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RESIDUALLY FINITE DIMENSIONAL C*-ALGEBRAS AND SUBQUOTIENTS OF THE CAR ALGEBRA

Marius Dadarlat

Abstract. It is proved that the cone of a separable nuclearly embeddable residually finite-dimensional C*-algebra embeds in the CAR algebra (the UHF algebra of type 2^∞). As a corollary we obtain a short new proof of Kirchberg's theorem asserting that a separable unital C*-algebra A is nuclearly embeddable if and only there is a semisplit extension $0 \to J \to E \to A \to 0$ with E a unital C*-subalgebra of the CAR algebra and the ideal J an AF-algebra. The new proof does not rely on the lifting theorem of Effros and Haagerup.

1. Introduction

Throughout the paper we let B denote the CAR algebra, $B \cong \bigotimes_{i=1}^{\infty} M_2(\mathbb{C})$. A C*-algebra A is called *nuclearly embeddable* if there is a nuclear faithful representation $\sigma : A \to \mathcal{L}(\mathcal{H})$, [Vo2]. S. Wassermann [W2] has shown that any nuclearly embeddable C*-algebra is exact. By a remarkable theorem of Kirchberg, the converse is also true: any exact C*-algebra is nuclearly embeddable [Ki2]. Having reduced the study of exact C*-algebras to that of nuclearly embeddable C*-algebras, Kirchberg proves the following.

Theorem 1.1. *(Kirchberg* [Ki2]*) Let A be a separable unital C*-algebra. The following conditions are equivalent: (i) A is nuclearly embeddable. (ii) There is a semisplit essential extension of C*-algebras $0 \to J \to E \to A \to 0$ such that E is a unital C*-subalgebra of the CAR algebra B and J is an AF-algebra stably isomorphic to B. (iii) There is an extension of C*-algebras $0 \to J \to E \to A \to 0$ such that E is a unital C*-subalgebra of the CAR algebra B.*

The equivalence between (i) and (iii) proves that nuclear embeddability (hence exactness) passes to quotients. On the other hand, using the equivalence between (i) and (ii), and his Weyl-von Neumann-Voiculescu type theorem, Kirchberg proved that any separable unital nuclearly embeddable C*-algebra embeds as a unital C*-subalgebra of the Cuntz algebra $\mathcal{O}_2$ [Ki4]. See also [KPh] for a different proof. Using ideas from [Ki1, Ki2], S. Wassermann [W2] gave a proof of Theorem 1.1 which is shorter and somewhat simpler than the original proof of [Ki2] as it avoids the use of Kirchberg's theory of normalizers of operator subsystems of C^*-algebras. Both proofs use techniques of operator spaces and

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they rely on the lifting results of Effros and Haagerup [EK] for establishing the equivalence of (i) and (ii).

A separable C*-algebra A is called *residually finite-dimensional* (abbreviated *RFD*) if it has a separating sequence of finite dimensional representations. Equivalently A embeds in a C*-algebra of the form $\prod_{n=1}^{\infty} M_{k(n)}$, where M_k stands for $M_k(\mathbb{C})$. In this paper we prove that the cone $CF = C_0(0,1] \otimes F$ of any separable residually finite-dimensional nuclearly embeddable C*-algebra F embeds in the CAR algebra (Theorem 2.6). Let A be a separable unital nuclearly embeddable C*-algebra. Using the quasidiagonality of CA proved by Voiculescu [Vo3], we observe that there is a semisplit extension $0 \to I \to \widetilde{CF} \to A \to 0$ with F a separable RFD nuclearly embeddable C*-algebra (Lemma 3.1). Combining Theorem 2.6 and Lemma 3.1, we obtain a short proof of the equivalence of (i) and (ii). The implication (iii) $\Rightarrow$ (ii) follows from [EK, Proposition 5.3 and Theorem 3.4], while (ii) $\Rightarrow$ (iii) is obvious. Since a subalgebra of a nuclear C*-algebra is nuclearly embeddable, the non nuclearly embeddable RFD algebras (such as the full C*-algebra of the free group on two generators) are not AF-embeddable. Thus one cannot infer AF-embeddability from just the mere abundance of finite dimensional representations. In a forthcoming paper [D4], we give more general results on the UHF-embeddability of a nuclearly embeddable RFD algebra. Thus we prove that if A is a separable nuclearly embeddable RFD algebra such that either the rational K-homology group $K^0(A) \otimes \mathbb{Q} = KK(A, \mathbb{C}) \otimes \mathbb{Q}$ is finitely generated (as a $\mathbb{Q}$-module) or A satisfies the universal coefficient theorem of [RS] (UCT) for the Kasparov groups, then A embeds in a UHF algebra. Since the proofs of those results rely on certain techniques of KK-theory [Ka, Sk, DE], we have chosen to present here a self-contained elementary proof of the UHF-embeddability of CA. Previous results on the AF-embeddability of nuclear RFD C*-algebras have appeared in [D2], for A homotopically dominated by an AF algebra, and [L], for A satisfying the UCT.

