 - \varphi(Q')\| < \delta < 1/4$. By functional calculus we find a projection $E \in B_2$

such that $\|\varphi(Q') - E\| < 2\delta$. Since φ is δ-multiplicative on $\widehat{F}'$ one shows that $\|E\varphi(x)E - \varphi(x)\| < 6\delta$ for all $x \in \widehat{F}'$. Then a straightforward estimate shows that $E\varphi E$ is 7δ-multiplicative on $\widehat{F}'$. Define $\varphi' \in Map(A_1, B_2)$ by $\varphi'(a) = E\varphi\gamma'(a)E$. Then

$$\|\varphi'(ab) - \varphi'(a)\varphi'(b)\| < 7\delta$$

for all $a, b \in \widehat{F}$. Next since $\|\gamma(Q') - \psi_2\varphi(Q')\| < \epsilon + 4w(F')$ we get $\|\gamma(Q') - \psi_2(E)\| < \epsilon + 2\delta + 4w(F') < 1$. By functional calculus there is a unitary $V \in U(A_1')$ such that $V\psi_2(E)V^* = \gamma(Q')$ and $\|V - 1_{A_1'}\| < 4(\epsilon + 2\delta + 4w(F'))^{1/2}$. Next we are going to perturb $\xi\xi'$ to a map with values in EB_2E. Since $\|\xi(P') - \varphi\psi_1'(P')\| < \epsilon + 4w(G')$, if $c = E\varphi\psi_1'(P')E$, then $\|\xi(P') - c\| < 6\delta + \epsilon + 4w(G')$. Recall that $E\varphi E$ is 7δ-multiplicative on $\widehat{F}'$. Thus $\|c^2 - c\| < 7\delta$. It follows that there is a selfadjoint projection $e \leq E$ such that $\|c - e\| < 14\delta$. Therefore $\|\xi(P') - e\| < 20\delta + \epsilon + 4w(G') < 1$. By functional calculus there is a unitary $U \in U(B_2)$ with $U\xi(P')U^* = e$ and $\|U - 1_{B_2}\| < 4(\epsilon + 20\delta + 4w(G'))^{1/2}$. Note that if ψ_1 is unital then $\psi_1'(P') = Q'$ hence we can take $e = E$. In this manner we obtain an approximately commuting diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma\gamma'} & \gamma(Q')A_1'\gamma(Q') \\ {\scriptstyle \psi_1}\uparrow & & \uparrow{\scriptstyle V\psi_2V^*} \\ B_1 & \xrightarrow[U\xi\xi'U^*]{} & EB_2E \end{array}$$

Indeed, by standard estimates one shows $\|\varphi'\psi_1(b) - U\xi\xi'(b)U^*\| < 6\delta + \epsilon + 4\,w(G_1) + 8(\epsilon + 20\delta + 4w(G_1))^{1/2}$ for all $b \in G_1$, and $\|V\psi_2\varphi'(a)V^* - \gamma\gamma'(a)\| < 6\delta + \epsilon + 4w(F_1) + 8(\epsilon + 2\delta + 4w(F_1))^{1/2}$ for all $a \in F_1$. The evaluations from the statement are obtained by changing ϵ and δ appropriately. $\square$

Lemma 2.4. *Let X, $Y = Y_0$, $Y_1, \ldots Y_s$ be finite connected CW complexes. Let $B_1 = \bigoplus_{j=1}^{s} P_j M_{k(j)}(C(Y_j))P_j$, $A_1 = QM_N(C(X))Q$ and let $\psi_1 : B_1 \to A_1$ be a $*$-homomorphism which is homotopic to a direct sum between an m-large $*$-homomorphism, $m > 4dim(X)$, and a $*$-homomorphism with finite dimensional image. Suppose that $dim(Y_j) \leq 3$ and $H^3(Y_j)$ is finite for $j = 0, \ldots, s$. Suppose that $K^*(X) \cong K^*(Y)$. Let $G_1 \subset B_1$, $F_1 \subset \widehat{F} \subset A_1$ be finite sets with all the elements of norm ≤ 1, such that $w(G_1) < 10^{-1}$ and $w(F_1) < 10^{-1}$. Then for any $\epsilon > 0$, $\delta > 0$ and $w > 0$ there exists a diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma} & A_1' \\ {\scriptstyle \psi_1}\uparrow & & \uparrow{\scriptstyle \psi_2} \\ B_1 & \xrightarrow[\xi]{} & B_2 \end{array}$$

