On the topology of the Kasparov groups and its applications

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Abstract

In this paper we establish a direct connection between stable approximate unitary equivalence for $*$-homomorphisms and the topology of the KK-groups which avoids entirely C*-algebra extension theory and does not require nuclearity assumptions. To this purpose we show that a topology on the Kasparov groups can be defined in terms of approximate unitary equivalence for Cuntz pairs and that this topology coincides with both Pimsner's topology and the Brown-Salinas topology. We study the generalized Rørdam group $KL(A,B) = KK(A,B)/\bar{0}$, and prove that if a separable exact residually finite dimensional C*-algebra satisfies the universal coefficient theorem in KK-theory, then it embeds in the UHF algebra of type 2^∞. In particular such an embedding exists for the C*-algebra of a second countable amenable locally compact maximally almost periodic group.

Key words: KK-theory, C*-algebras, amenable groups

1 Introduction

Two $*$-homomorphisms $\varphi, \psi : A \to B$ are unitarily equivalent if $u\varphi u^* = \psi$ for some unitary $u \in B$. They are approximately unitarily equivalent, written $\varphi \approx_u \psi$, if there is a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in B such that

$$\lim_{n\to\infty} \|u_n\varphi(a)u_n^* - \psi(a)\| = 0$$

for all $a \in A$. Stable approximate unitary equivalence is a more elaborated concept introduced in Def. 3.6. According to Glimm's theorem, any non type I separable C*-algebra has uncountably many non unitarily equivalent irreducible

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representations with the same kernel. In contrast, by Voiculescu's theorem, two irreducible representations of a separable C*-algebra have the same kernel if and only if they are approximately unitarily equivalent. A comparison of the above results suggests that the notion of unitary equivalence is sometimes too rigid and that for certain purposes one can do more things by working with approximate unitary equivalence. This point of view is illustrated by Elliott's intertwining argument: if $\varphi : A \to B$ and $\psi : B \to A$ are unital $*$-homomorphisms between separable C*-algebras such that $\varphi\psi \approx_u id_B$ and $\psi\varphi \approx_u id_A$, then A is isomorphic to B. It is therefore very natural to study approximate unitary equivalence of $*$-homomorphisms in a general context.

Two approximately unitarily equivalent $*$-homomorphisms $\varphi, \psi : A \to B$ induce the same map on K-theory with coefficients, but they may have different KK-theory classes. In order to handle this situation, Rørdam introduced the group $KL(A, B)$ as the quotient of $\mathrm{Ext}(SA, B)^{-1} \cong KK(A, B)$ by the subgroup $\mathrm{PExt}(K_{*-1}(A), K_*(B))$ of $\mathrm{Ext}(K_{*-1}(A), K_*(B))$ generated by pure group extensions [25]. This required the assumption that A satisfies the universal coefficient theorem (UCT) of [27]. Using a mapping cylinder construction, Rørdam showed that two approximately unitarily equivalent $*$-homomorphisms have the same class in $KL(A, B)$. On the other hand, a topology on the Ext-theory groups was considered by Brown-Douglas-Fillmore [4], and shown to have interesting applications in [3] and [28]. This topology, called hereafter the Brown-Salinas topology, is defined via approximate unitary equivalence of extensions. It was further investigated by Schochet in [31,32] and by the author in [7]. Schochet showed that the Kasparov product is continuous with respect to the Brown-Salinas topology for K-nuclear separable C*-algebras. An important idea from [31,32] is that one can use the continuity of the Kasparov product in order to transfer structural properties between KK-equivalent C*-algebras. As it turns out, the subgroup $\mathrm{PExt}(K_{*-1}(A), K_*(B))$ of $\mathrm{Ext}(SA, B)^{-1}$ coincides with the closure of zero in the Brown-Salinas topology under the assumption that A is nuclear and satisfies the UCT. It is then quite natural to define $KL(A, B)$ for arbitrary separable C*-algebras as $\mathrm{Ext}(SA, B)^{-1}/\bar{0}$ as proposed by H. Lin in [20]. Nevertheless, the study of $*$-homomorphisms from A to B via their class in $\mathrm{Ext}(SA, B)^{-1}$ is not optimal and leads to rather involved arguments as those in [18,20] and [7] where the Brown-Salinas topology of $\mathrm{Ext}(SA, B)^{-1}$ is related, in the nuclear case, to stable approximate unitary equivalence of $*$-homomorphisms from A to B.

Kasparov's KK-theory admits several equivalent descriptions. This deep feature enables one to choose working with the picture that is most effective in a given situation. Similarly, there are several (and as we are going to see, equivalent) ways to introduce a topology on the KK-groups. The Brown-Salinas topology was already mentioned. In a recent important paper [22], Pimsner defines a topology on the equivariant graded KK-theory and proves the continuity of the Kasparov product in full generality. The convergence of sequences in Pimsner's topology admits a particularly nice and simple algebraic description which leads to major simplifications of the theory (see Lemma 3.1). However, the previous descriptions of the topology of $KK(A,B)$ do not appear to be well adapted for the study of approximate unitary equivalence of $*$-homomorphisms.

In this paper we introduce a topology on $KK(A,B)$ in terms of Cuntz pairs and approximate unitary equivalence. We then show that this topology coincides with Pimsner's topology (Thms. 3.3 and 3.5). Our arguments rely on a result of Thomsen [35] and on our joint work with Eilers [12]. Two $*$-homomorphisms from A to B is the simplest instance of a Cuntz pair. However, since in general the Kasparov group $KK(A,B)$ is not generated by $*$-homomorphisms from A to B, it becomes necessary to work with Cuntz pairs. We revisit Rørdam's group $KK(A,B)/\bar{0}$ in our general setting and show that it is a polish group (cf. [31]) when endowed with the (quotient of) Pimsner's topology for arbitrary separable C*-algebras (see Prop. 2.8). Along the way we show that the Brown-Salinas topology coincides with Pimsner's topology (Cor. 6.3) and we give a series of applications which include:

(i) two $*$-homomorphisms are stably approximately unitarily equivalent if and only if their KK-theory classes are equal modulo the closure of zero (see Cor. 3.8, 3.7.)

(ii) If a separable C*-algebra A satisfies the universal coefficient theorem in KK-theory (UCT), then $KK(A,B)/\,\bar{0}$ is homeomorphic to $\mathrm{Hom}_{\Lambda}(\underline{K}(A),\underline{K}(B))$, where the latter group is endowed with the topology of pointwise convergence (see Thm. 4.1). Thus, in order to check that two KK-elements are close to each other, it suffices to verify that the maps they induce on the total K-theory group $\underline{K}(A)=\oplus_{n=0}^{\infty}K_*(A;\mathbb{Z}/n)$ agree on a sufficiently large finite subset.

(iii) If a separable exact residually finite dimensional C*-algebra satisfies the UCT then it embeds in the UHF algebra of type 2^∞; see Thm. 4.4. In particular the C*-algebra of a second countable amenable locally compact maximally almost periodic group embeds in the UHF algebra of type 2^∞.

(iv) We give a short proof of a theorem of H. Lin, [20], stating that two unital $*$-homomorphisms between Kirchberg C*-algebras are approximately unitarily equivalent if and only if their KL-classes coincide. This is used to show that a separable nuclear C*-algebra satisfies the approximate universal coefficient theorem of [20] if and only if it satisfies the UCT (Thm. 5.4), answering a question of H. Lin from [20].

For A in the bootstrap category of [27], one can derive (ii) from [32] and [11]. Its generalization to the nonnuclear case is necessary in view of applications such as (iii). The latter result was given a more complicated proof in an earlier preprint [5] which is now superseded by the present paper. A definition of the topology of $KK_{nuc}(A, B)$ has also appeared there, but it became a more useful tool after the emergence of [22]. The author is grateful to M. Pimsner for providing him with a draft of [22].

2 Metric structure

In this section we define an invariant pseudometric d on $KK(A, B)$ which makes $KK(A, B)$ a complete separable topological group. This is done by using a description of $KK(A, B)$ based on Cuntz pairs and the asymptotic unitary equivalence of [12].

The C*-algebras in this paper, denoted by A, B, C,... will be assumed to be separable. We only consider Hilbert B-bimodules, E, F,... that are countably generated. The notation H_B is reserved for the canonical Hilbert B-bimodule obtained as the completion of $\ell^2(\mathbb{N}) \otimes_{alg} B$. As in [15] we identify $M(B \otimes \mathcal{K})$ with $\mathcal{L}(H_B)$. A unital $*$-homomorphism $\gamma : A \to \mathcal{L}(H_B)$ is called *unitally absorbing* (for the pair of C*-algebras (A, B)) if for any unital $*$-homomorphism $\varphi : A \to \mathcal{L}(H_B)$ there is a sequence of unitaries $u_n \in \mathcal{L}(H_B, H_B \oplus H_B)$ such that for all $a \in A$:

(i) $\lim_{n\to\infty} \|u_n^* (\varphi(a) \oplus \gamma(a))\, u_n - \gamma(a)\| = 0$

(ii) $u_n^* (\varphi(a) \oplus \gamma(a))\, u_n - \gamma(a) \in \mathcal{K}(H_B)$

A $*$-homomorphism $\gamma : A \to \mathcal{L}(H_B)$ is called *absorbing* if its unitalization $\tilde{\gamma} : \tilde{A} \to \mathcal{L}(H_B)$ is unitally absorbing. The theorems of Voiculescu [37] and Kasparov [15] exhibit large classes of absorbing $*$-homomorphisms. Thomsen [35] proved the existence of absorbing $*$-homomorphisms for arbitrary separable C*-algebras.

Let $\mathcal{E}_c(A, B)$ denote the set of all Cuntz pairs (φ, ψ). They consists of $*$-homomorphisms $\varphi, \psi : A \to \mathcal{L}(H_B)$ such that $\varphi(a) - \psi(a) \in \mathcal{K}(H_B)$ for all $a \in A$. It is was shown by Cuntz that $KK(A, B)$ can be defined as the group of homotopy classes of Cuntz pairs. In our joint work with Eilers we proved that $KK(A, B)$ can be realized in terms of proper asymptotic unitary equivalence classes of Cuntz pairs:

Theorem 2.1 ([12]) *Let A, B be separable C*-algebras and let $(\varphi, \psi) \in \mathcal{E}_c(A, B)$ be a Cuntz pair. The following are equivalent:*

(i) $[\varphi, \psi] = 0$ in $KK(A, B)$.