2. Embedding RFD algebras in the CAR algebra

Proposition 2.1. *Let A, B be unital C*-algebras and let $\varphi_0, \varphi_1 : A \to B$ be two unital $*$-homomorphisms which are homotopic. Then for any $\mathcal{F} \subset A$ a finite subset and any $\epsilon > 0$ there exist $n \in \mathbb{N}$, a unital $*$-homomorphism $\eta : A \to M_{n-1}(B)$ and a unitary $u \in U_n(B)$ such that*

$$\|u(\varphi_0(a) \oplus \eta(a))u^* - \varphi_1(a) \oplus \eta(a)\| < \epsilon, \quad a \in \mathcal{F}. \tag{1}$$

Proof. By assumption there is a family of unital $*$-homomorphisms $(\varphi_t) : A \to B$ such that φ_0, φ_1 are equal to the given ones and for each $a \in A$, the map $t \mapsto \varphi_t(a)$ is norm-continuous on $[0,1]$. By uniform continuity we find an integer n such that

$$\|\varphi_{i+1/n}(a) - \varphi_{i/n}(a)\| < \epsilon, \quad 0 \leq i \leq n-1,\ a \in \mathcal{F}. \tag{2}$$

Define $\eta = \varphi_{1/n} \oplus \varphi_{2/n} \oplus \cdots \oplus \varphi_{n-1/n}$. Using (2)

(3)
$$\|\varphi_0(a) \oplus \eta(a) - \eta(a) \oplus \varphi_1(a)\| \leq \sup_{0 \leq i \leq n-1} \|\varphi_{i+1/n}(a) - \varphi_{i/n}(a)\| < \epsilon, \quad a \in \mathcal{F}.$$

If $u \in M_n(\mathbb{C}1_B)$ is the cyclic shift of order n, then $u(\eta(a) \oplus \varphi_1(a))u^* = \varphi_1(a) \oplus \eta(a)$. From this and (3) we obtain (1). □

The following proposition is an easy consequence of [DE, Theorem 3.8]. We give here an alternative proof which does not use KK-theory. Let $\mathcal{L}(\mathcal{H})$ denote the linear operators acting on a Hilbert space $\mathcal{H}$ and let $\mathcal{K}(\mathcal{H})$ denote the compact operators. We have $\mathcal{L}(\mathbb{C}^k) = \mathcal{K}(\mathbb{C}^k) \cong M_k$.

Proposition 2.2. *Let A be a unital separable C^*-algebra and let $\varphi, \psi : A \to M_m$ be two unital $*$-homomorphisms which are homotopic. Let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. Then for any faithful unital representation $\sigma : A \to \mathcal{L}(\mathcal{H})$ with $\sigma(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, there is a unitary $v \in \mathbb{C}1_{\mathbb{C}^m \oplus \mathcal{H}} + \mathcal{K}(\mathbb{C}^m \oplus \mathcal{H})$ such that*

$$\|v(\varphi(a) \oplus \sigma(a))v^* - \psi(a) \oplus \sigma(a))\| < \epsilon, \quad a \in \mathcal{F}. \tag{4}$$

Proof. By Proposition 2.1, there is a unital finite-dimensional representation $\eta : A \to M_r$ and a unitary $u \in M_{m+r}$ satisfying

$$\|u(\varphi(a) \oplus \eta(a))u^* - \psi(a) \oplus \eta(a)\| < \epsilon/3, \quad a \in \mathcal{F}. \tag{5}$$

Let z be the unitary $z = u \oplus 1_{\mathcal{H}} \in \mathbb{C}1_{\mathbb{C}^{m+r} \oplus \mathcal{H}} + \mathcal{K}(\mathbb{C}^{m+r} \oplus \mathcal{H})$. It follows from (5) that

$$\|z(\varphi(a) \oplus \eta(a) \oplus \sigma(a))z^* - \psi(a) \oplus \eta(a) \oplus \sigma(a)\| < \epsilon/3, \quad a \in \mathcal{F}. \tag{6}$$

By Voiculescu's Theorem [Vo1] there is a unitary $w : \mathcal{H} \to \mathbb{C}^r \oplus \mathcal{H}$ such that

$$\|w\sigma(a)w^* - \eta(a) \oplus \sigma(a)\| < \epsilon/3, \quad a \in \mathcal{F}. \tag{7}$$