where
$A_1' = Q_1 M_L(C(X))Q_1$, $B_2 = PM_S(C(Y))P \oplus M_p(\mathbb{C})$;
ξ, ψ_2, γ are $*$-homomorphisms , ψ_2 is unital and γ is a unital simple embedding; If ψ_1 is unital then one can arrange to have ξ unital.
$\varphi \in Map(A_1, B_2)$ is δ-multiplicative on $\widehat{F}$;
$\|\varphi\psi_1(b) - \xi(b)\| < \epsilon + 4\,w(G_1) + 16\,w(G_1)^{1/2}$ for all $b \in G_1$
$\|\psi_2\varphi(a) - \gamma(a)\| < 2\epsilon + 8w(F_1) + 16\,w(F_1)^{1/2}$ for all $a \in F_1$.

In addition there is a finite set $G_2 \subset B_2$ with all elements of norm ≤ 1, $\xi(G_1) \cup \varphi(F_1) \subset G_2$, $\xi(B_1)$ is contained in the C^*-subalgebra of B_2 generated by G_2, and $w(G_2) < w$.

Proof. Let $F_1 \subset A_1$ and $\epsilon > 0$ be as in the statement. With this data, let $F \subset A_1$ and δ_H be as provided by Lemma 1.7. We may assume that $\delta_H < \delta$. By applying Lemma 2.3 for G_1, F_1, $\widehat{F}_1 = \widehat{F} \cup F$, ϵ and δ_H, we obtain an approximately commuting diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma} & A_1' \\ {\scriptstyle \psi_1}\uparrow & & \uparrow{\scriptstyle \psi_2} \\ B_1 & \xrightarrow[\xi]{} & B_2 \end{array}$$

with properties as in the conclusion of Lemma 2.3. In particular φ is δ_H-multiplicative on $\widehat{F}_1$. Let $G_1' \subset B_1$ be a finite set that contains G_1 and generates B_1. Set $H_2 = \xi(G_1') \cup \varphi(F_1)$. Using Lemma 1.4, we find a simple unital embedding $\xi^0 : B_2 \to M_R(B_2)$ such that $w(\xi^0(H_2)) < w$. Let $n = rank(1_{B_2})$ and let $\lambda : B_2 \to M_n$ be an evaluation morphism. Define $\xi' : B_2 \to M_R(B_2) \oplus M_n$ by $\xi' = \xi^0 \oplus \lambda$. Then $w(\xi'(H_2)) < w$. Let $\gamma' : A_1' \to M_L(A_1')$ be a unital simple embedding. By taking L large enough, we can apply Lemma 1.2 to obtain a unital $*$-homomorphism $\psi : M_R(B_2) \oplus M_n \to M_L(A_1')$ such that $\psi\xi'$ is homotopic to $\gamma'\psi_2$. Let $\chi^{(s)}$ be the corresponding homotopy of $*$-homomorphisms. Set $A_1'' = M_L(A_1')$, $B_2' = M_R(B_2) \oplus M_n$, $\Phi^{(s)} = \chi^{(s)}\varphi$ and form the following diagram.

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma'\gamma} & A_1'' \\ {\scriptstyle \psi_1}\uparrow & & \uparrow{\scriptstyle \psi} \\ B_1 & \xrightarrow[\xi'\xi]{} & B_2' \end{array}$$

