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Continuous fields of Kirchberg C*-algebras

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Dedicated to George Elliott on his 60th birthday

Abstract

In this paper we study the C*-algebras associated to continuous fields over locally compact metrizable zero-dimensional spaces whose fibers are Kirchberg C*-algebras satisfying the UCT. We show that these algebras are inductive limits of finite direct sums of Kirchberg algebras and they are classified up to isomorphism by topological invariants.

MSC: 46L35; 46L80

Keywords: C*-algebra; Continuous field of C*-algebras; Kirchberg algebra; *K*-theory

1. Introduction

A purely infinite separable simple nuclear C*-algebra is called a Kirchberg algebra. Kirchberg [11] and Phillips [18] proved that two Kirchberg algebras A and B are stably isomorphic if and only if they are KK-equivalent. Consequently, if in addition A and B satisfy the universal coefficient theorem (UCT) of [22], then $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ if and only if $K_*(A) \cong K_*(B)$.

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Kirchberg [12] generalized the *KK*-theory isomorphism result to nonsimple C*-algebras. He showed that if A and B are nuclear separable C*-algebras with primitive ideal spectrum homeomorphic to some T_0-space X, then $A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ if and only if A is KK_X-equivalent to B, where KK_X is a suitable generalization of the Kasparov theory which preserves the primitive ideal spectra of A and B (or rather the lattice homomorphisms from the open subsets of X to the lattice of closed ideals of A and B). Unlike the case of simple C*-algebras, as observed in [14], one does not have a general algebraic criterion for recognizing when two C*-algebras are KK_X-equivalent, as we lack a generalization of the UCT for KK_X. Finding such a criterion seems to be a difficult problem even when the space X consists of finitely many points and is non-Hausdorff. The case when X consists of two points was solved by Rørdam [20].

One of the goals of the present paper is to propose an answer to the above question for the separable nuclear C*-algebras A whose primitive ideal spectrum $\mathrm{Prim}(A)$ is zero-dimensional and Hausdorff, under the assumption that all simple quotients of A satisfy the UCT. We introduce a homotopy invariant $\mathrm{Inv}(A)$ consisting of a preordered semigroup

$$P(A \otimes \mathcal{O}_2) \oplus \underline{K}(A),$$

together with the action of the Bockstein operations on $\underline{K}(A)$. Here $P(A \otimes \mathcal{O}_2)$ denotes the Murray–von Neumann semigroup of equivalence classes of projections in $A \otimes \mathcal{O}_2 \otimes \mathcal{K}$ and $\underline{K}(A)$ is the total K-theory group of A, see Section 4. It turns out that if A and B are separable nuclear C*-algebras with zero-dimensional Hausdorff primitive spectra and with all simple quotients satisfying the UCT, then $A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ if and only if $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$. Since we do not use Kirchberg's KK_X-theory, we deduce that for such algebras, A is KK_X-equivalent to B if and only if $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$. The examples constructed in [4] show that the action of the Bockstein operations is an essential part of the invariant. $\mathrm{Inv}(A)$ is an adaptation to purely infinite C*-algebras of an invariant introduced in [7,6]. A positive morphism $\mathrm{Inv}(A) \to \mathrm{Inv}(B)$ must preserve the filtration of the total K-theory group $\underline{K}(-)$ induced by ideals. Since all the ideals of the C*-algebras classified in this paper give rise to quasidiagonal extensions, the corresponding boundary maps vanish, and hence there is no need to include them in the classifying invariant as it was necessary to do in [20].

A related goal of this paper is to describe the structure and the classification of the C*-algebras associated to continuous fields over locally compact metrizable zero-dimensional spaces whose fibers are Kirchberg algebras satisfying the UCT. We show that these algebras are inductive limits of finite direct sums of Kirchberg algebras satisfying the UCT and that they are classified up to isomorphism by the topological invariant $\mathrm{Inv}_u(-) := (\mathrm{Inv}(-), P_u(-), \tau)$, see Sections 4 and 5. It is worth to note that their structure is obtained by proving first the classification result for a larger class of C*-algebras, using the same invariant. Since Inv_u is continuous and homotopy invariant, we deduce immediately that in this (larger) class of C*-algebras, the isomorphism and the shape equivalence are equivalent properties, see Section 5.

Let us describe how the paper is organized. Section 2 is devoted to preliminaries. Using the results of [11,18] we show in Section 3 that the C*-algebra associated to a continuous field of Kirchberg algebras satisfying the UCT over a metrizable zero-dimensional locally compact space admits local approximations by finite direct sums of Kirchberg algebras satisfying the UCT and having finitely generated K-theory. In Section 4 we introduce the invariants Inv and Inv_u and describe their basic properties. In Section 5 we prove that the C*-algebras A which admit local approximations as described above are classified up to stable isomorphism by $\mathrm{Inv}(A)$ and in fact they can be written as inductive limits of finite direct sums of Kirchberg algebras satisfying the UCT. In particular, these structure and classification results apply to the C*-algebras associated to continuous fields of Kirchberg algebras satisfying the UCT over a metrizable zero-dimensional locally compact space.

2. Preliminaries

Definition 2.1. A sequence (A_n) of C*-subalgebras of a C*-algebra A is called exhaustive if for any finite subset $\mathcal{F} \subset A$, any $m > 0$, and any $\varepsilon > 0$ one has $\mathcal{F} \subset_\varepsilon A_n$ for some $n > m$. The inclusion maps $A_n \hookrightarrow A$ are denoted by $\imath_n$ (respectively $\jmath_n : B_n \hookrightarrow B$ if (B_n) is an exhausting sequence for B).