If we set $v = (1_{\mathbb{C}^m} \oplus w^*)z(1_{\mathbb{C}^m} \oplus w) \in \mathbb{C}1_{\mathbb{C}^m \oplus \mathcal{H}} + \mathcal{K}(\mathbb{C}^m \oplus \mathcal{H})$, then (4) follows from (6) and (7). Indeed

$$\begin{aligned} \|v(\varphi(a) \oplus \sigma(a))v^* - \psi(a) \oplus \sigma(a)\| &= \|z(\varphi(a) \oplus w\sigma(a)w^*)z^* \\ &- \psi(a) \oplus w\sigma(a)w^*)\| \\ &\leq 2\|w\sigma(a)w^* - \eta(a) \oplus \sigma(a)\| \\ &+ \|z(\varphi(a) \oplus \eta(a) \oplus \sigma(a))z^* \\ &- \psi(a) \oplus \eta(a) \oplus \sigma(a)\| \\ &< 2\epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

□

Proposition 2.3 ([D3])**.** *Let A be a unital separable RFD C^*-algebra. Then A is nuclearly embeddable if and only if for any unital faithful representation $\sigma : A \to \mathcal{L}(\mathcal{H})$ with $\sigma(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, there exists a sequence of unital representations $\rho_n : A \to \mathcal{L}(\mathcal{H})$ whose images are contained in finite dimensional C^*-subalgebras of $\mathcal{L}(\mathcal{H})$ and such that for all $a \in A$, $\lim_{n \to \infty} \|\sigma(a) - \rho_n(a)\| = 0$*

Proof. This was proved in [D3]. A different proof is given the Appendix. □

Definition 2.4. Let A be a unital RFD C*-algebra. Let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. A unital representation $\pi : A \to M_k$ is called $(\mathcal{F}, \epsilon)$-admissible if there is a unital faithful representation $\sigma : A \to \mathcal{L}(\mathcal{H})$ with $\sigma(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, ($\mathcal{H} = \mathbb{C}^k \oplus \mathbb{C}^k \oplus \cdots$) such that if $\pi_\infty = \pi \oplus \pi \oplus \cdots$, then

$$\|\sigma(a) - \pi_\infty(a)\| < \epsilon \quad a \in \mathcal{F}. \tag{8}$$

Note that if π is $(\mathcal{F}, \epsilon)$-admissible, then so is $\pi \oplus \gamma$ for any unital finite dimensional representation γ. Moreover $\|\pi(a)\| \geq \|a\| - \epsilon$ for $a \in \mathcal{F}$. If A is separable nuclearly embeddable and RFD, then Proposition 2.3 guaranties the existence of $(\mathcal{F}, \epsilon)$-admissible representations for any finite set $\mathcal{F} \subset A$ and any $\epsilon > 0$. The following proposition is crucial for our embedding result. If n is a positive integer and π is a representation, then $n\pi$ will denote the representation $\pi \oplus \cdots \oplus \pi$ (n-times).

Proposition 2.5. *Let A be a separable unital nuclearly embeddable RFD C*-algebra. Let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. Then for any $(\mathcal{F}, \epsilon)$-admissible representation $\pi : A \to M_k$ and any two homotopic unital representations $\varphi, \psi : A \to M_m$, there exist a positive integer N and a unitary $u \in M_{m+Nk}$ such that*

$$\|u(\varphi(a) \oplus N\pi(a))u^* - \psi(a) \oplus N\pi(a)\| < 3\epsilon, \quad a \in \mathcal{F}.$$

Proof. By definition, π satisfies (8) for some unital faithful representation $\sigma : A \to \mathcal{L}(\mathcal{H})$ with $\sigma(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. By applying Proposition 2.2 to φ and ψ we find a unitary $v \in \mathbb{C}1_{\mathbb{C}^m \oplus \mathcal{H}} + \mathcal{K}(\mathbb{C}^m \oplus \mathcal{H})$ such that

$$\|v(\varphi(a) \oplus \sigma(a))v^* - \psi(a) \oplus \sigma(a)\| < \epsilon, \quad a \in \mathcal{F}. \tag{9}$$

From (8) and (9) we then obtain

$$\|v(\varphi(a) \oplus \pi_\infty(a))v^* - \psi(a) \oplus \pi_\infty(a)\| < 3\epsilon, \quad a \in \mathcal{F}. \tag{10}$$

Let $\mathcal{H}_n = \mathbb{C}^m \oplus \mathbb{C}^k \oplus \cdots \oplus \mathbb{C}^k \subset \mathbb{C}^m \oplus \mathcal{H}$ (n copies of $\mathbb{C}^k$) and let e_n denote the orthogonal projection of $\mathbb{C}^m \oplus \mathcal{H}$ onto $\mathcal{H}_n$. After a small perturbation of v we may assume that $v \in \mathbb{C}1_{\mathbb{C}^m \oplus \mathcal{H}} + \mathcal{K}(\mathcal{H}_N)$ for some large N. It is then clear that e_N commutes with v and with the images of $\varphi \oplus \pi_\infty$ and $\psi \oplus \pi_\infty$. Then $e_N(\varphi \oplus \pi_\infty)e_N = \varphi \oplus N\pi$, $e_N(\psi \oplus \pi_\infty)e_N = \psi \oplus N\pi$ and $u = e_N v e_N$ is a unitary in $\mathcal{L}(\mathcal{H}_N) \cong M_{m+Nk}$. We finish the proof by compressing by e_N in (10). □