(ii) There is a $*$-homomorphism $\gamma : A \to \mathcal{L}(H_B)$ and there is a continuous unitary valued map $t \mapsto u_t \in 1 + \mathcal{K}(H_B \oplus H_B)$, $t \in [0, \infty)$, such that for all

$a \in A$

$$\lim_{t\to\infty} \|u_t\left(\varphi(a)\oplus\gamma(a)\right)u_t^* - \psi(a)\oplus\gamma(a)\| = 0 \tag{1}$$

(iii) For any absorbing $*$-homomorphism $\gamma : A \to \mathcal{L}(H_B)$ there is a continuous unitary valued map $t \mapsto u_t \in I + \mathcal{K}(H_B \oplus H_B)$, $t \in [0,\infty)$ satisfying (1) for all $a \in A$.

This theorem suggests the following construction of a pseudometric on $KK(A,B)$.

Let $(a_i)_{i=1}^\infty$ be a dense sequence in the unit ball of A. If $\varphi, \psi : A \to \mathcal{L}(E)$ are $*$-homomorphisms, we define

$$\delta_0(\varphi,\psi) = \sum_{i=1}^\infty \frac{1}{2^i}\|\varphi(a_i)-\psi(a_i)\|, \text{ and}$$

$$\delta_\gamma(\varphi,\psi) = \inf\{\delta_0(\varphi\oplus\gamma, u(\psi\oplus\gamma)u^*) : u \in 1+\mathcal{K}(E\oplus F)\,\text{unitary}\,\},$$

where $\gamma : A \to \mathcal{L}(F)$ is an absorbing $*$-homomorphism. One verifies immediately that $\delta_\gamma(\varphi,\varphi)=0$, $\delta_\gamma(\varphi,\psi)=\delta_\gamma(\psi,\varphi)$ and $\delta_\gamma(\varphi,\eta)\le\delta_\gamma(\varphi,\psi)+\delta_\gamma(\psi,\eta)$. Moreover, if $\|\varphi_n(a)-\varphi(a)\|\to 0$ for all $a \in A$, then $\delta_\gamma(\varphi_n,\psi)\to\delta_\gamma(\varphi,\psi)$. If $\gamma_i : A \to \mathcal{L}(F_i)$, $i=1,2$ are $*$-homomorphisms, then we write $\gamma_1 \sim \gamma_2$ if there is a sequence of unitaries $w_n \in \mathcal{L}(F_1,F_2)$ such that for all $a \in A$

$$\lim_{n\to\infty}\|w_n\gamma_1(a)w_n^* - \gamma_2(a)\| = 0. \tag{2}$$

Lemma 2.2 *If $\gamma_1 \sim \gamma_2$, then $\delta_{\gamma_1}(\varphi,\psi)=\delta_{\gamma_2}(\varphi,\psi)$.*

PROOF. If $w \in \mathcal{L}(F_1,F_2)$ is a unitary, then $\delta_{\gamma_1}(\varphi,\psi)=\delta_{w\gamma_1 w^*}(\varphi,\psi)$, since conjugation by $1\oplus w$ maps $1+\mathcal{K}(E\oplus F_1)$ onto $1+\mathcal{K}(E\oplus F_2)$. Thus $\delta_{\gamma_1}(\varphi,\psi)=\delta_{w_n\gamma_1 w_n^*}(\varphi,\psi)\to\delta_{\gamma_2}(\varphi,\psi)$. $\square$

The assumption of Lemma 2.2 is automatically satisfied whenever γ_i are absorbing $*$-homomorphisms. Therefore we can define $\delta(\varphi,\psi)=\delta_\gamma(\varphi,\psi)$ for some absorbing $*$-homomorphism γ and this definition does not depend on γ.

Lemma 2.3 *With notation as above*

(a) If $w \in \mathcal{L}(E,F)$ is a unitary, then $\delta(w\varphi w^*, w\psi w^*)=\delta(\varphi,\psi)$,

(b) If $\eta : A \to \mathcal{L}(F)$ is a $*$-homomorphism, then $\delta(\varphi,\psi)=\delta(\varphi\oplus\eta,\psi\oplus\eta)=\delta(\eta\oplus\varphi,\eta\oplus\psi)$.

PROOF. For part (a) one argues as in the proof of the previous lemma. For part (b) one uses the observation that $\gamma \oplus \eta$ is absorbing whenever γ is absorbing and part (a). $\Box$

If $\varphi, \psi : A \to \mathcal{L}(E)$ are $*$-homomorphisms, we write $(\varphi) \approx (\varphi')$ if there is a sequence of unitaries $u_n \in 1 + \mathcal{K}(E)$ such that $\lim_{n\to\infty} \|u_n\varphi(a)u_n^* - \psi(a)\| = 0$ for all $a \in A$.

Lemma 2.4 *Let $\varphi, \psi : A \to \mathcal{L}(E)$ and $\varphi', \psi' : A \to \mathcal{L}(E)$ be $*$-homomorphisms. Assume that $(\varphi) \approx (\varphi')$ and $(\psi) \approx (\psi')$. Then $\delta(\varphi, \psi) = \delta(\varphi', \psi')$.*

PROOF. This is an immediate consequence of the definition of δ and the observation that if $u \in 1 + \mathcal{K}(E)$ is a unitary, then $\delta(u\varphi u^*, \psi) = \delta(\varphi, \psi)$. $\Box$

We are now ready to introduce a pseudometric d on $\mathcal{E}_c(A, B)$. A pseudometric satisfies all the properties of a metric except that $d(x, y) = 0$ may not imply $x = y$.

Definition 2.5 $d((\varphi, \psi), (\varphi', \psi')) = \delta(\varphi \oplus \psi', \psi \oplus \varphi')$.

Lemma 2.6 *If $x, x' \in \mathcal{E}_c(A, B)$ and $[x] = [x']$ in $KK(A, B)$ then $d(x, x') = 0$.*

PROOF. If $x = (\varphi, \psi)$ and $x = (\varphi', \psi')$ then $[x] - [x'] = [\varphi \oplus \psi', \psi \oplus \varphi'] = 0$. By Theorem 2.1 this implies $\delta(\varphi \oplus \psi', \psi \oplus \varphi') = 0$ hence $d(x, x') = 0$. $\Box$

Proposition 2.7 *d is a pseudometric on $\mathcal{E}_c(A, B)$ that descends to an invariant pseudometric on $KK(A, B)$ (denoted again by d!).*

PROOF. First we show that d is a pseudometric on $\mathcal{E}_c(A, B)$. Let $x = (\varphi, \psi), x' = (\varphi', \psi') \in \mathcal{E}_c(A, B)$. Then $d(x, x) = 0$ by Lemma 2.6. The equality $d(x, x') = d(x', x)$ is equivalent to $\delta(\varphi \oplus \psi', \psi \oplus \varphi') = \delta(\varphi' \oplus \psi, \psi' \oplus \varphi)$. The latter equality follows from Lemma 2.3(a) with w a permutation unitary and the symmetry of δ. In order to verify the triangle inequality for d, we first recall that if $\alpha, \alpha', \alpha'' : A \to \mathcal{L}(E)$ then

$$\delta(\alpha, \alpha') + \delta(\alpha', \alpha'') \geq \delta(\alpha, \alpha''). \tag{3}$$

Let $x'' = (\varphi'', \psi'') \in \mathcal{E}_c(A, B)$. The inequality $d(x, x') + d(x', x'') \geq d(x, x'')$ is equivalent to

$$\delta(\varphi \oplus \psi', \psi \oplus \varphi') + \delta(\varphi' \oplus \psi'', \psi' \oplus \varphi'') \geq \delta(\varphi \oplus \psi'', \psi \oplus \varphi'') \tag{4}$$

By Lemma 2.3

$\delta(\varphi \oplus \psi'', \psi \oplus \varphi'') = \delta(\varphi \oplus \psi'' \oplus \psi', \psi \oplus \varphi'' \oplus \psi') = \delta(\varphi \oplus \psi' \oplus \psi'', \psi \oplus \psi' \oplus \varphi'')$

and the latter term less than or equal to $\delta(\varphi \oplus \psi' \oplus \psi'', \psi \oplus \varphi' \oplus \psi'') + \delta(\psi \oplus \varphi' \oplus \psi'', \psi \oplus \psi' \oplus \varphi'')$ by (3). Finally $\delta(\varphi \oplus \psi' \oplus \psi'', \psi \oplus \varphi' \oplus \psi'') = \delta(\varphi \oplus \psi', \psi \oplus \varphi')$ and $\delta(\psi \oplus \varphi' \oplus \psi'', \psi \oplus \psi' \oplus \varphi'') = \delta(\varphi' \oplus \psi'', \psi' \oplus \varphi'')$ by Lemma 2.3. This proves the inequality (4).

Next we are going to verify that d descends to a metric on $KK(A,B)$. By symmetry, it suffices to prove that if $x, x', x'' \in \mathcal{E}_c(A,B)$ and $[x'] = [x'']$ in $KK(A,B)$, then $d(x,x'') \le d(x,x')$. By Lemma 2.6, $d(x',x'') = 0$. Since d is a pseudometric, $d(x,x'') \le d(x,x') + d(x',x'') = d(x,x')$.

It remains to verify the invariance of the pseudometric. We show that $d(x \oplus y, x' \oplus y) = d(x,x')$ for all $x, x', y \in \mathcal{E}_c(A,B)$. Let $\hat{d}([x],[x']) = d(x,x')$ denote (temporarily) the induced metric on $KK(A,B)$. We claim that $d(x,x') = \hat{d}([x]-[x'],0)$, which implies the invariance of d. To verify the claim note that if $x = (\varphi,\psi)$ and $x = (\varphi',\psi')$ then $d(x,x') = \delta(\varphi \oplus \psi', \psi \oplus \varphi')$ by definition, and $\hat{d}([x]-[x'],0) = d((\varphi \oplus \psi', \psi \oplus \varphi'),(0,0)) = \delta(\varphi \oplus \psi', \psi \oplus \varphi')$. □

Proposition 2.8 *Let A be B be separable C*-algebras. The topology of $KK(A,B)$ defined by the pseudometric d satisfies the second axiom of countability. If $\bar{0}$ denotes the closure of zero, then $KK(A,B)/\,\bar{0}$ is a polish group.*

PROOF. By a result of Thomsen [35, Thm. 3.2], every element of $KK(A,B)$ is represented by a Cuntz pair (α,γ), where $\gamma : A \to \mathcal{L}(H_B)$ is a fixed absorbing $*$-homomorphism. Therefore the image of each map α is contained in the separable C*-algebra $\gamma(A) + \mathcal{K}(H_B)$. This shows that the topology of $KK(A,B)$ satisfies the second axiom of countability.