Let A be a separable C*-algebra and let (A_n) be an exhaustive sequence for A. Let $\{x_1, x_2, \ldots, x_n, \ldots\} \subset A$ be a dense subset of A. After passing to a subsequence of (A_n) we may arrange that

$$\{x_1, x_2, \ldots, x_k\} \subset_{1/k} A_k, \quad \forall\, k \in \mathbb{N}. \tag{1}$$

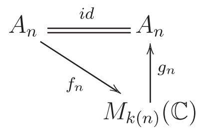
In the sequel we will always work with exhaustive sequences satisfying (1).

Lemma 2.2. *Let A be a separable C*-algebra with an exhaustive sequence (A_n) consisting of unital, nuclear C*-algebras. Then after passing to a subsequence of (A_n) satisfying* (1) *there is a sequence of completely positive contractions $\mu_n : A \to A_n$ which is asymptotically multiplicative and such that $\lim_{n\to\infty} \|\imath_n \mu_n(a) - a\| = 0$ for all $a \in A$, where $\imath_n : A_n \to A$ are the inclusion maps.*

Proof. Let $\mathcal{F} := \{x_1, x_2, \ldots, x_n, \ldots\}$ with $\overline{\mathcal{F}} = A$, and let $\mathcal{F}_k := \{x_1, x_2, \ldots, x_k\}(k \in \mathbb{N})$. After passing to a subsequence we may assume that (A_n) satisfies (1). Let $\mathcal{G}_k$ be a finite subset of A_k $(k \in \mathbb{N})$ such that

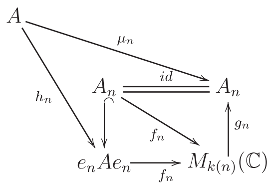
$$\mathcal{F}_k \subseteq_{1/k} \mathcal{G}_k, \quad k \in \mathbb{N}. \tag{2}$$

For each n, let e_n be the unit of A_n. Using that $\overline{\mathcal{F}} = A$ and (2) it is easy to check that (e_n) is a (*not necessarily increasing*) approximate unit of A. Since each A_n is nuclear and unital, it follows that we can find an approximate factorization of id_{A_n} by unital completely positive maps, on the finite set $\mathcal{G}_n$ within $1/n$:



$$\|g_n f_n(y) - y\| < 1/n, \quad y \in \mathcal{G}_n, \quad n \in \mathbb{N}. \tag{3}$$

Now, using Arveson's extension theorem for unital completely positive maps (see e.g. [21, Theorem 6.1.5]), we extend f_n to a unital completely positive map (denoted in the same way) $f_n : e_n A e_n \to M_{k(n)}(\mathbb{C})$ for every $n \in \mathbb{N}$. Define $h_n : A \to e_n A e_n$ by $h_n(a) = e_n a e_n$ and $\mu_n : A \to A_n$ by $\mu_n = g_n f_n h_n$. Hence we have a diagram:



Fix now an arbitrary $x \in A$ and $\varepsilon > 0$. Since $\overline{\mathcal{F}} = A$, it follows that there is a positive integer k such that $\|x - x_k\| < \varepsilon$. Since $x_k \in \mathcal{F}_k \subseteq \mathcal{F}_n$ for every $n \geqslant k$, it follows by (2) that there is an element $y_n \in \mathcal{G}_n \subset A_n$ such that $\|x_k - y_n\| < 1/n$. Hence

$$\|x - y_n\| < \varepsilon + 1/n, \quad \forall\, n \geqslant k. \tag{4}$$

Since as noticed earlier $\lim_{n\to\infty} \|e_n x e_n - x\| = 0$, there is $k_1 \geqslant k$ such that

$$\|e_n x e_n - x\| < \varepsilon, \quad \forall\, n \geqslant k_1. \tag{5}$$

Now, using (3)–(5) we can write for every $n \geqslant k_1$,

$$\begin{aligned}
\|\mu_n(x) - x\| &< \|\mu_n(x) - y_n\| + \varepsilon + 1/n = \|g_n f_n h_n(x) - y_n\| + \varepsilon + 1/n \\
&= \|g_n f_n(e_n x e_n) - y_n\| + \varepsilon + 1/n \\
&< \|g_n f_n(x) - y_n\| + 2\varepsilon + 1/n \\
&< \|g_n f_n(y_n) - y_n\| + 3\varepsilon + 2/n < 3\varepsilon + 3/n.
\end{aligned}$$

Hence $\lim_{n\to\infty} \|\imath_n \mu_n(x) - x\| = 0$. Note also that each $\mu_n : A \to A_n$ is a completely positive contraction as a composition of completely positive contractions. Finally, the fact that (μ_n) is asymptotically multiplicative follows easily since $\imath_n$ are $*$-monomorphisms and $\lim_{n\to\infty} \|\imath_n \mu_n(a) - a\| = 0$ for all $a \in A$. □

Definition 2.3. Let $\mathcal{A} = ((A(x))_{x\in X}, \Gamma)$ be a continuous field of C*-algebras over a metrizable locally compact space *X*. Here Γ consists of vector fields *a* on *X*, (i.e. $a \in \prod_{x\in X} A(x)$, $a(x) \in A(x)$) satisfying a number of natural axioms, including the continuity of the map $x \mapsto \|a(x)\|$, and the condition that Γ is closed under local uniform approximation [8, 10.1.2]. If *X* is compact, then Γ is a C*-algebra, called the C*-algebra associated to $\mathcal{A}$. If *X* is just locally compact, the C*-algebra associated to $\mathcal{A}$ consists of those vector fields $a \in \Gamma$ with the property that the map $x \mapsto \|a(x)\|$ is vanishing at infinity. Given a family of C*-algebras $(A(x))_{x\in X}$ there are in general many choices for Γ which makes $((A(x))_{x\in X}, \Gamma)$ a continuous field of C*-algebras with nonisomorphic associated C*-algebras.