The end points of Φ are $\psi\xi'\varphi$ and $\gamma'\psi_2\varphi$. By the previous application of Lemma 2.3 we have

$$\|\Phi^{(1)}(a) - \gamma'\gamma(a)\| < \epsilon + 4w(F_1) + 16w(F_1)^{1/2}$$

for all $a \in F_1$. The maps $\xi'\varphi$ and $\chi^{(s)}\varphi$ are δ_H-multiplicative on $F \subset \widehat{F}_1$. At this point we invoke Lemma 1.7. Thus there exist $k \in \mathbb{N}$, a $*$-homomorphism $\eta_1 : A_1 \to M_k(A_1'')$ with finite dimensional image and a unitary $u \in U_{k+1}(A_1'')$ such that

$$\|u(\psi\xi'\varphi(a) \oplus \eta_1(a))u^* - \gamma'\gamma(a) \oplus \eta_1(a)\| < 2\epsilon + 4w(F_1) + 16w(F_1)^{1/2}$$

for all $a \in F_1$. Let $\eta : A_1 \to M_m(B_2')$ be a unital $*$-homomorphism with finite dimensional image. We take m big enough so that Corollary 1.9 applies for η_1 and $\mu = (\psi \otimes id_m)\eta$. Hence there is a unitary $U \in U_{m+1}(A_1'')$ such that

$$\|U(\psi\xi'\varphi(a) \oplus (\psi \otimes id_m)\eta(a))U^* - \gamma'\gamma(a) \oplus (\psi \otimes id_m)\eta(a)\| < 2\epsilon + 8w(F_1) + 16\, w(F_1)^{1/2}$$

for all $a \in F_1$. We need the following notation. $\gamma_1 = \gamma'\gamma \oplus (\psi \otimes id_m)\eta$, $\sigma = U(\psi \otimes id_{m+1})U^*$, $\varphi_1 = \xi'\varphi \oplus \eta$, $\xi_1 = \xi'\xi \oplus \eta\psi_1$. With this notation the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma_1} & M_{m+1}(A_1'') \\ {\scriptstyle \psi_1}\uparrow & & \uparrow{\scriptstyle \sigma} \\ B_1 & \xrightarrow[\xi_1]{} & M_{m+1}(B_2') \end{array}$$

approximately commutes within the desired limits. More precisely

$$\|\varphi_1\psi_1(b) - \xi_1(b)\| < \epsilon + 4w(G_1) + 16w(G_1)^{1/2}$$

$$\|\sigma\varphi_1(a) - \gamma_1(a)\| < 2\epsilon + 8w(F_1) + 16\, w(F_1)^{1/2}$$

for all $b \in G_1$ and $a \in F_1$. In addition φ_1 is δ-multiplicative on $\widehat{F}$ since $\delta_H < \delta$. Next we set

$$G_2 = \{\xi_1(b) = \xi'\xi(b) \oplus \eta\psi_1(b) : \ b \in G_1'\} \cup \{\varphi_1(a) = \xi'\varphi(a) \oplus \eta(a) : \ a \in F_1\}.$$

It is clear that $G_2 \subset \xi'(H_2) \oplus \eta(A_1)$. Since $w(\xi'(H_2)) < w$ and η is a $*$-homomorphism with finite dimensional image it follows that $w(G_2) \leq w(\xi'(H_2)) < w$. Note that if we start with a unital ψ_1, then ξ_1 can be chosen to be unital. Since γ, γ' are simple embeddings it follows that γ_1 is a simple embedding. □

Remark 2.5 Consider the set up of Lemma 2.4. Let $q = rank(1_{A_1})$ and let $\lambda : A_1 \to M_q$ be an evaluation map. Then the diagram from the conclusion of Lemma 2.4 can be modified as follows.

$$\begin{array}{ccc} A_1 & \xrightarrow{\gamma\oplus\lambda} & A_1' \oplus M_q \\ \psi_1 \uparrow & & \uparrow \psi_2 \oplus id_q \\ B_1 & \xrightarrow[\xi\oplus\lambda\psi_1]{} & B_2 \oplus M_q \end{array}$$

Excepting for the form of B_2 and A_1' the conclusion of Lemma 2.4 remains true for the above diagram.

3. The main result

G. Elliott introduced a notion of approximate intertwining as a tool for producing isomorphism results for inductive limit C*-algebras [29]. This notion is especially effective for inductive systems of semiprojective C*-algebras (see [50], [41]). When dealing with more general inductive systems we need the following Proposition which involves approximate intertwinings consisting of approximate morphisms rather than $*$-homomorphisms. Other generalisations along the same lines are easily available. The idea of constructing $*$-homomorphisms as limits of approximate morphisms goes back to [42].