For a C*-algebra A we denote by CA the cone of A ($CA = C_0[0,1) \otimes A$) and by SA the suspension of A ($SA = C_0(0,1) \otimes A$). Let $\widetilde{A}$ denote the C*-algebra obtained by adding a unit to A.

Theorem 2.6. *Let A be a separable nuclearly embeddable RFD C*-algebra. Then CA and SA are embeddable in the CAR algebra B.*

Proof. We have that $SA \subset CA \subset \widetilde{CA}$ so that it suffices to show that $D = \widetilde{CA}$ embeds unitally in B. A key property of D is that any two unital $*$-homomorphisms $D \to M_k$ are homotopic. Let $(\mathcal{F}_n)$ be a sequence of increasing finite subsets of D whose union is dense in D and let $\epsilon_n = 2^{-n}$. We will construct inductively a sequence $(r(n))$ of powers of two and a sequence (γ_n) of representations $\gamma_n : D \to M_{k(n)}$, where $k(1) = r(1)$, $k(n) = k(n-1)r(n)$ for $n \geq 2$, such that

(i) γ_n is $(\mathcal{F}_n, \epsilon_n)$-admissible (in particular $\|\gamma_n(a)\| \geq \|a\| - \epsilon_n$ for $a \in \mathcal{F}_n$).
(ii) $\|\gamma_n(a) - r(n)\gamma_{n-1}(a)\| < 3\epsilon_{n-1}$, for $a \in \mathcal{F}_{n-1}$.

Let $\gamma_1 : D \to M_{k(1)}$ be an $(\mathcal{F}_1, \epsilon_1)$-admissible representation. Such representations exist by Proposition 2.3 since D is nuclearly embeddable and RFD. By adding one-dimensional representations to γ_1 we may arrange that $k(1)$ is a power of two. Suppose now that $\gamma_1, \dots, \gamma_n$ and $r(1), \dots, r(n)$ were constructed. Let $\pi : D \to M_k$ be an $(\mathcal{F}_{n+1}, \epsilon_{n+1})$-admissible representation. By adding one-dimensional representations to π we can assume that $k = sk(n)$ for some integer s. Then $\pi(1) = s\gamma_n(1)$ hence π and $s\gamma_n$ are homotopic. Since γ_n is $(\mathcal{F}_n, \epsilon_n)$-admissible, by Proposition 2.5 there is N and a unitary $u \in M_{k+Nk(n)}$ such that $\|u(\pi(a) \oplus N\gamma_n(a))u^* - s\gamma_n(a) \oplus N\gamma_n(a)\| < 3\epsilon_n$ for $a \in \mathcal{F}_n$. By increasing N we may arrange that $N+s$ is a power of two. We conclude the construction by defining $r(n+1) = N + s$ and $\gamma_{n+1}(a) = u(\pi(a) \oplus N\gamma_n(a))u^*$. Let $\iota_n : M_{k(n)} \hookrightarrow B$ be the canonical inclusion. Having the sequence γ_n available, we construct a unital embedding $\gamma : D \to \varinjlim M_{k(n)} \cong B$ by defining $\gamma(a)$, $a \in \cup_n \mathcal{F}_n$, to be the limit of the Cauchy sequence $(\iota_n\gamma_n(a))$ and then extend to D by continuity. Note that $\|\gamma(a)\| = \|a\|$ since $\|\gamma_n(a)\| \geq \|a\| - \epsilon_n$ for $a \in \mathcal{F}_n$. ☐

3. Subquotients of the CAR algebra

Lemma 3.1. *Let A be a separable unital nuclearly embeddable C*-algebra. Then there exists a semisplit essential extension $0 \to I \to \widetilde{CF} \to A \to 0$ with F a unital separable nuclearly embeddable RFD C*-algebra.*