Let $Y \subseteq X$ be a closed subspace and let Γ_Y be the restriction of Γ to *Y*. One verifies that $\mathcal{A}|_Y := ((A(x))_{x\in Y}, \Gamma_Y)$ is a continuous field of C*-algebras on *Y* [8, 10.1.12]. We shall denote by $A(Y)$ the C*-algebra associated to $\mathcal{A}|_Y$. Observe that $A(X)$ is the C*-algebra associated to $\mathcal{A}$.

Remark 2.4. Let $\mathcal{A} = ((A(x))_{x\in X}, \Gamma)$ be a continuous field of C*-algebras over a metrizable zero-dimensional locally compact space *X*. If $Y \subseteq X$ is closed, then

$$I(Y) = \{f \in A(X) : f|_Y = 0\}$$

is a closed, two-sided ideal of $A(X)$. Using [8, Proposition 10.1.12] one shows that $A(X)/I(Y) \cong A(Y)$. Let $(F_n)_{n=1}^\infty$ be a decreasing sequence of compact subsets of *X* forming a basis of neighborhoods of a point $x_0 \in X$. Then $A(x_0) \cong \varinjlim (A(F_n), \Phi_n)$, where each $\Phi_n : A(F_n) \to A(F_{n+1})$ is the restriction map ($\Phi_n(f) = f|_{F_{n+1}}$, $f \in A(F_n)$). Indeed, the $I(F_n)$'s form an increasing sequence of closed, two-sided ideals of $A(X)$ and $\overline{\bigcup_{n\in\mathbb{N}} I(F_n)} = I(x_0)$, since $\bigcap_{n\in\mathbb{N}} F_n = \{x_0\}$. Therefore

$$A(x_0) \cong A(X)/I(x_0) \cong \varinjlim (A(X)/I(F_n), \varphi_n) \cong \varinjlim (A(F_n), \Phi_n),$$

where $\varphi_n : A(X)/I(F_n) \to A(X)/I(F_{n+1})$ is induced by the inclusion $I(F_n) \subseteq I(F_{n+1})$.

Remark 2.5. Let $\mathcal{A} = ((A(x))_{x\in X}, \Gamma)$ be a continuous field of separable C*-algebras with a countable approximate unit of projections over a metrizable zero-dimensional locally compact space X. Let *U* be a closed subset of *X*. Then $A(U)$ has a countable approximate unit of projections. This follows from [16]. If all the $A(x)$'s have real rank zero, then $A(U)$ has real rank zero by [17, Theorem 2.1].

3. Continuous fields of Kirchberg algebras

In this section we establish an approximation property for C*-algebras associated to continuous fields of Kirchberg C*-algebras over zero-dimensional spaces, see Theorem 3.6.

Definition 3.1. A separable nuclear simple purely infinite C*-algebra is called a Kirchberg algebra.

We refer the reader to [21] for a background discussion of Kirchberg algebras.

Remark 3.2. Any Kirchberg algebra is either unital or of the form $A_0 \otimes \mathcal{K}$ where A_0 is a unital Kirchberg algebra and $[1_{A_0}] = 0$ in $K_0(A_0)$. In particular, all Kirchberg algebras admit an approximate unit consisting of projections.

We introduce here notation for certain classes of C*-algebras. This notation is used to shorten the statements of certain intermediate results.

Definition 3.3. $\mathcal{E}$ consists of unital Kirchberg algebras.

$\mathcal{E}_{\mathrm{uct}}$ consists of unital Kirchberg algebras satisfying the UCT.

$\mathcal{E}_{\mathrm{fg}}$ consists of unital Kirchberg algebras A with $K_*(A)$ finitely generated.

$\mathcal{E}_{\mathrm{fg-uct}} = \mathcal{E}_{\mathrm{fg}} \cap \mathcal{E}_{\mathrm{uct}}$

$\mathcal{B}$ consists of finite direct sums of unital Kirchberg algebras. One defines similarly $\mathcal{B}_{\mathrm{uct}}$, $\mathcal{B}_{\mathrm{fg}}$ and $\mathcal{B}_{\mathrm{fg-uct}}$. A (nuclear) separable C*-algebra is in the class $\mathcal{L}$ if it admits an exhaustive sequence (A_n) with each A_n in $\mathcal{B}$. One defines similarly $\mathcal{L}_{\mathrm{uct}}$, $\mathcal{L}_{\mathrm{fg}}$ and $\mathcal{L}_{\mathrm{fg-uct}}$.

Lemma 3.4. *Let $\mathcal{A} = ((A(x))_{x\in X}, \Gamma)$ be a continuous field of Kirchberg algebras over a metrizable zero-dimensional locally compact space X. Let A be the C*-algebra associated to $\mathcal{A}$. Then $A \cong A \otimes \mathcal{O}_\infty$.*

Proof. If we set $B := A \otimes \mathcal{K}$, then B is separable, stable, nuclear, and $\mathrm{Prim}(B) \cong \mathrm{Prim}(A) \cong X$ is Hausdorff and zero-dimensional. Every nonzero simple quotient of B is of the form $A(x_0) \otimes \mathcal{K}$, for some $x_0 \in X$. But $A(x_0)$ being a Kirchberg algebra is purely infinite and since the property of being purely infinite is invariant under stable isomorphism [13, Theorem 4.23] it follows that $A(x_0) \otimes \mathcal{K}$ is also purely infinite. Now, by [1, Theorem 1.5] we deduce that B is strongly purely infinite. Using again the fact that A and B are stably isomorphic, [14, Proposition 5.11(iii)] implies that A is strongly purely infinite. Hence the C*-algebra A is separable nuclear, has an approximate unit of projections (see Remark 2.5) and is strongly purely infinite. Then, by [14, Theorem 8.6] we have that $A \cong A \otimes \mathcal{O}_\infty$. □