Next we prove the completeness of $KK(A,B)$. Let (x_n) be a Cauchy sequence in $\mathcal{E}_c(A,B)$ where $x_n = (\alpha_n,\gamma)$ with $\gamma : A \to \mathcal{L}(H_B)$ as above. This means that $d(x_n,x_m) = \delta(\alpha_n \oplus \gamma, \gamma \oplus \alpha_m) \to 0$ as $m,n \to \infty$. Since $\delta(\alpha_m \oplus \gamma, \gamma \oplus \alpha_m) = d(x_m,x_m) = 0$, we have $\delta(\alpha_n \oplus \gamma, \alpha_m \oplus \gamma) \to 0$ as $m,n \to \infty$. Since $[\alpha_n,\gamma] = [\alpha_n \oplus \gamma, \gamma \oplus \gamma]$ in $KK(A,B)$, after replacing α_n by $\alpha_n \oplus \gamma$, we may assume that $\delta(\alpha_n,\alpha_m) \to 0$ as $m,n \to \infty$. After passing to a subsequence of (α_n), if necessary, we find a sequence of unitaries $u_n \in 1 + \mathcal{K}(H_B)$ such that $\delta_0(\alpha_n, u_{n+1}\alpha_{n+1}u_{n+1}^*) < 1/2^n$. Define $\alpha_n'(a) = (u_2 \cdots u_n)\alpha_n(a)(u_2 \cdots u_n)^*$ and note that (α_n') is a Cauchy sequence in $\mathrm{Hom}(A,\mathcal{L}(H_B))$ since $\delta_0(\alpha_n',\alpha_{n+1}') < 1/2^n$. Since $\mathrm{Hom}(A,\mathcal{L}(H_B))$ is complete, (α_n') converges to a $*$-homomorphism α with the property that $\alpha(a) - \gamma(a) \in \mathcal{K}(H_B)$ since $\alpha_n'(a) - \gamma(a) \in \mathcal{K}(H_B)$ for all $a \in A$. It follows that $[\alpha_n,\gamma] = [\alpha_n',\gamma]$ converges to $[\alpha,\gamma]$ in $KK(A,B)$. □

Proposition 2.8 does not follow from [31] since we do not assume A to be K-nuclear and we are working a priori with a different topology.

3 Approximate unitary equivalence and the topology of $KK(A,B)$

In this section we show that the approximate unitary equivalence of Cuntz pairs can be expressed in KK-theoretical terms, see Theorem 3.3. Consequently, the topology of $KK(A,B)$ defined by d coincides with Pimsner's topology, see Theorem 3.5. In the final part we apply these results to $*$-homomorphisms.

Let $\bar{\mathbb{N}} = \{1,2,\dots\} \cup \{\infty\}$ denote the one-point compactification of the natural numbers. We say that a topology on the KK-theory groups satisfies Pimsner's condition if the convergence of sequences is characterized as follows. A sequence (x_n) in $KK(A,B)$ converges to x_∞ if and only if there is $y \in KK(A, C(\bar{\mathbb{N}}) \otimes B)$ with $y(n) = x_n$ for $n \in \mathbb{N}$ and $y(\infty) = x_\infty$. Clearly a topology which satisfies the first axiom of countability and Pimsner's condition is unique. Pimsner made the following crucial observation.

Lemma 3.1 *[22] If a topology on the KK-groups satisfies the first axiom of countability and Pimsner's condition, then the Kasparov product is jointly continuous with respect to that topology.*

PROOF. By the functoriality of the cup product of Kasparov [16, 2.14], if $y \in KK(A, C(\bar{\mathbb{N}}) \otimes B)$ and $z \in KK(B, C(\bar{\mathbb{N}}) \otimes C)$, then the image $w \in KK(A, C(\bar{\mathbb{N}}) \otimes C)$ of the cup product $y \otimes_B z \in KK(A, C(\bar{\mathbb{N}} \times \bar{\mathbb{N}}) \otimes C)$ under the diagonal map satisfies $w(n) = z(n) \otimes y(n)$ for all $n \in \bar{\mathbb{N}}$. $\square$

We need some notation. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. If $\varphi : A \to \mathcal{L}_B(E)$ and $\psi : A \to \mathcal{L}_B(F)$ are two contractive completely positive maps, we write $\varphi \underset{\mathcal{F},\varepsilon}{\prec} \psi$ if there is an isometry $v \in \mathcal{L}_B(E,F)$ such that $\|\varphi(a) - v^*\psi(a)v\| < \varepsilon$ for all $a \in \mathcal{F}$. If v can be taken to be a unitary then we write $\varphi \underset{\mathcal{F},\varepsilon}{\sim} \psi$. We write $\varphi \prec \psi$ (respectively $\varphi \sim \psi$) if $\varphi \underset{\mathcal{F},\varepsilon}{\prec} \psi$ (respectively $\varphi \underset{\mathcal{F},\varepsilon}{\sim} \psi$) for all finite sets $\mathcal{F}$ and $\varepsilon > 0$. Note that if $\varphi \underset{\mathcal{F},\varepsilon_1}{\prec} \psi$ and $\psi \underset{\mathcal{F},\varepsilon_2}{\prec} \gamma$, then $\varphi \underset{\mathcal{F},\varepsilon_1+\varepsilon_2}{\prec} \gamma$.

Proposition 3.2 *Let A, B, C be separable C^*-algebras such that B stable and C is unital and nuclear. If $\gamma : A \to M(B)$ is an absorbing $*$-homomorphism for (A,B), then $\Gamma : A \to M(B \otimes C)$, $\Gamma(a) = \gamma(a) \otimes 1_C$, is an absorbing $*$-homomorphism for $(A, B \otimes C)$.*

PROOF. If $B = \mathcal{K}$ this is essentially Kasparov's absorption theorem [15]. By [10, Thm. 2.13] it suffices to prove that for any finite subset $\mathcal{F} \subset A$, any $\varepsilon > 0$ and any completely positive contraction $\sigma : A \to B \otimes C$ we have $\sigma \underset{\mathcal{F},\varepsilon}{\prec} \Gamma$. Since γ is an absorbing $*$-homomorphism for (A, B), we have $\Phi \prec \gamma$ and hence $\Phi \otimes 1_C \prec \gamma \otimes 1_C = \Gamma$ for any completely positive contraction $\Phi : A \to B$. Therefore it is enough to show that $\sigma \underset{\mathcal{F},\varepsilon}{\prec} \Phi \otimes 1_C$ for some completely positive contraction $\Phi : A \to B$. Since C is nuclear, as a consequence of Kasparov's theorem, $id_C \prec \delta \otimes 1_C$ where $\delta : C \to \mathcal{L}(H)$ is a unital faithful representation with $\delta(C) \cap \mathcal{K}(H) = \{0\}$. Therefore there is sequence of isometries $v_n \in \mathcal{L}_C(C, H_C)$ with

$$\lim_{n\to\infty} \|c - v_n^*(\delta(c) \otimes 1_C)v_n\| = 0$$

for all $c \in C$. Since H_C is the closure of $\oplus_{n=1}^{\infty} C$ one can perturb each v_n to a C-linear isometry $v_n : C \to C^{k(n)} \subset H_C$. Therefore if $\delta_n : C \to M_{k(n)}(\mathbb{C})$ denotes the completely positive contraction obtained by compressing δ to the subspace $\mathbb{C}^{k(n)}$ of H, we have

$$\lim_{n\to\infty} \|c - v_n^*(\delta_n(c) \otimes 1_C)v_n\| = 0$$

for all $c \in C$. If we set $V_n = \mathrm{id}_B \otimes v_n \in \mathcal{L}_{B\otimes C}(B \otimes C, (B \otimes C)^{k(n)})$ and $\Delta_n = \mathrm{id}_B \otimes \delta_n : B \otimes C \to B \otimes M_{k(n)}(\mathbb{C})$, then

$$\lim_{n\to\infty} \|x - V_n^*(\Delta_n(x) \otimes 1_C)V_n\| = 0$$

for all $x \in B \otimes C$. Consequently

$$\lim_{n\to\infty} \|\sigma(a) - V_n^*(\Delta_n(\sigma(a)) \otimes 1_C)V_n\| = 0 \tag{5}$$

for all $a \in A$. Note that $\Phi_n = \Delta_n\sigma : A \to M_{k(n)}(B) \cong B$ is a completely positive contraction. From (5) we see that $\sigma \underset{\mathcal{F},\varepsilon}{\prec} \Phi_n \otimes 1_C$ for some large enough n and this concludes the proof. $\square$

Theorem 3.3 *Let A, B be separable C^*-algebras and let $(\varphi_n, \psi_n)_{n\in\mathbb{N}}$ be a sequence of Cuntz pairs in $\mathcal{E}_c(A, B)$. The following are equivalent:*

(i) There is $y \in KK(A, C(\bar{\mathbb{N}}) \otimes B)$ such that $y(n) = [\varphi_n, \psi_n]$ for $n \in \mathbb{N}$ and $y(\infty) = 0$.

(ii) For any absorbing $*$-homomorphism $\gamma : A \to \mathcal{L}(H_B)$ there is a sequence of unitaries $u_n \in 1 + \mathcal{K}(H_B \oplus H_B)$ such that for all $a \in A$

$$\lim_{n\to\infty} \|u_n (\varphi_n(a) \oplus \gamma(a))\, u_n^* - \psi_n(a) \oplus \gamma(a)\| = 0 \tag{6}$$

(iii) The sequence $[\varphi_n, \psi_n]$ converges to zero in $(KK(A, B), d)$.

