Deformations of topological spaces predicted by E-theory

Marius Dădărlat *
Department of Mathematics
University of Maryland
College Park, MD 20742

Terry A. Loring †
Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131

1 Introduction

Let X be a locally compact space. By a *deformation* of X we mean a continuous field $\{A_t \mid t \in [0,1]\}$ of C^*-algebras with $A_0 \cong C_0(X)$, and

$$\{A_t \mid t \in (0,1]\} \cong B \times (0,1],$$

for a fixed C^*-algebra B. Replacing $C_0(X)$ by another C^*-algebra A, we generalize this to a deformation of one C^*-algebra to another. This is a basic interpretation of deformation—it reflects only the topology of X and omits more general fields of algebras—but is an important one. This importance is seen in the relation to E-theory and the examples [3, 8, 11, 12] that have arisen.

Deformations are, in fact, very common. About the only requirement for a C^*-algebra to arise as a deformed CW-complex is that it have the correct K-theory. This fact follows from our calculations in "unsuspended" E-theory [5]. We will explicitly describe one of the deformations predicted by these calculations: a deformation of a three-dimensional CW complex into a dimension-drop interval. We hope this example will further clarify the role of the dimension-drop interval as a building block in Elliott's inductive limits [6].

* current address: Department of Mathematics, Purdue University, West Lafayette, IN 47907
† partially supported by NSF grant DMS-9007347

Recall, from [3], that an *asymptotic morphism* $(\varphi_t) : A \to B$ between C^*-algebras is a collection of maps $\varphi_t : A \to B$ for $t \in [1, \infty)$ such that for $a, b \in A$ and $\alpha \in \mathbf{C}$, as $t \to \infty$,

$$\begin{aligned} \|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| &\to 0, \\ \|\varphi_t(a^*) - \varphi_t(a)^*\| &\to 0, \\ \|\varphi_t(\alpha a + b) - \alpha\varphi_t(a) - \varphi_t(b)\| &\to 0 \end{aligned}$$

and $t \mapsto \varphi_t(a)$ is continuous. We say that (φ_t) is *injective* if also $\limsup \|\varphi_t(a)\| > 0$ for all a.

Injective asymptotic morphisms correspond exactly to deformations. Thus, we will work in the context of asymptotic morphisms. See [3] for an explanation of this correspondence and the definitions of equivalence and homotopy for asymptotic morphisms.

The following result often gives the easiest way to show a given asymptotic morphism is injective. First, we recall how an asymptotic morphism $(\varphi_t) : A \to B$ induces maps on K-theory. Given a projection p in A, the class of $\varphi_*([p])$ in $K_0(B)$ is represented by any projection that is close to $\varphi_t(p)$ for some sufficiently large value of t. For projections, and unitaries, in $M_k(A)$ a similar construction is used.

Proposition 1 *Suppose $X \cup \{pt\}$ is a compact manifold. If an asymptotic morphism $(\varphi_t) : C_0(X) \to B$ induces an injective map on K-theory then (φ_t) is injective.*

Remark 2 This type of result holds more generally. In particular, it holds for the CW complex discussed in section 3.

Example 3 Our first exampe is an asymptotic morphism $(\alpha_t) : C_0(\mathbf{R}^2) \to \mathcal{K}$ which induces an isomorphism on K-theory. This well-known in several contexts.

We regard $C_0(\mathbf{R}^2)$ as the universal C^*-algebra generated by selfadjoint element h and normal element N subject to the relation $h = h^2 + N^*N$. so that the generator of $K_0(C_0(\mathbf{R}^2)) \cong \mathbf{Z}$ is just

$$\left[\begin{pmatrix} h & N^* \\ N & 1-h \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right].$$

An asymptotic morphism from A to B is given, up to equivalence, by a $*$-homomorphism from A to B_∞, where B_∞ is the C^*-algebra described in [3]. Therefore, if A is universal for a set of relations one need only define the paths in B that are to be the images of the generators. In this case, we need only define $\alpha_t(h)$ and $\alpha_t(N)$.