Lemma 3.5. *Let $\mathcal{A} = ((A(x))_{x\in X}, \Gamma)$ be a continuous field of unital Kirchberg algebras satisfying the UCT over a metrizable zero-dimensional locally compact space. Let A be the C*-algebra associated to $\mathcal{A}$. Let $\mathcal{F} \subset A$ be a finite set and let $\varepsilon > 0$. Then,* *for every $x \in X$, there is a clopen neighborhood U of x, and there is a unital Kirchberg algebra B satisfying the UCT, with $K_*(B)$ finitely generated and a $*$-homomorphism $\gamma : B \to A(U)$ such that $\mathrm{dist}(a|_U, \gamma(B)) < \varepsilon$ for all $a \in \mathcal{F}$. The distance is calculated in the* C*-*algebra $A(U)$. Another way of writing this is $\mathcal{F}|_U \subset_\varepsilon \gamma(B)$.*

Proof. Let us begin by writing $A(x)$ as inductive limit of a sequence of unital Kirchberg algebras with finitely generated K-theory satisfying the UCT, and with unital, injective connecting $*$-homomorphisms. This is possible by the classification theorem of Kirchberg and Phillips (see e.g. [21, Proposition 8.4.13]). Therefore, there is a unital Kirchberg subalgebra B with unital inclusion map $\imath : B \subset A(x)$ such that $K_*(B)$ is finitely generated, B satisfies the UCT and $\mathcal{F}|_x \subset_\varepsilon \imath(B)$. Write $\mathcal{F} = \{a_1, a_2, \ldots, a_r\}$ and let $b_1, b_2, \ldots, b_r \in B$ such that

$$\|a_i(x) - \imath(b_i)\| < \varepsilon, \quad 1 \leqslant i \leqslant r. \tag{6}$$

Let $(U_n)_{n\in\mathbb{N}}$ be a decreasing sequence (i.e. $U_{n+1} \subseteq U_n$, $n \in \mathbb{N}$), forming a basis system of compact and open neighborhoods of x. In particular $\bigcap_{n\in\mathbb{N}} U_n = \{x\}$. Therefore, by Remark 2.4, $A(x) = \varinjlim(A(U_n), \Phi_n)$, where each $*$-homomorphism $\Phi_n : A(U_n) \to A(U_{n+1})$ is the restriction map. Since $K_*(B)$ is finitely generated and B satisfies the UCT [22] implies that $KK(B, -)$ is continuous and hence there exist n and $\alpha \in KK(B, A(U_n))$ such that $[\pi_n]\alpha = [\imath] \in KK(B, A(x))$, where $\pi_n : A(U_n) \to A(x)$ is the restriction map. Now, by Lemma 3.4, $A(U_n) \cong A(U_n) \otimes \mathcal{O}_\infty$. Using this, the fact that $A(U_n)$ has an approximate unit of projections $(p_n)_{n\in\mathbb{N}}$ (see Remarks 2.5 and 3.2), and the fact that for every $k \in \mathbb{N}$, every projection in $M_k \otimes \mathcal{O}_\infty$ is (Murray–von Neumann) equivalent to a projection in $\mathcal{O}_\infty = (e_{11} \otimes 1)(M_k \otimes \mathcal{O}_\infty)(e_{11} \otimes 1)$, it follows that every projection in $M_k \otimes A(U_n) \cong M_k \otimes A(U_n) \otimes \mathcal{O}_\infty \cong \varinjlim M_k \otimes p_m A(U_n) p_m \otimes \mathcal{O}_\infty$ is equivalent to a projection in $\varinjlim p_m A(U_n) p_m \otimes \mathcal{O}_\infty \cong A(U_n) \otimes \mathcal{O}_\infty \cong A(U_n)$. Taking into account this observation and Kirchberg's theorem [21, Theorem 8.3.3], it follows that there is a $*$-homomorphism $\sigma : B \to A(U_n)$ with $[\sigma] = \alpha \in KK(B, A(U_n))$. From this we obtain that $\pi_n\sigma : B \to A(x)$ is a unital $*$-homomorphism which has the same *KK*-theory class as the unital $*$-homomorphism $\imath : B \to A(x)$. By the uniqueness part of Kirchberg's theorem, $\pi_n\sigma$ is approximately unitarily equivalent with $\imath$. In particular, there is a unitary $u_0 \in A(x)$ such that

$$\|u_0(\pi_n\sigma(b_i))u_0{}^* - \imath(b_i)\| < \varepsilon, \quad 1 \leqslant i \leqslant r. \tag{7}$$

From (6) and (7) we obtain

$$\|u_0(\pi_n\sigma(b_i))u_0{}^* - a_i(x)\| < 2\varepsilon, \quad 1 \leqslant i \leqslant r. \tag{8}$$

The above inequality holds in $A(x)$. After increasing n if necessary, we may assume that u_0 lifts to a unitary $u \in A(U_n)$, thus $\pi_n(u) = u(x) = u_0$. Using the continuity property of the norm for continuous fields of C*-algebras, we obtain from (3) after increasing n if necessary

$$\|u(y)\,\sigma(b_i)(y)\,u(y)^* - a_i(y)\| < 2\varepsilon, \quad y \in U_n, \quad 1 \leqslant i \leqslant r. \tag{9}$$

If we define $\gamma : B \to A(U_n)$ by $\gamma(b) = u\sigma(b)u^*$ for every $b \in B$ and if we let $a_i|_{U_n}$ denote the image of a_i in $A(U_n)$, then (9) becomes

$$\|\gamma(b_i) - a_i|_{U_n}\| < 2\varepsilon, \quad 1 \leqslant i \leqslant r \tag{10}$$

(since U_n is compact). Finally, note that (10) implies that $\mathcal{F}|_{U_n} \subset_{2\varepsilon} \gamma(B)$. □

Theorem 3.6. *Let $\mathcal{A} = ((A(x))_{x \in X}, \Gamma)$ be a continuous field of Kirchberg algebras satisfying the UCT over a metrizable zero-dimensional locally compact space X. Let A be the C*-algebra associated to $\mathcal{A}$. Then A admits an exhaustive sequence consisting of finite direct sums of unital Kirchberg algebras satisfying the UCT and having finitely generated K-theory groups.*