Remark 3.4 *It is easy to verify that condition (ii) is equivalent to asking that there is some* $*$*-homomorphism* $\gamma : A \to \mathcal{L}(H_B)$ *and there is a sequence of unitaries* $u_n \in I + \mathcal{K}(H_B \oplus H_B)$ *satisfying (6) for all* $a \in A$*. This is very similar to the proof of (ii)* $\Leftrightarrow$ *(iii) of Theorem 2.1.*

PROOF. Given two sequence of $*$-homomorphisms $\varphi_n, \psi_n : A \to \mathcal{L}(E_n)$, we write $(\varphi_n)_n \approx (\psi_n)_n$ if there is a sequence of unitaries $u_n \in 1 + \mathcal{K}(E_n)$ such that

$$\lim_{n\to\infty} \|u_n \varphi_n(a) u_n^* - \psi_n(a)\| = 0$$

for all $a \in A$. With this notation, the condition (6) reads $(\varphi_n \oplus \gamma)_n \approx (\psi_n \oplus \gamma)_n$. It is easy to verify that $\approx$ is an equivalence relation and that $(\varphi_n \oplus \varphi_n')_n \approx (\psi_n \oplus \psi_n')_n$ whenever $(\varphi_n)_n \approx (\psi_n)_n$ and $(\varphi_n')_n \approx (\psi_n')_n$.

We identify $\mathcal{L}(H_B)$ with $M(\mathcal{K} \otimes B)$ and $\mathcal{K}(H_B)$ with $\mathcal{K} \otimes B$. Therefore the set $\mathcal{E}_c(A, B)$ consists of pairs of $*$-homomorphisms $(\varphi, \psi) : A \to M(\mathcal{K} \otimes B)$ such that $\varphi(a) - \psi(a) \in \mathcal{K} \otimes B$ for all $a \in A$. Since $M(\mathcal{K} \otimes B \otimes C(\bar{\mathbb{N}})) \equiv C_s(\bar{\mathbb{N}}, M(\mathcal{K} \otimes B))$ (the set of strictly continuous functions from $\bar{\mathbb{N}}$ to $M(\mathcal{K} \otimes B)$) and $\mathcal{K} \otimes B \otimes C(\bar{\mathbb{N}}) = C(\bar{\mathbb{N}}, \mathcal{K} \otimes B)$, an element $(\Delta, \Gamma) \in \mathcal{E}_c(A, B \otimes C(\bar{\mathbb{N}}))$ is completely determined by a family $(\delta_n, \gamma_n)_{n \in \bar{\mathbb{N}}} \subset \mathcal{E}_c(A, B)$ such that

$$\lim_{n\to\infty} \delta_n(a) = \delta_\infty(a), \quad \lim_{n\to\infty} \gamma_n(a) = \gamma_\infty(a) \tag{7}$$

in the strict topology of $M(\mathcal{K} \otimes B)$ and such that

$$\lim_{n\to\infty} (\delta_n(a) - \gamma_n(a)) = \delta_\infty(a) - \gamma_\infty(a)$$

in the norm topology, for all $a \in A$. By [35, Thm. 3.2], each element of $KK(A, B)$ is represented by a pair $(\delta, \gamma) \in \mathcal{E}_c(A, B)$ where $\gamma : A \to M(\mathcal{K} \otimes B)$ is any given absorbing $*$-homomorphism. In view of Proposition 3.2, if $y \in KK(A, B \otimes C(\bar{\mathbb{N}}))$, then we can write $y = [\Delta, \Gamma]$ where $\Gamma = \gamma \otimes 1_{C(\bar{\mathbb{N}})}$ and $\gamma : A \to M(\mathcal{K} \otimes B)$ is a fixed absorbing $*$-homomorphism for (A, B). In other words Γ is given by a constant family $(\gamma_n)_{n \in \bar{\mathbb{N}}}$ with $\gamma_n = \gamma$. A crucial consequence of our choice of Γ is that $\delta_n(a) - \delta_\infty(a) \in \mathcal{K} \otimes B$ for all $a \in A$, since it is equal to $(\delta_n(a) - \gamma(a)) - (\delta_\infty(a) - \gamma(a))$ and therefore

$$\lim_{n\to\infty} \|\delta_n(a) - \delta_\infty(a)\| = 0 \tag{8}$$

for all $a \in A$. Therefore we are able to pass from strict convergence in (7) to norm convergence in (8). After this preliminary discussion we proceed with the proof of the theorem. The equivalence (ii)$\Leftrightarrow$ (iii) follows immediately from the definition of d and the separability of A.

(i) $\Rightarrow$ (ii) It is convenient to consider first the situation when (φ_n, ψ_n) is a sequence of Cuntz pairs where the second component ψ_n is fixed for all n and equal to some absorbing $*$-homomorphism γ as above. By assumption there is

$y \in KK(A, B \otimes C(\bar{\mathbb{N}}))$ such that $y(n) = [\varphi_n, \gamma]$ and $y(\infty) = 0$. Write $y = [\Delta, \Gamma]$ as above. Therefore $[\delta_n, \gamma] = [\varphi_n, \gamma]$ hence $[\delta_n, \varphi_n] = 0$ and $[\delta_\infty, \gamma] = 0$. Using Theorem 2.1 we obtain

$$(\delta_n \oplus \gamma)_n \approx (\varphi_n \oplus \gamma)_n, \quad (\delta_\infty \oplus \gamma)_n \approx (\gamma \oplus \gamma)_n.$$

In view of (8) this gives

$$(\varphi_n \oplus \gamma)_n \approx (\gamma \oplus \gamma)_n. \tag{9}$$

We now proceed with the general case with (φ_n, ψ_n) as in (i). Using [35, Thm. 3.2] again, we find a sequence $(\gamma_n, \gamma) \in \mathcal{E}_c(A, B)$ with $[\gamma_n, \gamma] = [\varphi_n, \psi_n]$ and γ absorbing. Since $[\varphi_n \oplus \gamma, \psi_n \oplus \gamma_n] = 0$, by Theorem 2.1 we obtain $(\varphi_n \oplus \gamma \oplus \gamma)_n \approx (\psi_n \oplus \gamma_n \oplus \gamma)_n$. By the first part of the proof, we have $(\gamma_n \oplus \gamma)_n \approx (\gamma \oplus \gamma)_n$. Altogether this gives $(\varphi_n \oplus \gamma \oplus \gamma)_n \approx (\psi_n \oplus \gamma \oplus \gamma)_n$. Since γ is absorbing, $(\gamma \oplus \gamma)_n \approx (\gamma)_n$ hence we obtain (ii): $(\varphi_n \oplus \gamma)_n \approx (\psi_n \oplus \gamma)_n$.

(ii) $\Rightarrow$ (i) Replacing φ_n by $\varphi_n \oplus \gamma$ and ψ_n by $\psi_n \oplus \gamma$ we may assume that there are unitaries $u_n \in I + \mathcal{K}(H_B)$ such that $\lim_{n\to\infty} \|u_n \varphi_n(a) u_n^* - \psi_n(a)\| = 0$ for all $a \in A$. Since $(u_n \varphi_n u_n^*, \psi_n)$ and (φ_n, ψ_n) have the same KK-class, after replacing φ_n by $u_n \varphi_n u_n^*$ we may further assume that $\lim_{n\to\infty} \|\varphi_n(a) - \psi_n(a)\| = 0$. Since both φ_n and γ are absorbing, there is a sequence of unitaries $w_n \in \mathcal{L}(H_B)$ such that $w_n \varphi_n(a) w_n^* - \gamma(a) \in \mathcal{K}(H_B)$ and $\lim_{n\to\infty} \|w_n \varphi_n(a) w_n^* - \gamma(a)\| = 0$. Define $*$-homomorphisms $\Phi, \Psi : A \to M(\mathcal{K} \otimes B \otimes C(\bar{\mathbb{N}}))$ by setting $\Phi_n = w_n \varphi_n w_n^*$, $\Phi_\infty = \gamma$, $\Psi_n = w_n \psi_n w_n^*$ and $\Psi_\infty = \gamma$. The family $(\Phi, \Psi) = (\Phi_n, \Psi_n)_{n \in \bar{\mathbb{N}}}$ defines an element y of $KK(A, C(\bar{\mathbb{N}}) \otimes B)$ such that $y(n) = [\Phi_n, \Psi_n] = [\varphi_n, \psi_n]$ for $n \in \mathbb{N}$ and $y(\infty) = [\Phi_\infty, \Psi_\infty] = [\gamma, \gamma] = 0$. $\square$

We collect the previous results of the section in the following form.

Theorem 3.5 *Let A be B be separable C^*-algebras. The topology of $KK(A, B)$ defined by the pseudometric d is separable and complete. A sequence $(x_n)_{n=1}^\infty$ converges to x_∞ in $KK(A, B)$ if and only if there is $y \in KK(A, C(\bar{\mathbb{N}}) \otimes B)$ with $y(n) = x_n$ for all $n \in \bar{\mathbb{N}}$. Therefore the topology defined by d satisfies Pimsner's condition and hence the Kasparov product is continuous. The topology defined by d coincides with Pimsner's topology.*

PROOF. The first part follows from Proposition 2.8 and Theorem 3.3. The second part follows from Lemma 3.1. $\square$

Let us see how the previous results can be applied to $*$-homomorphisms. The definition of stable approximate unitary equivalence for two $*$-homomorphisms $\varphi, \psi : A \to B$ is not quite straightforward. A naive definition that would require approximate unitary equivalence after taking direct sums with $*$-homomorphisms would not be satisfactory, due to a possible small supply of $*$-homomorphisms from A to B.

Definition 3.6 *Let A, B be separable C*-algebras. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are called stably approximately unitarily equivalent if there is a sequence of unitaries $v_n \in 1 + \mathcal{K}(B \oplus H_B)$ and an absorbing $*$-homomorphism $\gamma : A \to M(B \otimes \mathcal{K})$ such that for all $a \in A$*

$$\lim_{n\to\infty} \| v_n \left(\varphi(a) \oplus \gamma(a)\right) v_n^* - \psi(a) \oplus \gamma(a) \| = 0 \tag{10}$$

From Theorem 3.3 we obtain:

Corollary 3.7 *Let A, B be separable C*-algebras. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are stably approximately unitarily equivalent if and only if $[\varphi] - [\psi] \in \bar{0}$ in $KK(A, B)$, if and only if $d([\varphi], [\psi]) = 0$.*

This result becomes more useful when there are many $*$-homomorphisms from A to B or matrices over B. For illustration, we generalize [7, Thm. 5.1] and [20, Thm. 3.9]. Let A and B be unital separable C*-algebras such that either A or B is nuclear. Assume that there is a sequence of unital $*$-homomorphisms $\eta_n : A \to M_{k(n)}(B)$ such that for all nonzero $a \in A$ the closed two-sided ideal of $B \otimes \mathcal{K}$ generated by $\{\eta_n(a) : n \in \mathbb{N}\}$ is equal to $B \otimes \mathcal{K}$. We will also assume that each η_n appears infinitely many times in the sequence (η_n).