Proposition 3.1. *Consider a diagram*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\nu_{2,1}} & A_2 & \longrightarrow & \dots & \longrightarrow & A_n & \xrightarrow{\nu_{n+1,n}} & A_{n+1} & \longrightarrow & \dots \\ \psi_1 \uparrow & & \psi_2 \uparrow & & & & \psi_n \uparrow & & \uparrow \psi_{n+1} & & \\ B_1 & \xrightarrow[\xi_{2,1}]{} & B_2 & \longrightarrow & \dots & \longrightarrow & B_n & \xrightarrow[\xi_{n+1,n}]{} & B_{n+1} & \longrightarrow & \dots \end{array}$$

where A_n, B_n are C-algebras, $\xi_{n+1,n}$, $\nu_{n+1,n}$ are $*$-homomorphisms , and φ_n, ψ_n are linear, selfadjoint, contractive maps. Suppose that A_n, B_n are finitely generated and let $F_n \subset A_n$, $G_n \subset B_n$ be finite subsets with*

$$\nu_{n+1,n}(F_n) \cup \psi_{n+1}(G_{n+1}) \subset F_{n+1}, \quad \xi_{n+1,n}(G_n) \cup \varphi_n(F_n) \subset G_{n+1}.$$

Suppose that $\xi_{n+1,n}(B_n)$ *is contained in the* C^**-subalgebra of* B_{n+1} *generated by* G_{n+1} *and* $\nu_{n+1,n}(A_n)$ *is contained in the* C^**-subalgebra of* A_{n+1} *generated by* F_{n+1}*.*

Suppose that there is a sequence ϵ_n *of positive numbers with* $\sum_{n=1}^{\infty} \epsilon_n < \infty$*, such that* φ_n *is* ϵ_n*-multiplicative on* F_n*,* ψ_n *is* ϵ_n*-multiplicative on* G_n*, and*

$$\|\psi_{n+1}\varphi_n(a) - \nu_{n+1,n}(a)\| < \epsilon_n, \quad \|\varphi_n\psi_n(b) - \xi_{n+1,n}(b)\| < \epsilon_n$$

for all $a \in F_n$ *and* $b \in G_n$*. Then* A *is isomorphic to* B*.*

Proof. The proof is similar to the original argument in [29]. See also [50]. □

Theorem 3.2. *Let* A *be a* C^**-algebra of real rank zero. Suppose that* A *is an inductive limit* $A = \varinjlim(A_n, \nu_{n+1,n})$*,* $A_n = \bigoplus_{i=1}^{k_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i}$ *where* $X_{n,i}$ *are finite CW complexes. Suppose that either* A *has slow dimension growth or* $d = \sup_{n,i}(dim(X_{n,i})) < \infty$*. Then* A *is isomorphic to a* C^**-algebra* $B = \varinjlim(B_n, \xi_{n+1,n})$*,* $B_n = \bigoplus_{i=1}^{\ell_n} Q_{n,i} M_{\{n,i\}}(C(Y_{n,i}))Q_{n,i}$ *where* $Y_{n,i}$ *are finite connected CW complexes of dimension* ≤ 3 *with* $H^3(Y_{n,i})$ *finite.*

Proof. We construct inductively a diagram

$$\begin{array}{ccccccccc}
A_{s(1)} & \xrightarrow{\nu_{s(2),s(1)}} & A_{s(2)} & \longrightarrow \dots \longrightarrow & A_{s(n)} & \xrightarrow{\nu_{s(n+1),s(n)}} & A_{s(n+1)} & \longrightarrow & \dots \\
\psi_1 \uparrow & & \psi_2 \uparrow & & \psi_n \uparrow & & \uparrow \psi_{n+1} & & \\
B_1 & \xrightarrow[\xi_{2,1}]{} & B_2 & \longrightarrow \dots \longrightarrow & B_n & \xrightarrow[\xi_{n+1,n}]{} & B_{n+1} & \longrightarrow & \dots
\end{array}$$

where B_n are C*-algebras as in the statement of the Theorem, ψ_n are $*$-homomorphisms and φ_n are linear selfadjoint contractive maps.