Proof. By Remarks 2.5 and 3.2 it follows that A has an approximate unit of projections $(e_n)_{n \in \mathbb{N}}$. Since $A = \overline{\cup_{n \in \mathbb{N}} e_n A e_n} \cong \varinjlim e_n A e_n$, it clearly suffices to prove the statement for each $e_n A e_n$, and thus we may assume—and we shall—that A and all the $A(x)$'s are unital. Fix an arbitrary finite subset $\mathcal{F} = \{a_1, a_2, \ldots, a_m\}$ of A and an arbitrary $\varepsilon > 0$. For each $1 \leqslant i \leqslant m$, define $K_i := \{x \in X : \|a_i(x)\| \geqslant \varepsilon\}$. Then clearly each K_i is a compact subset of X and so is $K = \cup_{i=1}^m K_i$. Using Lemma 3.5 and the compactness of K we find clopen subsets of X denoted by $U_1, U_2, \ldots, U_k$, unital Kirchberg algebras $B_1, B_2, \ldots, B_k$ satisfying the UCT with $K_*(B_j)$ finitely generated, $*$-homomorphisms $\gamma_j : B_j \to A(U_j)$, and $b_{ij} \in B_j$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant k$ such that

$$K \subseteq U := \bigcup_{i=1}^{m} U_i \tag{11}$$

and

$$\|a_i|_{U_j} - \gamma_j(b_{ij})\| \leqslant \varepsilon, \quad 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant k. \tag{12}$$

Let $V_1, V_2, \ldots, V_k$ be mutually disjoint clopen subsets of $U_1, U_2, \ldots, U_k$ such that

$$\bigcup_{j=1}^{k} V_j = \bigcup_{j=1}^{k} U_j.$$

From (12) we easily get that

$$\|a_i|_{V_j} - r_j\gamma_j(b_{ij})\| \leqslant \varepsilon, \quad 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant k,$$

where $r_j : A(U_j) \to A(V_j)$ is the restriction $*$-homomorphism. Therefore, we may assume that $U_1, U_2, \ldots, U_k$ are mutually disjoint, and in particular

$$A(X) \cong A(X \setminus U) \oplus A(U_1) \oplus \cdots \oplus A(U_k).$$

Let $C_j = \gamma_j(B_j) \subseteq A(U_j)$ and $C = C_1 + \cdots + C_k \cong \bigoplus_{j=1}^k C_j$. Since B_j is simple, we have either $C_j \cong B_j$ or $C_j = 0$ hence $C \in \mathcal{B}_{\text{fg}-\text{uct}}$. Observe that since the U_j's are clopen, χ_{U_j} is a continuous function on X for every $1 \leqslant j \leqslant k$. Define

$$b_i = \sum_{j=1}^{k} \chi_{U_j}\gamma_j(b_{ij}) \in C, \quad 1 \leqslant i \leqslant m.$$

Now, note that (12) implies that

$$\|(a_i - b_i)|_U\| \leqslant \varepsilon, \quad 1 \leqslant i \leqslant m. \tag{13}$$

Now, if $x \in X \setminus U$ then $b_i(x) = 0$, and $x \in X \setminus K$ and hence, using the definition of K, we get $\|a_i(x)\| < \varepsilon\,(1 \leqslant i \leqslant m)$. This shows that for $x \in X \setminus U$, we have

$$\|a_i(x) - b_i(x)\| = \|a_i(x)\| < \varepsilon, \quad 1 \leqslant i \leqslant m \tag{14}$$

In conclusion, (13) and (14) taken together show that

$$\|a_i - b_i\| \leqslant \varepsilon, \quad 1 \leqslant i \leqslant m,$$

where each $b_i \in C$. Since we already argued that $C \in \mathcal{B}_{\text{fg}-\text{uct}}$, this concludes the proof. □

Corollary 3.7. *Let $\mathcal{A} = ((A(x))_{x \in X}, \Gamma)$ be a continuous field of simple separable nuclear* C*-*algebras satisfying the UCT over a metrizable zero-dimensional locally compact space X. Let A be the C*-algebra associated to $\mathcal{A}$. Then A satisfies the UCT.*

Proof. Since $\mathcal{O}_\infty$ is KK-equivalent to $\mathbb{C}$, we may replace A by $A \otimes \mathcal{O}_\infty$. By [15, Corollary 2.8] $A \otimes \mathcal{O}_\infty$ satisfies the assumptions of Theorem 3.6. By [3], any nuclear separable C*-algebra which admits an exhausting sequence of C*-subalgebras satisfying the UCT will also satisfy the UCT. □

Remark 3.8. Let X be a metrizable, zero-dimensional, locally compact space and let $\mathcal{A} = ((A(x))_{x\in X}, \Gamma)$ be a continuous field of separable C*-algebras such that each fiber $A(x)$ admits an exhaustive sequence $(A_n(x))_n$ of simple semiprojective C*-algebras. Let A be the C*-algebra associated to $\mathcal{A}$. Then A is the inductive limit of a sequence $(A_n)_n$, where each A_n is a finite direct sum of C*-algebras of the form $A_m(x)$ (m and x may vary). The proof of this statement is similar with the proof of Theorem 3.6 but it is much simpler.

4. The invariant and basic properties

We will use K-theory with coefficients. For each $m \geqslant 2$ let W_m be the Moore space obtained by attaching a two-cell to the circle by a degree m-map. Fix a base point $*$ in each of the spaces W_m. Let $C(W)$ denote the C*-algebra obtained by adding a unit to $\bigoplus_{m=2}^{\infty} C_0(W_m \setminus *)$. Similarly, let $C(W_M)$ denote the unitalization of $\bigoplus_{m=2}^{M} C_0(W_m \setminus *)$, where M is an integer $\geqslant 2$. Define

$$C = C(\mathbb{T}) \otimes C(W) \quad \text{and} \quad C_M = C(\mathbb{T}) \otimes C(W_M).$$

Note that we have natural embeddings $C(\mathbb{T}) \otimes C(W_m) \subset C_M \subset C$ $(m \leqslant M)$.