Corollary 3.8 *Let A and B be unital separable C*-algebras such that either A or B is nuclear. Assume that (η_n) is as above and let φ, ψ be two unital $*$-homomorphisms from A to B. Then $[\varphi] - [\psi] \in \bar{0}$ if and only if there exist a sequence on integers $(m(n))$ and unitaries (u_n) in matrices over B (of appropriate size) such that*

$$\lim_{n\to\infty} \| u_n(\varphi(a) \oplus \gamma_n(a)) u_n^* - \psi(a) \oplus \gamma_n(a) \| = 0 \tag{11}$$

for all $a \in A$, where $\gamma_n = \eta_1 \oplus \cdots \oplus \eta_{m(n)}$.

PROOF. We verify only the nontrivial implication ($\Rightarrow$). To simplify notation, we give the proof in the case when all $k(n) = 1$, i.e. $\eta_n : A \to B$. The condition $[\varphi] - [\psi] \in \bar{0}$ is equivalent to the condition that 0 belongs to the closure of $[\varphi] - [\psi]$. If $\gamma : A \to M(B \otimes \mathcal{K})$ is defined by

$$\gamma(a) = \mathrm{diag}(\eta_1(a), \eta_2(a), \cdots),$$

then γ is a unitally absorbing representation by a result of [13]. If $\widehat{\gamma} : A \to \mathcal{L}(H_B \oplus H_B) \cong \mathcal{L}(H_B)$ is defined by $\widehat{\gamma} = \gamma \oplus 0$, then $\widehat{\gamma}$ is absorbing. By

Theorem 3.3 there is a sequence of unitaries $v_n \in 1 + \mathcal{K}(B \oplus H_B)$ such that for all $a \in A$

$$\lim_{n\to\infty} \|v_n (\varphi(a) \oplus \widehat{\gamma}(a))\, v_n^* - \psi(a) \oplus \widehat{\gamma}(a)\| = 0 \tag{12}$$

If $e_m = 1_B \oplus \cdots \oplus 1_B$ (m-times), then $\lim_{m\to\infty} \|[v_n, e_m]\| = 0$ for all m. For each n let $m(n)$ be such that $\|[v_n, e_{m(n)}]\| < 1/n$. By functional calculus, there are unitaries $u_n \in M_{m(n)}(B)$ with $\lim_{n\to\infty} \|u_n - e_{m(n)} v_n e_{m(n)}\| = 0$. With these choices we derive (11) by compressing in (12) by $e_{m(n)}$. $\square$

The following result is derived by a similar argument.

Corollary 3.9 *Let A, B and (η_n) be as in Cor. 3.8 and let φ, ψ be two unital $*$-homomorphisms from A to B. For any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is $\delta > 0$ such that if $d([\varphi], [\psi]) < \delta$ then $\varphi \oplus \gamma_n \underset{\mathcal{F},\varepsilon}{\sim} \psi \oplus \gamma_n$ for some n.*

4 The UCT, K-theory with coefficients and applications

Let A and B be separable C*-algebras. The total K-theory group $\underline{K}(A) = \oplus_{n=0}^{\infty} K_*(A; \mathbb{Z}/n)$ has a natural action of the Bockstein operations Λ of [30]. In this section we show that if A satisfies the UCT, then $KK(A,B)/\bar{0}$ is isomorphic as a topological group with $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ endowed with the topology of pointwise convergence. This is extremely useful since in order to check that two KK-elements are close to each other it suffices to show that the maps they induce on $\underline{K}(-)$ agree on some (sufficiently large) finite subset. By a result of J.L. Tu [36], the C*-algebra of an a-T-menable locally compact second countable groupoid with Haar system satisfies the UCT. This shows that there are large natural classes of non-nuclear C*-algebras satisfying the UCT. As an application we show that the C*-algebra of a second countable amenable locally compact maximally almost periodic group embeds in the UHF algebra of type 2^∞.

If d_* is the metric on $\underline{K}(B)$ with $d_*(x,y) = 1$ for $x \neq y$, then $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ becomes a polish group with respect to the metric

$$d(\mu, \nu) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_*(\mu(x_n), \nu(x_n)),$$

where $\{x_1, x_2, \dots\}$ is an enumeration $\underline{K}(A)$.

A separable C*-algebra satisfies the UCT of [27] if and only if is KK-equivalent to a commutative C*-algebra, if and only it satisfies the following universal multi-coefficient exact sequence of [11]:

$$0 \to \mathrm{PExt}(K_{*-1}(A), K_*(B)) \to KK(A,B) \xrightarrow{\Gamma} \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \to 0. \tag{13}$$

Here PExt stands for the subgroup of Ext corresponding to pure extensions. We refer the reader to the monograph [33] for an excellent introduction to PExt. The map Γ is induced by the Kasparov product and therefore is continuous. This is also easily seen directly since if two projections are close to each other then they have the same K-theory class.

If $x \in KK(A,B)$ we denote $\Gamma(x)$ by $\underline{x}$. The following result can be deduced from [32] for nuclear C*-algebras A in the bootstrap category of [29], modulo the identification of Pimsner's topology with the Brown-Salinas topology. The idea of using the continuity of the Kasparov product in its proof is borrowed from [32].

Theorem 4.1 *Let A and B be separable C*-algebras and assume that A satisfies the UCT. Then*

(a) $x_n \to x$ in $KK(A,B)$ if and only if $\underline{x_n} \to \underline{x}$ in $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$.

(b) The map $KK(A,B)/\,\bar{0} \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ is an isomorphism of topological groups. In particular $KK(A,B)/\,\bar{0}$ is totally disconnected.

PROOF. Part (a) is an immediate consequence of (b). Since the Kasparov product is continuous, multiplication by a KK-invertible element $y \in KK(A, A')$ induces a commutative diagram

$$\begin{array}{ccc} KK(A',B) & \longrightarrow & \mathrm{Hom}_\Lambda(\underline{K}(A'), \underline{K}(B)) \\ \downarrow & & \downarrow \\ KK(A,B) & \longrightarrow & \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \end{array}$$

where the horizontal maps are continuous and the vertical maps are homeomorphisms. Therefore, after replacing A by a KK-equivalent C*-algebra (as in [32]), we may assume that A is the closure of an increasing sequence (A_n) of nuclear C*-subalgebras of A satisfying the UCT and with the property that each $K_*(A_n)$ is finitely generated. In particular the map $\Gamma_n : KK(A_n, B) \to \mathrm{Hom}_\Lambda(\underline{K}(A_n), \underline{K}(B))$ is an isomorphism by (13). By the open mapping theorem all we need to prove is that $\ker(\Gamma) = \bar{0}$. The inclusion $\ker(\Gamma) \supset \bar{0}$ follows from the continuity of Γ. Conversely let $[\alpha, \gamma] \in \ker(\Gamma)$ with γ absorbing. Let $\mathcal{F}_n \subset A_n$ be a finite subset such that the union of $(\mathcal{F}_n)$ is dense in A. Since the diagram

$$\begin{array}{ccc} KK(A,B) & \longrightarrow & \mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(B)) \\ \downarrow & & \downarrow \\ KK(A_n,B) & \longrightarrow & \mathrm{Hom}_\Lambda(\underline{K}(A_n),\underline{K}(B)) \end{array}$$

is commutative, we have that $[\alpha,\gamma]=0$ when regarded as an element of $KK(A_n,B)$. By Theorem 2.1 there is a unitary $u_n \in 1+\mathcal{K}(H_B\oplus H_B)$ such that for all $a_n \in \mathcal{F}_n$

$$\|u_n\,(\alpha(a)\oplus\gamma(a))\,u_n^* - \gamma(a)\oplus\gamma(a)\| < 1/n.$$

Therefore

$$\lim_{n\to\infty}\|u_n\,(\alpha(a)\oplus\gamma(a))\,u_n^* - \gamma(a)\oplus\gamma(a)\| = 0$$

for all $a\in A$, hence $d([\alpha,\gamma],0)=0$ and $[\alpha,\gamma]\in\bar{0}$. □

Proposition 4.2 *Let A and B be separable C*-algebras and assume that A satisfies the UCT and that the group $K_*(B)$ is finitely generated. Then for any subgroup G of $KK(A,B)$ and any $\varepsilon>0$ there is a finitely generated subgroup H of G which is ε-dense in G, i.e. for every $x\in G$ there is $y\in H$ such that $d(x,y)<\varepsilon$.*

PROOF. Let $U=\{z\in KK(A,B) : d(z,0)<\varepsilon\}$. Since the map

$$\Gamma : KK(A,B)\to \mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(B))$$

is open, there exists an integer $m\geq 0$ and $t_1,\dots,t_n\in\underline{K}(A)_m$ such that

$$\{\alpha\in\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(B)) : \alpha(t_1)=\cdots=\alpha(t_n)=0\}\subset\Gamma(U).$$

Here $\underline{K}(A)_m$ denotes the subgroup of $\underline{K}(A)$ generated by $K_*(A;\mathbb{Z}/k)$ with $k\leq m$. Let $\Gamma_n : G\to\prod_{i=1}^n\underline{K}(B)_m$ be defined by $\Gamma_n(x)=(\underline{x}(t_1),\dots,\underline{x}(t_n))$. Since $K_*(B)$ is abelian and finitely generated so is $\underline{K}(B)_m$ and its subgroup $\Gamma_n(G)$. Therefore there is a finitely generated subgroup H of G such that $\Gamma_n(G)=\Gamma_n(H)$. In particular for any $x\in G$ there is $y\in H$ such that $\underline{x}(t_i)=\underline{y}(t_i)$ for all i, $1\leq i\leq n$. Therefore $\underline{x}-\underline{y}\in\Gamma(U)$, hence $x-y\in U+\bar{0}$. We conclude that $d(x,y)=d(x-y,0)<\varepsilon$. □

Let us recall that a C*-algebra is called nuclearly embeddable if it has a faithful nuclear representation on a Hilbert space. Kirchberg proved that a separable C*-algebra is nuclearly embeddable if and only if is exact. A C*-algebra A is called residually finite dimensional (abbreviated RFD) if the finite dimensional representations of A separate the points of A. Using notation introduced before Proposition 3.2 we have:

Theorem 4.3 *Let A be a separable unital exact RFD C^*-algebra satisfying the UCT. For any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there are unital finite dimensional irreducible $*$-representations $\pi_1, \dots, \pi_r$ such that for any unital finite dimensional $*$-representation $\pi : A \to \mathcal{L}(H_\pi)$,*

$$\pi \oplus m_1\pi_1 \oplus \cdots \oplus m_r\pi_r \underset{\mathcal{F},\varepsilon}{\sim} k_1\pi_1 \oplus \cdots \oplus k_r\pi_r$$

for some nonnegative integers $m_1, \dots m_r, k_1, \dots k_r$.