Moreover we construct finite sets $G_n \subset B_n$, $F_n \subset A_{s(n)}$ with elements of norm ≤ 1, such that

$\nu_{s(n+1),s(n)}(F_n) \cup \psi_{n+1}(G_{n+1}) \subset F_{n+1}$,

$\xi_{n+1,n}(G_n) \cup \varphi_n(F_n) \subset G_{n+1}$,

$\nu_{s(n+1),s(n)}(A_{s(n)})$ is contained in the C*-subalgebra of $A_{s(n+1)}$ generated by F_{n+1},

$\xi_{n+1,n}(B_n)$ is contained in the C*-subalgebra of B_{n+1} generated by G_{n+1},

$w(F_n) < 10^{-6n-6}$, $w(G_n) < 10^{-6n-6}$,

φ_n is 2^{-n}-multiplicative on F_n,

$\|\varphi_n\psi_n(b) - \xi_{n+1,n}(b)\| < 2^{-n}$ for all $b \in G_n$,

$\|\psi_{n+1}\varphi_n(a) - \nu_{s(n+1),s(n)}(a)\| < 2^{-n}$ for all $a \in F_n$.

Each C*-algebra A_s is of the form $A_s = \bigoplus_{j=1}^{k_s} A_{s,j}$ with $A_{s,j} = P_{s,j} M_{[s,j]}(C(X_{s,j}))P_{s,j}$. We may assume that each $X_{s,j}$ is connected. We also want each partial $*$-homomorphism $\psi_n^{j,i} : B_{n,i} \to A_{s(n),j}$ to be homotopic to a direct sum between a $(4dim(X_{s(n),j})+1)$-large $*$-homomorphism, and a $*$-homomorphism with finite dimensional image. This property will enable us to apply Lemma 2.4.

We start the inductive process with $A_{s(1)} = \{0\}$ and $B_1 = \{0\}$. Let us assume that we have accomplished the construction up to the n^{th}-stage. Let $F_{n,j} = F_n \cap A_{s(n),j}$ and let $\psi^j : B_n \to A_{s(n),j}$ be the corresponding partial $*$-homomorphisms of ψ_n. Let $\epsilon = \delta = w = 10^{-12n-12}$. Let Y_j be a finite connected CW complex of dimension ≤ 3, such that $H^3(Y_j)$ is finite and $K^*(Y_j) \cong K^*(X_{s(n),j})$. Applying Lemma 2.4 for ψ^j, G_n, $F_{n,j}$, ϵ, δ, w we obtain a diagram

$$(*) \qquad \begin{array}{ccc} A_{s(n),j} & \xrightarrow{\gamma^j} & A'_{s(n),j} \\ \psi^j \uparrow & & \uparrow \psi_2^j \\ B_n & \xrightarrow[\xi^j]{} & B(j) \end{array}$$

where the spectrum of $A'_{s(n),j}$ is $X_{s(n),j}$ and the spectrum of $B(j)$ is $Y_j \cup \{pt\}$. Here ψ_2^j is a unital $*$-homomorphism and γ^j is a unital simple embedding. By replacing γ^j by $\gamma\gamma^j$ and ψ_2^j by $\gamma\psi_2^j$ where $\gamma : A'_{s(n),j} \to A'$ is a large unital simple embedding, we can assume that γ^j is $(4\,dim(X_{s(n),j})+1)$-large . If ψ^j is unital, then ξ^j is unital. In addition

$$\|\varphi^j\psi^j(b) - \xi^j(b)\| < \epsilon + 4w(G_n) + 16w(G_n)^{1/2} < 10^{-3n}$$

$$\|\psi_2^j\varphi^j(a) - \gamma^j(a)\| < 2\epsilon + 8w(F_{n,j}) + 16w(F_{n,j})^{1/2})^{1/2} < 10^{-3n}$$

for all $b \in G_n$ and $a \in F_{n,j}$. The map φ^j is 2^{-n-1}-multiplicative on $F_{n,j}$. We also obtain a finite set $G_{n+1,j} \subset B(j)$ corresponding to the set G_2 of Lemma 2.4.