Let S denote the unilateral shift and, for $t \in [1, \infty)$, let D_t denote the diagonal operator whose diagonal corresponds to the sequence

$$1/t, 2/t, \ldots, [t]/t, 1, 1, \ldots$$

Set $\alpha_t(h) = 1 - D_t$ and $\alpha_t(N) = \sqrt{D_t - D_t^2}S$. Since the required relations hold asymptotically, this determines α_t. The fact that this induces an isomorphism on K_0 follows from the calculation, in [2] or [7], of the spectrum of

$$\begin{bmatrix} 1 - D_t & \sqrt{D_t - D_t^2}S^* \\ \sqrt{D_t - D_t^2}S & D_t \end{bmatrix}.$$

See [9] for more details and a modification of this example that produces deformations of $\mathbf{RP}^2$ and the Klein bottle.

2 Unsuspended E-theory

Let A and B be C^*-algebras. For convenience, we shall assume that A and B are separable and nuclear. We will use the notation

$$\begin{aligned} [A, B] &= \text{homotopy classes of } *\text{-homomorphisms}, \\ [[A, B]] &= \text{homotopy classes of asymptotic morphisms.} \end{aligned}$$

We will use the following isomorphisms, from [3],

$$\begin{aligned} KK(A, B) &\cong E(A, B) \\ &\cong [[SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]] \\ &\cong [[S^2A \otimes \mathcal{K}, S^2B \otimes \mathcal{K}]]. \end{aligned}$$

We now arrive at our main result. By $X \cup \{\text{pt}\}$ we mean the one-point compactification of a locally compact space X. Combined with Proposition 1 this result guarantees the existence of many deformations.

Proposition 4 *If $X \cup \{pt\}$ is a connected, finite CW complex then the suspension map*

$$[[C_0(X), B \otimes \mathcal{K}]] \to KK(C_0(X), B)$$

is an isomorphism.

The proof of this will be given in [5]. The inverse map may be described as follows. Let

$$\beta : C_0(\mathbf{R}^1) \to C_0(\mathbf{R}^3) \otimes \mathcal{K}$$

be a $*$-homomorphism inducing an isomorphism on K-theory. By [4, Corollary 3.1.8] there exists a map

$$\beta_X : C_0(X) \to S^2 C_0(X) \otimes \mathcal{K}$$

whose suspension is homotopic to $\beta \otimes \mathrm{id}_{C_0(X)}$. Composition, on appropriate sides, by β_X and the asymptotic morphism

$$1 \otimes \alpha : S^2 B \otimes \mathcal{K} \to B \otimes \mathcal{K}$$

(see Example 3) defines the inverse mapping

$$[[S^2 C_0(X) \otimes \mathcal{K}, S^2 B \otimes \mathcal{K}]] \to [[C_0(X), B \otimes \mathcal{K}]].$$

Using the universal coefficient theorem we obtain a corollary.

Corollary 5 *If $X \cup \{pt\}$ is a finite CW complex and $\eta : K^*(X) \to K_*(B)$ is an isomorphism then there exists a deformation of X to $B \otimes \mathcal{K}$ which induces η.*

3 Matricial torsion

Consider the three-dimensional CW complex obtained by attaching, with degree two, the boundary of a three-cell B^3 to a two-sphere S^2. Remove the base-point (which sits in the copy of S^2) and call the result X. That is,

$$X \cup \{\mathrm{pt}\} = B^3 \cup_\zeta S^2$$

where $\zeta : \partial B^3 \to S^2$ has degree 2. Thus $K_0(C_0(X)) = 0$ and $K_1(C_0(X)) \cong \mathbf{Z}/2$.

Let B denote the non-unital dimension-drop interval, that is,

$$B = \{ f \in C_0((0,1], M_2) \mid f(1) \text{ is scalar } \}.$$

One may compute $K_0(B) = 0$ and $K_1(B) \cong \mathbf{Z}/2$.

We know, by Corollary 5, that there is an asymptotic morphism

$$(\psi_t) : C_0(X) \to B \otimes \mathcal{K}$$

inducing an isomorphism on K-theory. This is an example of topological torsion being "quantized" into matricial torsion. Our goal is to find ψ explicitly.