PROOF. Let $\mathrm{fdr}(A)$ denote the set of unital finite dimensional $*$-representations of A. If $\pi \in \mathrm{fdr}(A)$, we denote by $[\pi]$ its class in $KK(A, \mathbb{C})$. From the definition of the metric d we derive the following observation. Given $\mathcal{F}$ and ε as in the statement, there is $\varepsilon_0 > 0$ such that if π and π' are unital finite dimensional $*$-representations of A on the same space H_π with $d([\pi], [\pi']) < \varepsilon_0$ then for any unitally absorbing $*$-homomorphism $\gamma : A \to \mathcal{L}(H)$ there is a unitary $u \in 1 + \mathcal{K}(H_\pi \oplus H)$ such that

$$\|\pi(a) \oplus \gamma(a) - u(\pi'(a) \oplus \gamma(a))u^*\| < \varepsilon$$

for all $a \in \mathcal{F}$. Since A is separable there is a sequence $(\pi_n)_{n=1}^\infty$ in $\mathrm{fdr}(A)$ whose unitary orbit is dense in $\mathrm{fdr}(A)$ in the point-norm topology. This means that for any $\pi \in \mathrm{fdr}(A)$, any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$, $\pi \underset{\mathcal{F},\varepsilon}{\sim} \pi_n$ for some n. Consequently it suffices to prove the theorem only for representations π that appear in the sequence $(\pi_n)_{n=1}^\infty$. We may assume that each π_n is repeating infinitely many times. Let G be the subgroup of $KK(A, \mathbb{C})$ generated by the set $\{[\pi_n] : n \geq 1\}$. By Proposition 4.2 there is a finitely generated subgroup H of G that is ε_0-dense in G. Therefore there is r such that H is generated by $[\pi_1], \dots, [\pi_r]$. Fix a unitally absorbing $*$-homomorphism $\gamma : A \to \mathcal{L}(H)$. Since A is nuclearly embeddable, by enlarging r, we may arrange that

$$\gamma \underset{\mathcal{F},\varepsilon}{\sim} \infty \cdot (\pi_1 \oplus \cdots \oplus \pi_r) \tag{14}$$

by an approximation result of [6]; see also [8, Prop. 6.1] for a more direct proof. Let π be as in the statement of the theorem. We may assume that π appears in the sequence $(\pi_n)_{n=1}^\infty$ and therefore its K-homology class $[\pi]$ belongs to G. It follows that there is $h \in H$ with $d([\pi], h) < \varepsilon_0$. Thus there are positive integers $m_1, \dots m_r, k_1, \dots k_r$ such that

$$d([\pi \oplus m_1\pi_1 \oplus \cdots \oplus m_r\pi_r], [k_1\pi_1 \oplus \cdots \oplus k_r\pi_r]) < \varepsilon_0.$$

By our choice of ε_0 this implies that there is a unitary u of the form 1+*compact* such that

$$\|\pi(a) \oplus m_1\pi_1(a) \oplus \cdots \oplus m_r\pi_r(a) \oplus \gamma(a) - u(k_1\pi_1(a) \oplus \cdots \oplus k_r\pi_r(a) \oplus \gamma(a))u^*\| < \varepsilon$$

for all $a \in \mathcal{F}$. Using (14) and compressing by a suitable finite dimensional projection e we obtain that there exist a positive integer N and a unitary v close to eue such that, if $M_i = m_i + N$ and $K_i = k_i + N$, then

$$\|\pi(a) \oplus M_1\pi_1(a) \oplus \cdots \oplus M_r\pi_r(a) - v(K_1\pi_1(a) \oplus \cdots \oplus K_r\pi_r(a))v^*\| < 3\varepsilon$$

This concludes the proof. □

If A is unital, the subgroup of $K_0(\mathbb{C}) = \mathbb{Z}$ generated by $\{[\pi(1_A)] : \pi \in \mathrm{fdr}(A)\}$ is isomorphic to $d\mathbb{Z}$ for some integer $d \geq 1$. The number d is a topological invariant of A and is denoted by $d(A)$.

Theorem 4.4 *Let A be a separable exact RFD C*-algebra satisfying the UCT. Then A embeds in the UHF C*-algebra of type 2^∞ denoted by B. If A is unital then it embeds as a unital C*-subalgebra in $M_{d(A)}(B)$.*

PROOF. By adding a unit to A (whether or not A has already a unit) we have $d(\tilde{A}) = 1$. Thus it suffices to prove only the second part of the theorem. Let $(\mathcal{F}_n)_{n=1}^\infty$ be an increasing sequence of finite subsets of A whose union is dense in A and let $\varepsilon_n = 1/2^n$. By Theorem 4.3 there exist a sequence $(\pi_n)_{n=1}^\infty$ in $\mathrm{fdr}(A)$ and integers $0 < r(1) < r(2) < \cdots < r(n) < \ldots$, such that if $\mathcal{R}_n \subset \mathrm{fdr}(A)$ consists of all unital representations unitarily equivalent to representations of the form $k_1\pi_1 \oplus \cdots \oplus k_{r(n)}\pi_{r(n)}$ with $k_i > 0$, then for any $\pi \in \mathrm{fdr}(A)$ there are $\alpha, \beta \in \mathcal{R}_n$ with $\pi \oplus \alpha \underset{\mathcal{F}_n\varepsilon_n}{\sim} \beta$. After changing notation if necessary, we may assume that there is $\gamma_1 \in \mathcal{R}_1$, $\gamma_1 : A \to M_{k(1)}(\mathbb{C})$ such that $k(1) = 2^m d(A)$ for some positive integer m. We will construct inductively a sequence of unital $*$-homomorphisms $\gamma_n : A \to M_{k(n)}(\mathbb{C})$ with $\gamma_n \in \mathcal{R}_n$ and such that $\|\gamma_{n+1}(a) - m(n)\gamma_n(a)\| < \varepsilon_n$ for all $a \in \mathcal{F}_n$, where $m(n)$ is some power of 2 and $k(n+1) = m(n)k(n)$. Note that γ_n will satisfy $\lim_{n\to\infty} \|\gamma_n(a)\| = \|a\|$ for all $a \in A$ since the sequence $(\pi_n)_{n=1}^\infty$ separates the elements of A. Suppose that $\gamma_1, \ldots \gamma_n$ were constructed. Pick some $\pi \in \mathcal{R}_{n+1}$. Then $\pi \oplus \alpha \underset{\mathcal{F}_n\varepsilon_n}{\sim} \beta$ for some $\alpha, \beta \in \mathcal{R}_n$. Since $\gamma_n \in \mathcal{R}_n$, there exists a power of 2 denoted by $m(n)$ and $\beta' \in \mathcal{R}_n$ such that $\beta \oplus \beta'$ is unitarily equivalent to $m(n)\gamma_n$ hence $\pi \oplus \alpha \oplus \beta' \underset{\mathcal{F}_n\varepsilon_n}{\sim} m(n)\gamma_n$. It follows that there is a finite dimensional unitary u such that $\|u(\pi \oplus \alpha \oplus \beta')(a)u^* - m(n)\gamma_n(a)\| < \varepsilon_n$ for all $a \in \mathcal{F}_n$. Setting $\gamma_{n+1} = u(\pi \oplus \alpha \oplus \beta')u^*$ we complete the induction process.

Let $\iota_n : M_{k(n)}(\mathbb{C}) \hookrightarrow \varinjlim M_{k(n)}(\mathbb{C}) \cong M_{d(A)}(B)$ be the canonical inclusion. Having the sequence γ_n available, we construct a unital embedding $\gamma : A \to M_{d(A)}(B)$ by defining $\gamma(a)$, $a \in \cup_{n=1}^\infty \mathcal{F}_n$, to be the limit of the Cauchy sequence $(\iota_n\gamma_n(a))$ and then extend to A by continuity. □

Remark 4.5 *The AF-embeddability of a separable nuclear RFD C*-algebra satisfying the UCT was proved in [19]. The approximation property given by Theorem 4.3 is a stronger property than UHF-embeddability. It is significant that it holds for exact C*-algebras since as noted in [8] the UHF-embeddability of the cone of an exact separable RFD C*-algebra (which satisfies the UCT by virtue of being contractible) implies Kirchberg's fundamental characterization of exact separable C*-algebras as subquotients of UHF algebras [17]. Subsequently Ozawa proved that AF-embeddability of separable exact C*-algebras is a homotopy invariant [21].*

A locally compact group G is called maximally almost periodic (abbreviated MAP) if it has a separating family of finite dimensional unitary representations. Residually finite groups are examples of MAP groups. If G is a second countable amenable locally compact MAP group, then $C^*(G)$ is residually finite dimensional by [2] and satisfies the UCT by [36]. By Theorem 4.4 we have the following.