The total K-theory group of a C*-algebra A is given by

$$\underline{K}(A) = K_*(A) \oplus \bigoplus_{m=2}^{\infty} K_*(A; \mathbb{Z}/m) \cong K_0(A \otimes C).$$

This group is acted on by the set of coefficient and Bockstein operations denoted by Λ. It is useful to consider the following direct summand of $\underline{K}(A)$:

$$\underline{K}(A)_M = K_*(A) \oplus \bigoplus_{m=2}^{M} K_*(A; \mathbb{Z}/m) \cong K_0(A \otimes C_M).$$

Remark 4.1. Assume that A is a separable C*-algebra satisfying the UCT. If M annihilates the torsion part of $K_*(A)$, i.e. $M \operatorname{Tors} K_*(A) = 0$, then the map

$$\operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \to \operatorname{Hom}_\Lambda(\underline{K}(A)_M, \underline{K}(B)_M)$$

induced by the restriction map $\underline{K}(A) \to \underline{K}(A)_M$ is bijective for any σ-unital C*-algebra B [7, Corollary 2.11]. If moreover $K_*(A)$ is a finitely generated group, then $\underline{K}(A)$ is finitely generated as a Λ-module [6, Proposition 4.13]. More precisely there are $x_1, \ldots, x_r \in \underline{K}(A)_M$, such that for any $x \in \underline{K}(A)$, there exist $k_1, \ldots, k_r \in \mathbb{Z}$ and $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that

$$x = k_1\lambda_1(x_1) + \cdots + k_r\lambda_r(x_r).$$

If A is a C*-algebra, we denote by $P(A)$ the Murray–von Neumann abelian semigroup consisting of equivalence classes of projections in $A \otimes \mathcal{K}$.

We introduce two homotopy invariants, $\mathrm{Inv}(A)$ which will be used for stable C*-algebras A and $\mathrm{Inv}_u(A)$ in the general case. To this purpose we consider the pair $(J(A), J(A)^+)$ consisting of a semigroup $J(A)$ together with a distinguished subsemigroup $J(A)^+ \subset J(A)$ (called the positive subsemigroup of $J(A)$). This is defined as follows. Consider the map

$$\rho_0 : P(A \otimes C) \to P(A \otimes \mathcal{O}_2)$$

induced by the unital $*$-homomorphism $C \to \mathcal{O}_2$, $f \to \omega(f)1_{\mathcal{O}_2}$ (where ω is a fixed character of the commutative C*-algebra C). Since the spectrum of C is path connected, it follows that ρ_0 is independent on the choice of the character ω. We define

$$J(A) := P(A \otimes \mathcal{O}_2) \oplus \underline{K}(A)$$

and define $J(A)^+$ to be the image of the map

$$\rho : P(A \otimes C) \to P(A \otimes \mathcal{O}_2) \oplus \underline{K}(A) = J(A),$$

$$\rho[p] := \rho_0[p] \oplus [p]_{\underline{K}(A)}.$$

$J(A)$ is a semigroup with unit $(0, 0)$ and is also a Λ-module (Bockstein and coefficient operations), where Λ acts only on the second component $\underline{K}(A)$. The pair $(J(A), J(A)^+)$ together with the action of Λ on $J(A)$ is denoted by $\mathrm{Inv}(A)$.

In abstract terms, the invariant consists of a $\mathbb{Z}/2$-graded abelian semigroup $J = J^{(0)} \oplus J^{(1)}$ where $J^{(1)}$ is in fact a graded group acted by Λ, together with a distinguished subsemigroup $J^+ \subset J^{(0)} \oplus J^{(1)}$. For a C*-algebra A we have $J(A)^{(0)} = P(A \otimes \mathcal{O}_2)$ and $J(A)^{(1)} = \underline{K}(A)$.

Note that a $*$-homomorphism $\varphi : A \to B$ induces a morphism of the invariant

$$\mathrm{Inv}(\varphi) : \mathrm{Inv}(A) \to \mathrm{Inv}(B)$$

in the sense that $\mathrm{Inv}(\varphi)$ is a morphism of graded semigroups which is Λ-linear and preserves the positive subsemigroups. We express these properties by saying that $\mathrm{Inv}(\varphi)$ is positive and Λ-linear. More precisely:

Definition 4.2. A map $\alpha : \mathrm{Inv}(A) \to \mathrm{Inv}(B)$ is said to be positive and Λ-linear if it has two components

$$\alpha^{(0)} : P(A \otimes \mathcal{O}_2) \to P(B \otimes \mathcal{O}_2)$$

with $\alpha^{(0)}$ a unit preserving morphism of semigroups ($\alpha^{(0)}(0) = 0$) and

$$\alpha^{(1)} : \underline{K}(A) \to \underline{K}(B),$$

where $\alpha^{(1)}$ is graded morphism of Λ-modules and $\alpha(J(A)^+) \subseteq J(B)^+$.

It is clear that $\mathrm{Inv}(A)$ does not distinguish between a C*-algebra A and its stabilization $A \otimes \mathcal{K}$. Following [19], we consider the set $P_u(A)$ consisting of unitary equivalence classes of projections in A, where the unitaries are from the unitalization $\widetilde{A}$ if A is nonunital. As in [19], $P_u(A)$ is equipped with its family of all finite orthogonal sets. A finer invariant is obtained by enlarging $\mathrm{Inv}(A)$ to

$$\mathrm{Inv}_u(A) = (\mathrm{Inv}(A), P_u(A), \tau),$$

where $\tau : P_u(A) \to \mathrm{Inv}(A)$ is given by the composition of the maps $P_u(A) \to P(A \otimes C)$, $[p]_u \mapsto [p \otimes 1_C]$, and $\rho : P(A \otimes C) \to \mathrm{Inv}(A)$.