Proof. Since $\widetilde{CA}$ is homotopic to $\mathbb{C}$, any unital representation $\sigma : D \to \mathcal{L}(\mathcal{H})$ of $D = \widetilde{CA}$ is homotopic to a representation with image contained in $\mathbb{C}1_{\mathcal{H}}$. By [Vo3, Proposition 3] (its proof rather than its statement), there is a unital $*$-monomorphism $j : D \to \prod_{n=1}^{\infty} M_{k(n)}/\sum_{n=1}^{\infty} M_{k(n)}$ which admits a unital completely positive lifting $\eta : D \to \prod M_{k(n)}$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sum M_{k(n)} & \longrightarrow & \prod M_{k(n)} & \xrightarrow{\pi} & \prod M_{k(n)}/\sum M_{k(n)} & \longrightarrow & 0 \\ & & \| & & \Phi\uparrow & & \uparrow j & & \\ 0 & \longrightarrow & \sum M_{k(n)} & \longrightarrow & F & \xrightarrow{\pi'} & D & \longrightarrow & 0 \end{array}$$

where the extension at the bottom is the pull-back of the extension at the top. We are going to show that the unital $*$-monomorphism Φ is nuclear, hence F is a nuclearly embeddable C*-algebra. Since D is nuclearly embeddable, the map

η is nuclear by [D1, Proposition 3.3]. Note that η induces a unital completely positive map $\eta' : D \to F$ such that $\pi'\eta' = id_D$ and $\Phi\eta' = \eta$. Let e_n be the unit of $M_{k(1)} \oplus \cdots \oplus M_{k(n)}$. Then (e_n) is an approximate unit of projections of $\sum M_{k(n)}$ which is central in $\prod M_{k(n)}$ hence it is central in F. For any $z \in F$, $z = e_n z e_n + (1-e_n)z(1-e_n)$ and

$$\begin{aligned}\lim_{n\to\infty} \|z - e_n z e_n - (1-e_n)\eta'(\pi'(z))(1-e_n)\| &= \\ \lim_{n\to\infty} \|(1-e_n)(z - \eta'(\pi'(z)))(1-e_n)\| &= 0\end{aligned}$$

as $z - \eta'(\pi'(z)) \in \sum M_{k(n)}$. Since $\Phi\eta' = \eta$, we obtain

$$\lim_{n\to\infty} \|\Phi(z) - e_n\Phi(z)e_n - (1-e_n)\eta(\pi'(z))(1-e_n)\| = 0.$$

This proves that Φ is nuclear since the maps η and $e_n\Phi(-)e_n$ are nuclear. Note that F is RFD as it embeds in $\prod M_{k(n)}$.

To finish the proof we need the following observation. Suppose that G is a unital C*-algebra. Then $\widetilde{CG} \cong \{f \in C([0,1], G) : f(1) \in \mathbb{C}1_G\}$. Let λ be a state of G. The surjection $\pi_G : \widetilde{CG} \to G$, $\pi_G(f) = f(0)$ admits a unital completely positive right inverse given by $\eta_G(a)(t) = t\lambda(a)1_G + (1-t)a$, $(a \in G, t \in [0,1])$. It follows that $\eta_F \circ \eta' \circ \eta_A$ is a unital completely positive right inverse for the composition

$$\widetilde{CF} \xrightarrow{\pi_F} F \xrightarrow{\pi'} \widetilde{CA} \xrightarrow{\pi_A} A$$

Moreover it is clear that $\widetilde{CF}$ is nuclearly embeddable and RFD since F is so. Finally we set $I = \ker(\pi_A \circ \pi' \circ \pi_F)$ and notice that I is an essential ideal of $\widetilde{CF}$ as it contains SF. □

4. Proof of Theorem 1.1

(i) $\Rightarrow$ (ii) Let $0 \to I \to \widetilde{CF} \xrightarrow{\nu} A \to 0$ be the extension given Lemma 3.1 and let η be a unital completely positive right inverse of ν. By Theorem 2.6 $\widetilde{CF}$ embeds unitally in B . Let $J = \overline{IBI}$ be the hereditary C*-subalgebra of B generated by I. Then J is a two-sided closed ideal of the C*-algebra $E = \widetilde{CF} + J$ and it is easy to check that we still have a semisplit essential unital extension $0 \to J \to E \xrightarrow{\pi} A \to 0$. Indeed, the composition of $\widetilde{CF} \hookrightarrow E$ with η defines a unital completely positive right inverse of π. To check that J is essential in E, let $x \in \widetilde{CF}$, $y \in J$ be such that $(x+y)J = 0$. Let (h_n) be an approximate unit of I. Then $h_n(x+y)h_n = 0$ hence $y = -\lim_{n\to\infty} h_n x h_n \in I$. Now $(x+y)I = 0$ implies $x + y = 0$ since I is essential in $\widetilde{CF}$. Since J is a hereditary C*-subalgebra of B and B is simple, J is stably isomorphic to B by Brown's theorem [Br]. The rest of the proof is taken from [W3]. We include it for the sake of completeness. (ii) $\Rightarrow$ (i) Let $\eta : A \to E$ be a unital completely positive right inverse of $\pi : E \to A$. Since $\pi\eta = id_A$, η is a unital complete isometry hence it is a complete order embedding. This means that its inverse $\eta^{-1} : \eta(A) \to A$ is unital and completely positive. Let $\sigma : A \to \mathcal{L}(\mathcal{H})$ be a unital faithful representation. By Arveson's extension theorem $\sigma\eta^{-1}$ extends to a unital completely positive map $\theta : B \to \mathcal{L}(\mathcal{H})$. Then $\sigma = \theta\eta$ is nuclear since it factorises through the nuclear algebra B (ii) $\Rightarrow$ (iii) is obvious. (iii) $\Rightarrow$ (ii) If E embeds in B, then so does $\widetilde{CE}$ since $C_0[0,1) \subset B$ and $B \otimes B \cong B$. After replacing the map $E \to A$ by the composition $\widetilde{CE} \to E \to A$ whose kernel is an essential ideal of $\widetilde{CE}$, we may assume that the given extension is essential. Arguing as in the last part of (i) $\Rightarrow$ (ii), we may arrange that J is an essential AF-ideal. Since B has property (C) of Archbold and Batty and $E \subset B$, E has property (C) [AB]. One concludes the proof by applying the lifting results of Effros and Haagerup. Indeed, by [EK, Proposition 5.3 (3) and Theorem 3.4] any extension $0 \to J \to E \to A \to 0$ is semisplit if E has property (C) and J is an AF-ideal. □