We first must be more explicit about the attaching map and the associated $*$-homomorphism $\theta : C_0(\mathbf{R}^2) \to C_0(\mathbf{R}^2)$. Using the generators and relations of Example 3, we determine θ by setting

$$\begin{aligned} \theta(h) &= f(h), \\ \theta(N) &= g_1(h)N + g_2(h)N^* \end{aligned}$$

where f, g_1 and g_2 are functions of the form

which satisfy $g_1 g_2 = 0$ and $f(t) = f(t)^2 + (g_1(t)^2 + g_2(t)^2)\,(t - t^2)$.

We will also need $\iota : \mathcal{K} \to \mathcal{K} \otimes M_2$ given by $\iota(T) = T \otimes I$. With the additional notation of δ_1 indicating evaluation at 1, we have two pull-back diagrams:

	$C_0(X)$			$B \otimes \mathcal{K}$	
$C_0(0,1] \otimes C_0(\mathbf{R}^2)$		$C_0(\mathbf{R}^2)$	$C_0(0,1] \otimes \mathcal{K} \otimes M_2$		$\mathcal{K}$
δ_1		θ	δ_1		ι
	$C_0(\mathbf{R}^2)$			$\mathcal{K} \otimes M_2$	

Lemma 6 *There exists an asymptotic morphism* $(\varphi_t) : C_0(\mathbf{R}^2) \to M_2(\mathcal{K})$ *such that, for all* t,

$$\text{image}(\varphi_t \circ \theta) \subseteq \left\{ \begin{bmatrix} T & \\ & T \end{bmatrix} \middle| \, T \in \mathcal{K} \right\}$$

which induces an isomorphism on K*-theory.*

We defer the proof until after we see how the lemma is used.

Consider the following commutative diagram (commuting exactly for each t):

$$\begin{array}{ccc} C_0(0,1] \otimes C_0(\mathbf{R}^2) & \overset{\text{id} \otimes \varphi_t}{} & C_0(0,1] \otimes M_2(\mathcal{K}) \\ \delta_1 & & \delta_1 \\ C_0(\mathbf{R}^2) & \varphi_t & M_2(\mathcal{K}) \\ \theta & & \iota \\ C_0(\mathbf{R}^2) & \eta_t & \mathcal{K} \end{array}$$

Here η_t is the unique solution to $\iota \circ \eta_t = \varphi_t \circ \theta$. Since φ induces an isomorphism on K-theory, η must as well.

Since these maps are not $*$-homomorphisms we cannot immediately invoke the pull-back property. However, simply restricting $\text{id}_{C_0(0,1]} \otimes \varphi_t$ produces $\psi_t : C_0(X) \to B \otimes \mathcal{K}$. Now considering the K-theory of the commuting diagram

$$\begin{array}{ccc} C_0(0,1) \otimes C_0(\mathbf{R}^2) & \overset{\text{id} \otimes \varphi_t}{} & C_0(0,1) \otimes M_2(\mathcal{K}) \\ & & \\ C_0(X) & \psi_t & B \otimes \mathcal{K} \end{array}$$

it is easy to see that ψ induces an isomorphism on K-theory.

Proof (Lemma 6) In order to specify $(\varphi_t) : C_0(\mathbf{R}^2) \to M_2(\mathcal{K})$, it suffices to specify where the generators h and N are sent. At $t = n \in \mathbf{N}$, we shall have $\varphi_n(h) = H_n$ and $\varphi_n(N) = N_n$ where H_n and N_n are the following elements of $M_2(M_{2^n})$ (we regard M_{2^n} as a corner of $\mathcal{K}$

and so $M_2(M_{2^n}) \subseteq M_2(\mathcal{K})$):

$$H_n = \left[\begin{array}{cccc|cccc} \alpha_1 & & & & & & & \\ & \alpha_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \alpha_{2^n} & & & & \\ \hline & & & & 1-\alpha_1 & & & \\ & & & & & 1-\alpha_2 & & \\ & & & & & & \ddots & \\ & & & & & & & 1-\alpha_{2^n} \end{array}\right],$$

$$N_n = \left[\begin{array}{cccc|cccc} 0 & & & & & & & \\ \beta_1 & 0 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \beta_{2^n-1} & 0 & & & & \\ \hline & & & & 0 & \beta_1 & & \\ & & & & & 0 & \ddots & \\ & & & & & & \ddots & \beta_{2^n-1} \\ & & & \beta_{2^n} & & & & 0 \end{array}\right]$$

where, for $j = 1, \ldots, 2^n$, $\alpha_j = j/2^{n+1}$ and $\beta_j = \sqrt{\alpha_j - \alpha_j^2}$.