A morphism $\alpha : \mathrm{Inv}_u(A) \to \mathrm{Inv}_u(B)$ consists, in addition to the components $\alpha^{(i)}$ from Definition 4.2, of a map $\alpha_u : P_u(A) \to P_u(B)$ which maps orthogonal sets to orthogonal sets and makes the following diagram commutative:

$$\begin{array}{ccc} P_u(A) & \xrightarrow{\alpha_u} & P_u(B) \\ \tau \downarrow & & \downarrow \tau \\ \mathrm{Inv}(A) & \xrightarrow[(\alpha^{(0)}, \alpha^{(1)})]{} & \mathrm{Inv}(B) \end{array}$$

Lemma 4.3. *If A is a Kirchberg algebra, then*

(a) $P(A \otimes \mathcal{O}_2) = P(\mathcal{O}_2) = \{0, \infty\}$,

(b) $J(A) = \{0, \infty\} \oplus \underline{K}(A)$,

(c) $J(A)^+ = \{(0, 0)\} \cup \{(\infty, x) : x \in \underline{K}(A)\}$,

(d) *the map $\rho : P(A \otimes C) \to J(A)$ is injective. Moreover, ρ is injective even when A is a finite direct sum of Kirchberg algebras and even when $A \in \mathcal{L}$.*

Proof. (a) A is either unital or $A \cong A_0 \otimes \mathcal{K}$, where A_0 is a unital Kirchberg algebra (see Remark 3.2). Since $P(B) = P(B \otimes \mathcal{K})$ for every C*-algebra B, if follows that we may assume that A is a unital Kirchberg algebra. But then, since A is simple, separable, unital and nuclear, a remarkable result of Kirchberg ([21, Theorem 7.1.2]) implies that $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. Therefore $P(A \otimes \mathcal{O}_2) = P(\mathcal{O}_2)$. Since any two nonzero projections in $\mathcal{O}_2 \otimes \mathcal{K}$ are equivalent, it follows that $P(\mathcal{O}_2) = \{0, \infty\}$. In conclusion, $P(A \otimes \mathcal{O}_2) = P(\mathcal{O}_2) = \{0, \infty\}$.

(b) This follows immediately from (a).

(c) Observe that if B is a simple C*-algebra and X is a compact, connected space, then every nonzero projection p in $B \otimes C(X)$ is full. To show that $J(A)^+ := \rho(P(A \otimes C)) \subseteq \{(0, 0)\} \cup \{(\infty, x) : x \in \underline{K}(A)\}$, note that if $\rho_0[p] = 0$ for some $p \in P(A \otimes C)$, then

$p = 0$. To prove the opposite inclusion, observe first that $(0, 0) = \rho([0]) \in J(A)^+$. Now fix an arbitrary $x \in \underline{K}(A)$. Since $A \otimes \mathcal{O}_\infty \cong A$, it follows that $(A \otimes C) \otimes \mathcal{O}_\infty \cong A \otimes C$ and hence $A \otimes C$ is purely infinite (use [13, Proposition 4.5]). Let p be an arbitrary nonzero projection of A. Then, by [13, Theorem 4.16] it follows that $p \otimes 1_C$ is a properly infinite projection in $A \otimes C$. Moreover, since A is simple, it follows that $p \otimes 1_C$ is full in $A \otimes C$. Then, by a result of Cuntz [2] (see also [1, Lemma 4.15] it follows that there is a full (and properly infinite) projection e in $A \otimes C$ such that $[e]_{\underline{K}(A)} = x \in \underline{K}(A) = K_0(A \otimes C)$. Observe also that since e is full in $A \otimes C$, we have $\rho_0[e] = \infty$. In conclusion, we have $\rho[e] = \rho_0[e] \otimes [e]_{\underline{K}(A)} = (\infty, x)$.

(d) First we consider the case when A is a Kirchberg algebra. Let p, q be projections in $A \otimes C \otimes \mathcal{K}$ such that $\rho[p] = \rho[q]$. Since $(A \otimes C \otimes \mathcal{K}) \otimes \mathcal{O}_\infty \cong A \otimes C \otimes \mathcal{K}$ (because $A \otimes \mathcal{O}_\infty \cong A$) and since $A \otimes C \otimes \mathcal{K}$ has an approximate unit of projections, it follows (as in the proof of Lemma 3.5) that p and q are equivalent with projections in $A \otimes C$. Hence, we may assume that $p, q \in A \otimes C$. Since $\rho_0[p] = \rho_0[q]$ it follows that p and q are simultaneously zero or nonzero. It suffices to consider the case when they are both nonzero. In the proof of (c), we showed that $A \otimes C$ is purely infinite. Then, by [13, Theorem 4.16] it follows that p and q (being nonzero) are properly infinite projections in $A \otimes C$. Since, as observed above, they are also full, results of Cuntz [2] (see also [1, Lemma 4.15]) allow us to conclude from $[p]_{\underline{K}(A)} = [p]_{K_0(A \otimes C)} = [q]_{K_0(A \otimes C)} = [q]_{\underline{K}(A)}$ that p and q are equivalent in $A \otimes C$, i.e. $[p] = [q]$ in $P(A \otimes C)$. Hence ρ is injective.