Applications

Following Kirchberg, let us review the main applications of Theorem 1.1.

1. *Any separable unital nuclearly embeddable C*-algebra embeds as a unital C*-subalgebra of the Cuntz algebra $\mathcal{O}_2$.* We use the implication (i) $\Rightarrow$ (ii) of Theorem 1.1 so that we do not rely on [EK]. Let $0 \to J \to E \to A \to 0$ be a semisplit essential extension given by Theorem 1.1. Since B embeds unitally in $\mathcal{O}_2$ [Cu1, 1.5], after replacing J by the hereditary C*-subalgebra J' of $\mathcal{O}_2$ generated by J and replacing E by $E' = E + J'$ we obtain a semisplit essential extension $0 \to J' \to E' \to A \to 0$ where J' is stably isomorphic to $\mathcal{O}_2$ by [Br]. In particular J' is non-unital and hence stable by [Zh, Theorem 1.2] so that $J' \cong \mathcal{O}_2 \otimes \mathcal{K}(\mathcal{H})$. By the Weyl-von Neumann-Voiculescu type theorem of Kirchberg [Ki4], the extension $0 \to J' \to E' \to A \to 0$ is unitally absorbing. Since $Ext^{-1}(A, \mathcal{O}_2) = 0$, (as $id_{\mathcal{O}_2} \oplus id_{\mathcal{O}_2}$ is homotopic to $id_{\mathcal{O}_2}$ by [Cu2] and $Ext^{-1}(A, -)$ is homotopy invariant by [Ka]) we conclude that the extension $0 \to J' \to E' \to A \to 0$ splits. Therefore there is a unital $*$-monomorphism $\gamma : A \to E' \subset \mathcal{O}_2$.

2. *Any quotient of a unital separable nuclearly embeddable C*-algebra A is nuclearly embeddable.* This is an obvious consequence of the equivalence between (i) and (iii), whose proof relies essentially on [EK].

5. Appendix

Here we prove a generalization of Proposition 2.3 which clarifies the role of nuclear embeddability in our context. Let A be a C*-algebra, let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. If $\varphi : A \to \mathcal{L}(\mathcal{H}_\varphi)$ and $\psi : A \to \mathcal{L}(\mathcal{H}_\psi)$ are two maps, we write $\varphi \underset{\mathcal{F},\epsilon}{\prec} \psi$ if there is an isometry $v : \mathcal{H}_\varphi \to \mathcal{H}_\psi$ such that $\|\varphi(a) - v^*\psi(a)v\| < \epsilon$ for all $a \in \mathcal{F}$. If v can be chosen to be a unitary, then we write $\varphi \underset{\mathcal{F},\epsilon}{\sim} \psi$. We write $\varphi \prec \psi$ ($\varphi \sim \psi$) if $\varphi \underset{\mathcal{F},\epsilon}{\prec} \psi$ (respectively $\varphi \underset{\mathcal{F},\epsilon}{\sim} \psi$) for all $\mathcal{F}$ and ϵ. Note that $\varphi \underset{\mathcal{F},\epsilon}{\sim} \psi \Leftrightarrow \psi \underset{\mathcal{F},\epsilon}{\sim} \varphi$ and if $\varphi \underset{\mathcal{F},\epsilon_1}{\sim} \psi$, $\psi \underset{\mathcal{F},\epsilon_2}{\sim} \gamma$, then $\varphi \underset{\mathcal{F},\epsilon_1+\epsilon_2}{\sim} \gamma$. We let φ_∞ denote the infinite direct sum $\varphi \oplus \varphi \oplus \cdots$.