Corollary 4.6 *The C*-algebra of a second countable amenable locally compact MAP group G is embeddable in the UHF C*-algebra of type 2^∞.*

Remark 4.7 *If in addition we assume that G is discrete, then G injects in the unitary group of B. Note that this result is non-trivial even for the discrete Heisenberg group $\mathbb{H}_3$, since $\mathbb{H}_3$ does not have injective finite dimensional unitary representations. Indeed if $\pi : \mathbb{H}_3 \to U(n)$ is an irreducible representation and s, t are generators of $\mathbb{H}_3$ such that $r = s^{-1}t^{-1}st$ generates the center of $\mathbb{H}_3$, then $\pi(r) = \lambda 1_n$, $\lambda \in \mathbb{C}$, hence $\lambda^n = \det(\pi(r)) = \det(\pi(s^{-1}t^{-1}st)) = 1$.*

5 From KL-equivalence to KK-equivalence

In this section we address the question of when the Hausdorff quotient of $KK(A, B)$ admits an algebraic description. The following definition due to H. Lin appears in [20], except that the topology considered there is the Brown-Salinas topology, which we will show to coincide with Pimsner's topology in the next section. A separable C*-algebra A satisfies the AUCT if the natural map

$$\frac{KK(A,B)}{\bar{0}} \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$$

is a bijection for all separable C*-algebras B.

Let $KL(A, B)$ denote the quotient group $KK(A, B)/\bar{0}$. Since the Kasparov product is continuous, it descends to an associative product $KL(A, B) \times KL(B, C) \to KL(A, C)$. The group $KL(A, B)$ was first introduced by Rørdam [24] as the quotient of $KK(A, B)$ by $\mathrm{PExt}(K_{*-1}(A), K_*(B))$. The assumption that A satisfies the UCT was necessary in order to make $\mathrm{PExt}(K_{*-1}(A), K_*(B))$ a subgroup of $KK(A,B)$ via the inclusion

$$\mathrm{PExt}(K_{*-1}(A), K_*(B)) \hookrightarrow \mathrm{Ext}(K_{*-1}(A), K_*(B)) \hookrightarrow K(A,B).$$

In Section 4 we showed that if A satisfies the UCT then $\mathrm{PExt}(K_{*-1}(A), K_*(B))$ coincides with the closure of zero, hence the terminology is consistent. Two separable C*-algebras A and B are *KK-equivalent*, written $A \sim_{KK} B$, if there exist $\alpha \in KK(A,B)$ and $\beta \in KK(B,A)$ such that

$$\alpha\beta = [\mathrm{id}_A], \quad \beta\alpha = [\mathrm{id}_B].$$

Similarly, A is *KL-equivalent* to B, written $A \sim_{KL} B$ if there exist $\alpha \in KK(A,B)$ and $\beta \in KK(B,A)$ such that

$$\alpha\beta - [\mathrm{id}_A] \in \bar{0}, \quad \beta\alpha - [\mathrm{id}_B] \in \bar{0}.$$

Equivalently, $A \sim_{KL} B$ if and only if there exist $\alpha \in KL(A,B)$ and $\beta \in KL(B,A)$ such that

$$\alpha\beta = [\mathrm{id}_A], \quad \beta\alpha = [\mathrm{id}_B].$$

by $KL(A,B)$. Note that KL-equivalence corresponds to the notion of isomorphism in the category with objects separable C*-algebras and morphisms from A to B given by $KL(A,B)$.

A separable simple unital purely infinite nuclear C*-algebra A is called a Kirchberg C*-algebra [26, 4.3.1]. One says that A is in standard form if $[1_A] = 0$ in $K_0(A)$. The following result is due to H. Lin, except that he works with the Brown-Salinas topology.

Theorem 5.1 ([20]) *Let A and B be unital Kirchberg C*-algebras.*

(a) Let $\varphi, \psi : A \to B$ be unital $*$-homomorphisms. If $[\varphi] = [\psi]$ in $KL(A,B)$ then $\varphi \approx_u \psi$.

(b) Assume that A and B are in standard form. If $A \sim_{KL} B$ then A is isomorphic to B.

PROOF. We include a new simple proof. (a) Since the constant sequence $[\psi]$ converges to $[\varphi]$ there is $y \in KK(A, C(\bar{\mathbb{N}}) \otimes B)$ such that $y(n) = [\psi]$ for $n \in \mathbb{N}$ and $y(\infty) = [\varphi]$. Since $B \cong B \otimes \mathcal{O}_\infty$ by Kirchberg's theorem [26, 7.2.6], it follows by Phillips' classification theorem [26, Thm. 8.2.1] and by [26, Prop. 4.1.4] that there is a unital $*$-homomorphism $\Psi : A \to C(\bar{\mathbb{N}}) \otimes B$ with $y = [\Psi]$. Note that Ψ is given by a family of $*$-homomorphisms, $\Psi = (\psi_n)_{n \in \bar{\mathbb{N}}}$ satisfying

$$\lim_{n\to\infty} \|\psi_n(a) - \psi_\infty(a)\| = 0 \tag{15}$$

for all $a \in A$. Since $[\psi_n] = y(n) = [\psi]$ it follows from [26, Thm. 8.2.1] that $\psi_n \approx_u \psi$ for all $n \in \mathbb{N}$ and similarly $\psi_\infty \approx_u \varphi$ since $[\psi_\infty] = y_\infty = [\varphi]$. In combination with (15) this gives $\varphi \approx_u \psi$. The converse follows from Theorem 3.3.

(b) Let α and β be as in the definition of KL-equivalence. Applying [26, Thm. 8.3.3] again we lift α and β to unital $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $[\varphi\psi] - [\mathrm{id}_B] \in \bar{0}$ and $[\psi\varphi] - [\mathrm{id}_A] \in \bar{0}$. From part (a) we have $\varphi\psi \approx_u \mathrm{id}_B$ and $\psi\varphi \approx_u \mathrm{id}_A$. It follows that A is isomorphic to B by Elliott's intertwining argument [26, 2.3.4]. □

Corollary 5.2 *Two separable nuclear C*-algebras are KK-equivalent if and only if they are KL-equivalent.*

PROOF. Any separable nuclear C*-algebra is KK-equivalent to a unital Kirchberg algebra in standard form [26, Prop. 8.4.5]. We conclude the proof by applying Theorem 5.1. □

It is known that the validity of UCT for all nuclear separable C*-algebras is equivalent to the statement that $KK(A, A) = 0$ for all nuclear separable C*-algebras A with $K_*(A) = 0$ (see [27] and [34, Prop. 5.3]). The following answers an informal question of Larry Brown and shows that if A fails to satisfy the UCT then $KK(A, A)/\bar{0} \neq 0$.

Corollary 5.3 *Let A be a separable nuclear C*-algebra. If $KK(A, A) = \bar{0}$ then A satisfies the UCT and in fact $A \sim_{KK} 0$.*

Next we show that a nuclear separable C*-algebra satisfies the AUCT if and only if it satisfies the UCT. This answers a question of H. Lin [20].

Theorem 5.4 *Let A be a separable nuclear C*-algebra. The following assertions are equivalent.*

(i) A satisfies the UCT.

(ii) A satisfies the AUCT.

(iii) A is KL-equivalent to a commutative C-algebra.*

(iv) A is KK-equivalent to a commutative C-algebra.*

PROOF. (i) $\Rightarrow$ (ii) follows from Theorem 4.1. (ii) $\Rightarrow$ (iii) Assume that A satisfies the AUCT. Let C be a separable commutative C*-algebra with $K_*(C) \cong K_*(A)$. Since C satisfies the UCT, there is $\alpha \in KK(C, A)$ such that the induced map $\alpha_* : K_*(C) \to K_*(A)$ is a bijection. Then $\Gamma(\alpha) : \underline{K}(C) \to \underline{K}(A)$ is a bijection by the five lemma. We denote by $\dot\alpha$ the image of α in $KL(C,A)$. For a separable C*-algebra B, consider the commutative diagram

$$\begin{array}{ccc} KL(A,B) & \longrightarrow & \mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(B)) \\ \downarrow & & \downarrow \\ KL(C,B) & \longrightarrow & \mathrm{Hom}_\Lambda(\underline{K}(C),\underline{K}(B)) \end{array}$$

where the vertical maps are $x \mapsto \dot\alpha x$ and composition with $\Gamma(\alpha)$. The top horizontal map is bijective by assumption and the bottom horizontal map is bijective by Theorem 4.1. Thus the map $KL(A,B) \to KL(C,B)$ is a bijection for all separable C*-algebras B. By the usual "category theory" argument it follows that $\dot\alpha$ has an inverse $\dot\beta \in KL(A,C)$.

(iii) ⇒ (iv) follows from Corollary 5.2. (iv) ⇒ (i) was proved in [27]. □

Finally let us we mention that similar methods were used to prove that if a nuclear separable C*-algebra A can be approximated by C*-subalgebras satisfying the UCT, then A satisfies the UCT (see [9]).

6 KK-topology versus Ext-topology

For separable C*-algebras A, B, Kasparov [14] has established an isomorphism

$$KK(A,B) \cong \mathrm{Ext}(SA,B)^{-1}.$$

These two groups come with natural topologies, Pimsner's topology and respectively the Brown-Salinas topology. In these section we show that Kasparov's isomorphism is a homeomorphism. The following result and its proof is an adaptation of [23, Thm. 3.3].