We are interested, more generally, in matrices $A, B \in M_2(M_k)$ such that the following relations hold:

$$\begin{array}{rcl} \|[A,B]\|, \|[B,B^*]\|, \|[A,B^*]\| & \leq & \epsilon \\ \|A - A^*\| & \leq & \epsilon \\ \|A^2 + B^*B - A\| & \leq & \epsilon \\ f(A) & \in & M_k \otimes I \\ g_1(A)B + g_2(A)B^* & \in & M_k \otimes I \end{array} \tag{1}$$

It may be checked that H_n and N_n satisfy (1), for some $\epsilon_n \geq 0$, with $\epsilon_n \to 0$ as $n \to \infty$.

We will need some auxiliary matrices in $M_2(M_{2^n})$:

$$\tilde{H}_n = \left[\begin{array}{ccc|ccc} 1/2 & & & & & \\ & 1/2 & & & & \\ & & 1/2 & & & \\ \hline & & & 1/2 & & \\ & & & & 1/2 & \\ & & & & & 1/2 \end{array}\right], \ \tilde{N}_n = \left[\begin{array}{ccc|ccc} 0 & & 1/2 & & & \\ 1/2 & 0 & & & & \\ & 1/2 & 0 & & & \\ \hline & & & 0 & 1/2 & \\ & & & & 0 & \\ & & & & & 1/2 \\ & & & 1/2 & & 0 \end{array}\right]$$

A path of unitaries, multiplied by 1/2, in the lower-right-hand corner will create a path satisfying (1) from $\tilde{H}_n, \tilde{N}_n$ to

$$\left[\begin{array}{ccc|ccc} 1/2 & & & & & \\ & 1/2 & & & & \\ & & 1/2 & & & \\ \hline & & & 1/2 & & \\ & & & & 1/2 & \\ & & & & & 1/2 \end{array}\right], \left[\begin{array}{ccc|ccc} 0 & & 1/2 & & & \\ 1/2 & 0 & & & & \\ & 1/2 & 0 & & & \\ \hline & & & 0 & & 1/2 \\ & & & 1/2 & 0 & \\ & & & & 1/2 & 0 \end{array}\right].$$

Now, deforming the pair of scalars $(1/2, 1/2)$ to $(0, 0)$ appropriately continues this path to the pair of matrices $0, 0$. By this argument, we have reduced the construction of (φ_t) to being able to connect $H_n \oplus \tilde{H}_n, N_n \oplus \tilde{N}_n$ to H_{n+1}, N_{n+1} via pairs satisfying (1).

Let

$$A_n = \begin{bmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_{2^n} \end{bmatrix}, \ B_n = \begin{bmatrix} 0 & & \\ \beta_1 & 0 & \\ & \beta_{2^n-1} & 0 \end{bmatrix}, \ C_n = \begin{bmatrix} 0 & & \\ & 0 & \\ & & \beta_{2^n} \end{bmatrix}$$

so that

$$H_n = \left[\begin{array}{c|c} A_n & \\ \hline & 1-A_n \end{array}\right], \; N_n = \left[\begin{array}{c|c} B_n & 0 \\ \hline C_n & B_n^* \end{array}\right].$$

Let

$$\tilde{A}_n = \begin{bmatrix} 1/2 & & & \\ & 1/2 & & \\ & & \ddots & \\ & & & 1/2 \end{bmatrix}, \; \tilde{B}_n = \begin{bmatrix} 0 & & & 1/2 \\ 1/2 & 0 & & \\ & & \ddots & \\ & & 1/2 & 0 \end{bmatrix}$$

so that

$$\tilde{H}_n = \left[\begin{array}{c|c} \tilde{A}_n & \\ \hline & 1-\tilde{A}_n \end{array}\right], \; \tilde{N}_n = \left[\begin{array}{c|c} \tilde{B}_n & \\ \hline & \tilde{B}_n^* \end{array}\right].$$