Lemma 5.1. *Let A be a unital C^*-algebra, let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. There exist $\mathcal{G} \subset A$ a finite subset and $\delta > 0$ such that if $\varphi : A \to \mathcal{L}(\mathcal{H}_\varphi)$, $\psi : A \to \mathcal{L}(\mathcal{H}_\psi)$ are selfadjoint contractions with $\|\varphi(a^*a) - \varphi(a^*)\varphi(a)\| < \delta$, $\|\psi(a^*a) - \psi(a^*)\psi(a)\| < \delta$, $a \in \mathcal{G}$, then we have the following. (i) If $\varphi_\infty \underset{\mathcal{G},\delta}{\prec} \psi$, then $\varphi \oplus \psi \underset{\mathcal{F},\epsilon}{\sim} \psi$. (ii) If $\varphi_\infty \underset{\mathcal{G},\delta}{\prec} \psi$ and if $\psi_\infty \underset{\mathcal{G},\delta}{\prec} \varphi$, then $\varphi \underset{\mathcal{F},\epsilon}{\sim} \psi$.*

Proof. This goes along the lines of the proof of Voiculescu's theorem [Vo1], [Ar]. It suffices to prove only part (i), since (i) $\Rightarrow$ (ii). Let $\mathcal{G} = \{ab : a, b \in \mathcal{F} \cup \mathcal{F}^*\} \cup \mathcal{F} \cup \mathcal{F}^*$ and let $\delta > 0$ be small enough so that $2\delta + 8\delta^{1/2}(2M+3)^{1/2} < \epsilon$, where $M = \max\{\|a\| : a \in \mathcal{F}\}$. Define $\phi = \varphi_\infty$. By assumption there is an isometry $v : \mathcal{H}_\phi \to \mathcal{H}_\psi$ such that

$$\|\phi(x) - v^*\psi(x)v\| < \delta, \quad x \in \mathcal{G}. \tag{11}$$

From (11) and the identity

$$\begin{aligned}(v\phi(a) - \psi(a)v)^*(v\phi(a) - \psi(a)v) \;&=\; \phi(a^*)(\phi(a) - v^*\psi(a)v) + (\phi(a^*) \\ &-\; v^*\psi(a^*)v)\phi(a) + v^*(\psi(a^*)\psi(a) \\ &-\; \psi(a^*a))v + (v^*\psi(a^*a)v \\ &-\; \phi(a^*a)) + (\phi(a^*a) - \phi(a^*)\phi(a))\end{aligned}$$

we obtain

$$\|v\phi(a) - \psi(a)v\| < \delta^{1/2}(2M+3)^{1/2}, \quad a \in \mathcal{F} \cup \mathcal{F}^*. \tag{12}$$

If $p = vv^*$, then since ϕ is selfadjoint

$$[\psi(a), p] = (\psi(a)v - v\phi(a))v^* + v(v\phi(a^*) - \psi(a^*)v)^*. \tag{13}$$

From (12) and (13) we obtain $\|\psi(a)p - p\psi(a)\| < 2\delta^{1/2}(2M+3)^{1/2}$ for all $a \in \mathcal{F}$, hence

$$\|\psi(a) - p\psi(a)p - (1-p)\psi(a)(1-p)\| < \delta_1 = 4\delta^{1/2}(2M+3)^{1/2}, \quad a \in \mathcal{F}. \tag{14}$$

Regarding v as a unitary from $\mathcal{H}_\phi$ to $p\mathcal{H}_\psi$, we obtain from (11)

$$\phi \underset{\mathcal{G},\delta}{\sim} p\psi p. \tag{15}$$

Combining (14) with (15) and setting $\lambda(a) = (1-p)\psi(a)(1-p)$ we have $\phi \oplus \lambda \underset{\mathcal{G},\delta}{\sim} p\psi p \oplus (1-p)\psi(1-p) \underset{\mathcal{F},\delta_1}{\sim} \psi$, hence

$$\varphi_\infty \oplus \lambda = \phi \oplus \lambda \underset{\mathcal{F},\epsilon/2}{\sim} \psi$$

since $\mathcal{F} \subset \mathcal{G}$ and $\delta + \delta_1 = \delta + 4\delta^{1/2}(2M+3)^{1/2} < \epsilon/2$ by our choice of δ. Therefore

$$\psi \underset{\mathcal{F},\epsilon/2}{\sim} \varphi_\infty \oplus \lambda \sim \varphi \oplus \varphi_\infty \oplus \lambda \underset{\mathcal{F},\epsilon/2}{\sim} \varphi \oplus \psi,$$

hence $\psi \underset{\mathcal{F},\epsilon}{\sim} \varphi \oplus \psi$. ∎

6. Proof of Proposition 2.3

The result is a consequence of the following.