First we deal with the case when $d < \infty$. Let us now fix the index j, $1 \leq j \leq k_{s(n)}$. For $r > s(n)$ and $1 \leq \ell \leq k_r$, let $E^{\ell,j} = \nu^{\ell,j}_{r,s(n)}(1_{A_{s(n),j}})$, where $\nu^{\ell,j}_{r,s(n)}$ is a partial $*$-homomorphism of $\nu_{r,s(n)}$. Since A has real rank zero, by Theorem 2.5 in [49] and Remark 1.4.6 in [31], as r increases, the spectral variation of $\nu_{r,s(n)}$ becomes arbitrarily small. Let q denote the rank of the unit of $A_{s(n),j}$ and let q' denote the rank of the unit of $A'_{s(n),j}$. By using Lemma 2.3 and Remark 2.5 in [31], we can find $r > s(n)$ and a partition of $\{1, \ldots, k_r\}$ into two sets L_1 and L_2 with the the following properties. If $\ell \in L_1$, then $rank(E^{\ell,j}) > 5dq'$. If $\ell \in L_2$, then there is a $*$-homomorphism with finite dimensional image $\mu^{\ell,j} : A_{s(n),j} \to E^{\ell,j}A_{r,\ell}E^{\ell,j}$, such that

$$\|\mu^{\ell,j}(a) - \nu^{\ell,j}_{r,s(n)}(a)\| < \epsilon$$

for all $a \in F_{n,j}$.

For $\ell \in L_2$ the diagram

$$(**) \qquad \begin{array}{ccc} A_{s(n),j} & \xrightarrow{\nu^{\ell,j}_{r,s(n)}} & E^{\ell,j}A_{r,\ell}E^{\ell,j} \\ \psi^j \uparrow & & \uparrow \\ B_n & \xrightarrow[\mu^{\ell,j}\psi^j]{} & \mu^{\ell,j}(A_{s(n),j}) \end{array}$$

approximately commutes to within ϵ on $F_{n,j}$, G_n. Let $G_{n+1,j,\ell} = \mu^{\ell,j}(\psi^j(G_n) \cup F_{n,j})$. Then $w(G_{n+1,j,\ell}) = 0$.

For $\ell \in L_1$, we modify the diagram $(*)$ as indicated in Remark 2.5 and obtain a diagram

$$\begin{array}{ccc} A_{s(n),j} & \xrightarrow{\gamma^\ell} & A(\ell) \\ \psi^j \uparrow & & \uparrow \psi_2^\ell \\ B_n & \xrightarrow[\xi^\ell]{} & B(\ell) \end{array}$$

where $B(\ell) = B(j) \oplus M_q$, $A(\ell) = A'_{s(n),j} \oplus M_q$ and $\gamma^\ell = \gamma^j \oplus \lambda$. The new diagram inherits all the properties of diagram $(*)$. We also obtain a finite subset $G(\ell) \subset B(\ell)$, with $w(G(\ell)) < w$, corresponding to the set G_2 of Lemma 2.4. Recall that we have arranged that $q' > 4dq$ and $rank(E^{\ell,j}) > 5dq'$. This enables us to use Lemma 1.2 since γ^j is a unital simple embedding. Thus there is a unital $*$-homomorphism $\sigma^\ell : A(\ell) \to E^{\ell,j} A_{r,\ell} E^{\ell,j}$ such that $\sigma^\ell \gamma^\ell$ is homotopic to $\nu^{\ell,j}_{r,s(n)}$. By Theorem 2.29 and Remark 2.30 in [31], after increasing r, and changing the notation appropriately, we find a unitary $u \in U(E^{\ell,j} A_{r,\ell} E^{\ell,j})$ such that

$$\|u\sigma^\ell\gamma^\ell(a)u^* - \nu^{\ell,j}_{r,s(n)}(a)\| \leq 70w(F_n) < 70 \times 10^{-3n}$$