By Berg's technique, [1, 10], there exists a unitary $W \in M_{2^{n+1}}$ such that, for some constant C, independent of n,

$$\left\| W \left[\begin{array}{c|c} A_n & \\ \hline & \tilde{A}_n \end{array}\right] W^* - \left[\begin{array}{c|c} A_n & \\ \hline & \tilde{A}_n \end{array}\right] \right\| \le C2^{-n/2},$$

$$\left\| W \left[\begin{array}{c|c} B_n & \\ \hline & \tilde{B}_n \end{array}\right] W^* - B_{n+1} \right\| \le C2^{-n/2}$$

and (by keeping W trivial except for vectors in a "segment" avoiding the "last" basis vector in each copy of $\mathbf{C}^{2^n}$)

$$W \left[\begin{array}{c|c} 0 & \\ \hline & C_n \end{array}\right] W^* = \left[\begin{array}{c|c} 0 & \\ \hline & C_n \end{array}\right].$$

Let $\hat{W} = W \otimes I_2$. It follows from above that

$$\hat{W} \left[\begin{array}{cc|cc} A_n & & & \\ & \tilde{A}_n & & \\ \hline & & 1-A_n & \\ & & & 1-\tilde{A}_n \end{array}\right] \hat{W}^* \approx \left[\begin{array}{cc|cc} A_n & & & \\ & \tilde{A}_n & & \\ \hline & & 1-A_n & \\ & & & 1-\tilde{A}_n \end{array}\right]$$

and

$$\hat{W}\left[\begin{array}{cc|cc} B_n & & & \\ & \tilde{B}_n & & \\ \hline C_n & & B_n^* & \\ & 0 & & \tilde{B}_n^* \end{array}\right]\hat{W}^* \approx \left[\begin{array}{c|c} B_{n+1} & \\ \hline C_{n+1} & B_{n+1}^* \end{array}\right].$$

Notice that the two matrices on the left satisfy (1) By taking a path of unitaries from W to I, we get paths satisfying (1) from $H_n \oplus \tilde{H}_n, N_n \oplus \tilde{N}_n$ to this pair. Linear interpolation from this pair to the pair on the right-hand side gives a path satisfying (1), perhaps after increasing ϵ_n. Finally, it is a simple matter to slide the scalars which are on the diagonal of

$$\left[\begin{array}{cc|cc} A_n & & & \\ & \tilde{A}_n & & \\ \hline & & 1-A_n & \\ & & & 1-\tilde{A}_n \end{array}\right]$$

to connect the right-hand pair to H_{n+1}, N_{n+1}. We may conclude that (φ_t) exists with the properties specified in the lemma and $\varphi_n(h) = H_n$ and $\varphi_n(N) = N_n$. □

References

[1] I. D. Berg, *On operators which almost commute with the shift,* J. Operator Theory, 11 (1984) 365-377.

[2] M.-D. Choi, *Almost commuting matrices need not be nearly commuting,* Proc. Amer. Math. Soc., 102 (1988), 529-533.

[3] A. Connes and N. Higson, *Deformations, morphismes asymptotiques et K-theory bivariante,* C. R. Acad. Sci. Parie, t. 310 Série I (1990) 101-106.

[4] M. Dădărlat and A. Nemethi, *Shape theory and connective K-theory,* J. Operator Theory, 23 (1990), 207-291.

[5] M. Dădărlat and T. A. Loring, in preparation.

[6] G. A. Elliott, *On the classification of C^*-algebras of the real rank zero,* preprint.

[7] R. Exel and T. A. Loring, *Invariants of almost commuting unitaries,* J. Funct. Anal., 95 (1991) 364-376.

[8] S. Klimek and A. Lesniewski, *Quantum Riemann surfaces I, The unit disk,* preprint.

[9] T. A. Loring, *Deformations of nonorientable surfaces as torsion E-theory elements,* preprint.

[10] T. A. Loring, *Berg's technique for pseudo-actions with applications to AF embeddings,* Canad. J. Math., 43 (1991) 119-157.

[11] M. A. Rieffel, *Deformation quantization for actions of* $\mathbf{R}^d$, preprint.

[12] S. L. Woronowicz, *Twisted SU(2) group. An example of a non-commutative differential calculus,* Publ. RIMS, Kyoto Univ., 23 (1987), 117-181.