Proposition 6.1. *Let A be a unital separable nuclearly embeddable C^*-algebra and let (χ_n) be a sequence of unital representations separating the elements of A. Then for any $\mathcal{F} \subset A$ a finite subset and any $\epsilon > 0$ there is a representation π of A of the form $\pi = \chi_{i_1} \oplus \chi_{i_2} \oplus \cdots \oplus \chi_{i_m}$ such that if $\sigma : A \to \mathcal{L}(\mathcal{H})$ is any unital faithful representation with $\sigma(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, then $\sigma \underset{\mathcal{F},\epsilon}{\sim} \pi_\infty$.*

Proof. By Voiculescu's theorem [Vo1] it suffices to prove the statement for a fixed representation σ. With $\mathcal{F}$ and ϵ as above let $\mathcal{G} \subset A$ and $\delta > 0$ be given by Lemma 5.1. In the first part of the proof we find π of the desired form such that $\sigma \underset{\mathcal{G},\delta}{\prec} \pi_\infty$. Since A is nuclearly embeddable, the representation σ is nuclear. Thus we find unital completely positive maps $\alpha : A \to \mathcal{L}(\mathbb{C}^k)$ and $\beta : \mathcal{L}(\mathbb{C}^k) \to \mathcal{L}(\mathcal{H})$ such that

$$\|\sigma(a) - \beta\alpha(a)\| < \delta/2, \quad a \in \mathcal{G}. \tag{16}$$

We may assume that each representation in the sequence (χ_n) repeats itself infinitely many times. Thus $\chi = \chi_1 \oplus \chi_2 \oplus \dots$ is a unital faithful representation of A of infinite multiplicity. If π_α is the Stinespring dilation of α, then we have $\alpha \prec \pi_\alpha \prec \pi_\alpha \oplus \chi$. Therefore $\alpha \prec \chi$ since $\pi_\alpha \oplus \chi \sim \chi$ by Voiculescu's theorem. By a standard perturbation argument we obtain $\alpha \underset{\mathcal{G},\delta/2}{\prec} \chi_1 \oplus \cdots \oplus \chi_n$ for some large enough n. Thus if we set $\pi = \chi_1 \oplus \cdots \oplus \chi_n$, then there is an isometry $w : \mathbb{C}^k \to \mathcal{H}_\pi$ with

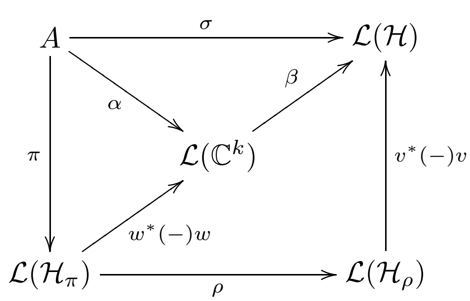
$$\|\alpha(a) - w^*\pi(a)w\| < \delta/2, \quad a \in \mathcal{G}. \tag{17}$$

By Stinespring's theorem, the unital completely positive map $x \mapsto \beta(w^*xw)$ can be dilated to a unital representation $\rho : \mathcal{L}(\mathcal{H}_\pi) \to \mathcal{L}(\mathcal{H}_\rho)$. Thus we find an isometry $v : \mathcal{H} \to \mathcal{H}_\rho$ with

$$\beta(w^*xw) = v^*\rho(x)v, \quad x \in \mathcal{L}(\mathcal{H}_\pi). \tag{18}$$

From (16), (17) and (18) we obtain

$$\|\sigma(a) - v^*\rho(\pi(a))v\| = \|\sigma(a) - \beta(w^*\pi(a)w)\|$$
$$\leq \|\sigma(a) - \beta\alpha(a)\| + \|\beta\alpha(a) - \beta(w^*\pi(a)w)\| < \delta/2 + \delta/2 = \delta, \quad a \in \mathcal{G}.$$



This gives $\sigma \underset{\mathcal{G},\delta}{\prec} \rho\pi$. It is clear from the representation theory of $\mathcal{L}(\mathcal{H}_\pi)$ that $\rho\pi \sim \pi_\infty$. Therefore $\sigma \underset{\mathcal{G},\delta}{\prec} \pi_\infty$ hence $\sigma_\infty \underset{\mathcal{G},\delta}{\prec} \pi_\infty$ as $(\pi_\infty)_\infty \sim \pi_\infty$. By Voiculescu's theorem, we have $\sigma \oplus \pi_\infty \sim \sigma$ hence $(\pi_\infty)_\infty \prec \sigma$. By Lemma 5.1(ii) it follows that $\sigma \underset{\mathcal{F},\epsilon}{\sim} \pi_\infty$. □

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Department of Mathematics, Purdue University, West Lafayette IN 47907.
E-mail address: `mdd@math.purdue.edu`