for all $a \in F_{n,j}$. It follows that the diagram

$$(***) \qquad \begin{array}{ccc} A_{s(n),j} & \xrightarrow{\nu^{\ell,j}_{r,s(n)}} & E^{\ell,j} A_{r,\ell} E^{\ell,j} \\ \psi^j \Big\uparrow & & \Big\uparrow u\sigma^\ell\psi_2^\ell u^* \\ B_n & \xrightarrow[\xi^\ell]{} & B(\ell) \end{array}$$

approximately commutes to within $70 \times 10^{-3n} + 10^{-3n} < 2^{-n-1}$ on $F_{n,j}$ and to within $10^{-3n} < 2^{-n-1}$ on G_n. Finally by assembling the diagrams $(**)$ and $(***)$ we obtain a diagram

$$\begin{array}{ccc} A_{s(n)} & \xrightarrow{\nu_{r,s(n)}} & A_r \\ \psi_n \Big\uparrow & & \Big\uparrow \psi_{n+1} \\ B_n & \xrightarrow[\xi_{n+1,n}]{} & B_{n+1} \end{array}$$

which approximately commutes to within the desired limits. The set G_{n+1} is obtained as the union of the sets $G(\ell)$ and $G_{n+1,j,\ell}$. Let F'_n be a finite set of norm-one generators of $A_{s(n)}$ that contains F_n, and set $F_{n+1} = \nu_{r,s(n)}(F'_n) \cup \psi_{n+1}(G_{n+1})$. By increasing r we can arrange to have $w(F_{n+1}) < 10^{-6n-12}$ (see Theorem 1.4.14 in [31]). Moreover ψ_{n+1} will become homotopic to a direct sum between a $(4d+1)$-large $*$-homomorphism, and a $*$-homomorphism with finite dimensional image (see Lemma 2.3 and Remark 2.5 in [31]). Finally we set $s(n+1) = r$.

The case when A has slow dimension growth is dealt with in a similar way. Actually that case is even simpler since the set L_2 is void and only diagrams of type $(***)$ are involved. Lemma 2.3 of [31] is replaced by the condition of slow dimension growth. One uses Theorem 2.29 and Remark 2.32 in [31].

If the inductive system for A has all the connecting maps unital, then the inductive system for B has the same property. □

Remark 3.3.

Let A be a C*-algebra as in the statement of Theorem 3.2. Let $\mathcal{S}$ be a a set consisting of points and finite connected CW complexes Y with $dim(Y) \leq 3$ such that for any space $X_{n,i}$ there is a space in $\mathcal{S}$ with the same K-theory. Then the C*-algebra B from Theorem 3.2 can be chosen such that the spaces $Y_{n,i}$ are in $\mathcal{S}$. For instance if all $X_{n,i}$ are torsion free, then we can take $\mathcal{S}$ to consists of points, wedges of circles and two dimensional spheres. By Theorem 5.28 in [31], A is isomorphic to a circle C*-algebra hence is classified by K-theory [29]. This gives a new proof to a very recent result of Gong [35], see also [21]. This remark can be also used to shorten the last part of the proof of Theorem 5.8 in [31].

Theorem 3.4. *Let A, B be two simple C*-algebras of real rank zero. Suppose that A and B are inductive limits* $A = \varinjlim A_n$ *and* $B = \varinjlim B_n$, $A_n = \bigoplus_{i=1}^{k_n} Q_{n,i} M_{[n,i]}(C(X_{n,i}))Q_{n,i}$, $B_n = \bigoplus_{i=1}^{l_n} P_{n,i} M_{\{n,i\}}(C(Y_{n,i}))P_{n,i}$. *Suppose that* $X_{n,i}$, $Y_{n,i}$ *are finite CW complexes. Suppose that A and B have slow dimension growth. Then A is isomorphic to B if and only if*

$$(K_*(A), K_*(A)_+, \Sigma_*(A)) \cong (K_*(B), K_*(B)_+, \Sigma_*(B)).$$

Proof. For spaces $X_{n,i}$, $Y_{n,i}$ of dimension at most three the result is given in Theorem 5.8 of [31]. Theorem 3.4 follows by combining this special case with Theorem 3.2 from above. Recall that for simple C*-algebras the condition of slow dimension growth is weaker than the condition $sup_n(dim(Spectrum(A_n))) < \infty$. The order structure on K_* was introduced in [24] and [29]. □

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