Proposition 6.1 *Let A, B be separable C*-algebras and let X be a compact metrizable space. Then any element $y \in \mathrm{Ext}(A, C(X)\otimes B)^{-1}$ is represented by a $*$-homomorphism $\sigma : A \to Q(C(X)\otimes B\otimes\mathcal{K})$ which lifts to a completely positive contraction $\varphi : A \to C(X)\otimes M(B\otimes\mathcal{K}) \subset M(C(X)\otimes B\otimes\mathcal{K})$.*

PROOF. Since y is an invertible extension, y is represented by some $*$-homomorphism $\tau : A \to Q(C(X)\otimes B\otimes\mathcal{K})$ which lifts to a completely positive contraction

$$\phi : A \to M(C(X)\otimes B\otimes\mathcal{K}) \cong \mathcal{L}(H_{C(X)\otimes B}) \cong C_s(X, \mathcal{L}(H_B)).$$

By [15, Thm. 3], ϕ dilates to a $*$-homomorphism $\rho : A \to C_s(X, \mathcal{L}(H_B \oplus H_B))$ of the form

$$\rho(a) = \begin{pmatrix} \phi(a) & \alpha(a) \\ \beta(a) & \psi(a) \end{pmatrix},$$

such that $\alpha(a), \beta(a) \in C_s(X, \mathcal{K}(H_B))$ for all $a \in A$. After replacing ρ by $\rho \oplus \gamma$ for some $(A, C(X) \otimes B)$-absorbing $*$-homomorphism γ, we may assume that ρ itself is $(A, C(X) \otimes B)$-absorbing. Let $H = H' = H_B$ and consider the maps

$$\widetilde{\phi} : A \to C_s(X, \mathcal{L}(H \oplus (H \oplus H') \oplus (H \oplus H') \oplus \cdots))$$

defined by

$$\widetilde{\phi}(a) = \phi(a) \oplus \rho(a) \oplus \rho(a) \cdots$$

and

$$\widetilde{\rho} : A \to C_s(X, \mathcal{L}((H \oplus H') \oplus (H \oplus H') \oplus \cdots))$$

defined by

$$\widetilde{\rho}(a) = \rho(a) \oplus \rho(a) \oplus \cdots .$$

Consider also the constant unitary operator

$$G \in C_s(X, \mathcal{L}((H \oplus H') \oplus (H \oplus H') \oplus \cdots, H \oplus (H \oplus H') \oplus (H \oplus H') \oplus \cdots))$$

defined by

$$G(x)((h_1 \oplus h_1') \oplus (h_2 \oplus h_2') \oplus \cdots) = h_1 \oplus (h_2 \oplus h_1') \oplus (h_3 \oplus h_2') \oplus \cdots .$$

Let $S \in \mathcal{L}(H \oplus H \oplus \cdots)$ be the shift operator $S(h_1 \oplus h_2 \oplus \cdots) = 0 \oplus h_1 \oplus h_2 \oplus \cdots$. If $U \in C_s(X, \mathcal{L}(H \oplus H', H))$ is a unitary operator, let us define

$$\widetilde{U} \in C_s(X, \mathcal{L}((H \oplus H') \oplus (H \oplus H') \oplus \cdots, H \oplus H \oplus \cdots))$$

by $\widetilde{U} = U \oplus U \oplus \cdots$. The following identity was verified in the proof of [23, Thm. 3.3]:

$$\begin{aligned} \widetilde{U} G^* \widetilde{\phi}(a) G \widetilde{U}^* = \widetilde{U} \widetilde{\rho}(a) \widetilde{U}^* &- [U(\alpha(a) + \beta(a)U^* \oplus U(\alpha(a) + \beta(a))U^* \oplus \cdots] \\ &+ [U\beta(a)U^* \oplus U\beta(a)U^* \oplus \cdots] \circ S^* \\ &+ [U\alpha(a)U^* \oplus U\alpha(a)U^* \oplus \cdots] \circ S. \end{aligned} \tag{16}$$

Let $\gamma : A \to \mathcal{L}(H)$ be an (A, B)-absorbing $*$-homomorphism and let us define $\rho_0 : A \to C_s(X, \mathcal{L}(H))$ by $\rho_0(a)(x) = \gamma(a)$ for all $a \in A$ and $x \in X$. By Proposition 3.2, ρ_0 is an $(A, C(X) \otimes B)$-absorbing $*$-homomorphism. Since both ρ and ρ_0 are absorbing, there is a unitary $U \in C_s(X, \mathcal{L}(H \oplus H', H))$ such that $U\rho(a)U^* - \rho_0(a) \in C(X, \mathcal{K}(H))$ for all $a \in A$. This shows that

$$\widetilde{U} \widetilde{\rho}(a) \widetilde{U}^* = U\rho(a)U^* \oplus U\rho(a)U^* \oplus \cdots$$

is a norm-continuous function of $x \in X$. Since $\alpha(a), \beta(a) \in C(X, \mathcal{K}(H))$, and since the map $x \mapsto U(x)$ is strictly continuous, we see that the other three terms appearing on the right hand side of equation (16) are also norm-continuous functions of x. Therefore

$$\widetilde{U}G^*\widetilde{\phi}(a)G\widetilde{U}^* \in C(X, \mathcal{L}(H \oplus H \oplus \cdots)) \cong C(X) \otimes \mathcal{L}(H_B).$$

We conclude the proof by noting $\widetilde{U}G^*\widetilde{\phi}(\cdot)G\widetilde{U}^*$ defines the same element $y \in \mathrm{Ext}(A, C(X) \otimes B)^{-1}$ as ϕ.

Theorem 6.2 *Let A, B be separable C^*-algebras and let (x_n) and x_∞ be elements of* $\mathrm{Ext}(A, B)^{-1}$. *Then $x_n \to x_\infty$ in the Brown-Salinas topology if and only if there is $y \in \mathrm{Ext}(A, C(\bar{\mathbb{N}}) \otimes B)^{-1}$ such that $y(n) = x_n$ for all $n \in \bar{\mathbb{N}}$.*

PROOF. First we prove the implication ($\Rightarrow$). The elements of $\mathrm{Ext}(A, B)^{-1}$ are represented by $*$-homomorphisms

$$\sigma : A \to Q(B \otimes \mathcal{K}) = M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$$

which admit completely positive contractive liftings $A \to M(B \otimes \mathcal{K})$. Such a map σ is called liftable. Let (σ_n), σ_∞ be liftable $*$-homomorphisms with $x_n = [\sigma_n]$ and $x_\infty = [\sigma_\infty]$. Since $x_n \to x_\infty$ in the Brown-Salinas topology, if $\gamma : A \to M(B \otimes \mathcal{K})$ is an absorbing $*$-homomorphism, then there is a sequence of unitaries $u_n \in Q(B \otimes \mathcal{K})$ liftable to unitaries in $M(B \otimes \mathcal{K})$ such that

$$\lim_{n\to\infty} \|u_n(\sigma_n(a) \oplus \dot{\gamma}(a))u_n^* - \sigma_\infty(a) \oplus \dot{\gamma}(a)\| = 0$$

for all $a \in A$. Since $[\sigma_n] = [u_n(\sigma_n \oplus \dot{\gamma})u_n^*]$ and $[\sigma_\infty] = [\sigma_\infty \oplus \dot{\gamma}]$ in $\mathrm{Ext}(A, B)^{-1}$, without loss of generality we may assume that

$$\lim_{n\to\infty} \|\sigma_n(a) - \sigma_\infty(a)\| = 0 \tag{17}$$

for all $a \in A$. Define a $*$-homomorphism

$$\eta : A \to C(\bar{\mathbb{N}}) \otimes Q(B \otimes \mathcal{K}) \subset Q(C(\bar{\mathbb{N}}) \otimes B \otimes \mathcal{K}),$$

by $\eta(a)(n) = \sigma_n(a)$, $n \in \bar{\mathbb{N}}$. We want to show that η is liftable. For $k \in \mathbb{N}$ define $\eta^{(k)} : A \to C(\bar{\mathbb{N}}) \otimes Q(B \otimes \mathcal{K}) \subset Q(B \otimes C(\bar{\mathbb{N}}) \otimes \mathcal{K})$ by $\eta^{(k)}(a)(n) = \sigma_n(a)$ if $n \leq k$ and $\eta^{(k)}(a)(n) = \sigma_\infty(a)$ if $n > k$. Note that $\eta^{(k)}$ lifts to a completely positive contraction $A \to C(\bar{\mathbb{N}}) \otimes M(B \otimes \mathcal{K})$. Since

$$\lim_{k\to\infty} \|\eta^{(k)}(a) - \eta(a)\| = \lim_{k\to\infty} \sup_{n>k} \|\sigma_n(a) - \sigma_\infty(a)\| = 0$$

by (17), it follows by a result of Arveson, [1, Thm. 6], that η is liftable and hence $y = [\eta] \in \mathrm{Ext}(A, C(\bar{\mathbb{N}}) \otimes B)^{-1}$. It is clear that $y(n) = x_n$ for all $n \in \bar{\mathbb{N}}$. Let us prove the converse implication ($\Leftarrow$). By Proposition 6.1 every element $y \in \mathrm{Ext}(A, C(\bar{\mathbb{N}}) \otimes B)^{-1}$ is represented by a $*$-homomorphism

$$\Phi : A \to C(\bar{\mathbb{N}}) \otimes Q(B \otimes \mathcal{K}) \subset Q(C(\bar{\mathbb{N}}) \otimes B \otimes \mathcal{K}). \tag{18}$$

Therefore, if $(\Phi_n)_{n\in\bar{\mathbb{N}}}$ are the components of Φ, then

$$\lim_{n\to\infty} \|\Phi_n(a) - \Phi_\infty(a)\| = 0$$

for all $a \in A$, and hence $y(n) = [\Phi_n]$ converges to $y(\infty) = [\Phi_\infty]$ in the Brown-Salinas topology. $\square$

Let β be a generator of $KK^1(S\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$. The Kasparov product

$$KK^1(S\mathbb{C}, \mathbb{C}) \otimes KK(A, B) \to KK^1(SA, B)$$

induces a natural isomorphism

$$\chi : KK(A, B) \ni \alpha \mapsto \beta \otimes \alpha \in KK^1(SA, B) \cong \mathrm{Ext}(SA, B)^{-1}.$$

Corollary 6.3 *Let A, B be separable C*-algebras. The map $\chi : KK(A, B) \to \mathrm{Ext}(SA, B)^{-1}$ is a homeomorphism, when $KK(A, B)$ is given the Pimsner topology and $\mathrm{Ext}(SA, B)^{-1}$ is endowed with the Brown-Salinas topology.*

PROOF. The evaluation map at $n \in \bar{\mathbb{N}}$ induces a commutative diagram

$$\begin{array}{ccc} KK(A, C(\bar{\mathbb{N}}) \otimes B) & \xrightarrow{\chi} & \mathrm{Ext}(SA, C(\bar{\mathbb{N}}) \otimes B)^{-1} \\ \downarrow & & \downarrow \\ KK(A, B) & \xrightarrow{\chi} & \mathrm{Ext}(SA, B)^{-1} \end{array}$$

Since χ is a bijection, the result follows from Theorems 3.5 and 6.1 $\square$

7 Open questions

1. Let A, B be separable C*-algebras and assume that A is nuclear. Is the polish group $KK(A, B)/\bar{0}$ totally disconnected?

2. Let A be a separable nuclear C*-algebra. Fix an invariant metric for the topology of $K^0(A) = KK(A, \mathbb{C})$. Is it true that for any $\varepsilon > 0$ there is a finitely generated subgroup of $K^0(A)$ which is ε-dense in $K^0(A)$?

Both questions have positive answers if one assumes that A satisfies the UCT, as seen in Section 4.

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