In the case $A = \oplus_{i=1}^n A_i$ where each A_i is a Kirchberg algebra, the fact that the map $\rho : P(A \otimes C) \to J(A)$ is injective follows from the fact that ρ is injective if A is simple (the above case) and the observation that ρ and P are additive with respect to direct sums:

$$\rho = \oplus_{i=1}^n \rho_i : P(A \otimes C) = \oplus_{i=1}^n P(A_i \otimes C) \to J(A) = \oplus_{i=1}^n J(A_i).$$

Let B be a C*-algebra with an exhausting sequence (B_n) of C*-subalgebras. Using functional calculus, one shows that any partial isometry in B can be approximated by partial isometries in B_n's. Applying this observation to $A \otimes C$ and using the previous cases we obtain that ρ is also injective for $A \in \mathcal{L}$. □

Lemma 4.4. *If $A = A_1 \oplus \cdots \oplus A_m$ with A_i Kirchberg algebras, then*

(a) $J(A)^+ = \{((r_1, x_1), \ldots, (r_m, x_m)) : r_i \in P(A_i \otimes \mathcal{O}_2), x_i \in \underline{K}(A_i), r_i = 0 \Rightarrow x_i = 0\}$.

In particular this shows that $J(A)^+$ is Λ-invariant.

(b) *If A satisfies the UCT and $K_*(A)$ is finitely generated, then* $\mathrm{Inv}(A)$ *has the following semi-projectivity like property. Let $B = B_1 \oplus \cdots \oplus B_n$ with B_i Kirchberg algebras. Let $\alpha : J(A) \to J(B)$ be a Λ-linear morphism of graded semigroups. Then there are finitely many elements $\{t_1, \ldots, t_r\}$ in $J(A)^+$ with the property that if $\alpha(t_j) \in J(B)^+$ for all $1 \leqslant j \leqslant r$, then $\alpha(J(A)^+) \subset J(B)^+$.*

Proof. The first part follows from Lemma 4.3. To argue for the second part, it suffices to assume that both A and B are Kirchberg algebras and A satisfies the UCT. Since $K_*(A)$ is finitely generated, it follows that $\underline{K}(A)$ is finitely generated as a Λ-module [6, Proposition 4.13]. Let $\{x_1, \ldots, x_r\}$ be a list of generators. Then $t_j := (\infty, x_j) \in J(A)^+$ will satisfy the required property. Indeed, if $t \in J(A)^+, t \neq (0, 0)$, then $t = (\infty, x)$ for some $x \in \underline{K}(A)$ and $x = \sum_j \lambda_j x_j$ for some $\lambda_j \in \Lambda$. Thus

$$t = (\infty, x) = \left(\infty, \sum_j \lambda_j x_j\right) = \sum_j \lambda_j (\infty, x_j) = \sum_j \lambda_j t_j$$

and

$$\alpha(t) = \sum_j \lambda_j \alpha(t_j) \in J(B)^+$$

since $\alpha(t_j) \in J(B)^+$ (by hypothesis) and $\lambda_j \alpha(t_j) \in J(B)^+$, because $J(B)^+$ is Λ-invariant. Since $\alpha((0, 0)) = (0, 0) \in J(B)^+$, the proof is complete. □

Lemma 4.5. *Let B be a separable C*-algebra with exhaustive sequence (B_n). For any $z_1, \ldots, z_k \in J(B)^+$, there exist m and $y_1, \ldots, y_k \in J(B_m)^+$ such that map $J(\jmath_m) : J(B_m) \to J(B)$ satisfies $J(\jmath_m)(y_i) = z_i$.*

Proof. This follows by functional calculus like in the proof of continuity of K_0. □

Lemma 4.6. *If A is a separable C*-algebra with an exhausting sequence (A_n) and all A_n have finitely generated K-theory and satisfy the UCT, then there is a subsequence $(A_{r(n)})$ of (A_n) and Λ-linear morphisms $v_n : \underline{K}(A_{r(n)}) \to \underline{K}(A_{r(n+1)})$ such that such that for each n the diagram*

$$\begin{array}{ccc} \underline{K}(A_{r(n)}) & \xrightarrow{\nu_n} & \underline{K}(A_{r(n+1)}) \\ & \searrow^{\underline{K}(\imath_{r(n)})} & \downarrow{\scriptstyle \underline{K}(\imath_{r(n+1)})} \\ & & \underline{K}(A) \end{array}$$

is commutative and the induced map $\varinjlim(\underline{K}(A_{r(n)}), v_n) \to \underline{K}(A)$ is an isomorphism of Λ-modules.

Proof. If A is a C*-algebra with $K_*(A)$ finitely generated and satisfies the UCT, then $\underline{K}(A)$ is finitely generated as a Λ-module, actually generated by the group generators of $\underline{K}(A)_M$ for some M, see Remark 4.1. On the other hand, only finitely many elements of Λ act on $\underline{K}(A)_M$. Hence $\underline{K}(A)_M$ is a finitely presented Λ-module. That is, one has the group relations among generators and a finite number of relations involving finitely many Bockstein operations. These observations apply to $A_{r(n)}$. Therefore one proves this lemma similarly to the proof of Lemma 4.5. At each stage one produces a diagram as in the statement which commutes when restricted to the $\underline{K}(A_{r(n)})_{M(n)}$, where $M(n)\,\mathrm{Tors}\,K_*(A_{r(n)}) = 0$. Finally one applies Remark 4.1. □

Lemma 4.7. *Let A be a C*-algebra which admits an exhaustive sequence (A_n) such that each A_n is a finite direct sum of Kirchberg algebras satisfying the UCT and with finitely generated K-theory. Then after passing to a subsequence of (A_n) (if necessary) there is an inductive system*

$$\cdots (J(A_n), J(A_n)^+) \xrightarrow{\nu_n} (J(A_{n+1}), J(A_{n+1})^+) \cdots$$

such that for each n, ν_n is positive and Λ-linear, the diagram

$$\begin{array}{ccc} J(A_n) & \xrightarrow{\nu_n} & J(A_{n+1}) \\ & \searrow^{J(\imath_n)} & \downarrow{\scriptstyle J(\imath_{n+1})} \\ & & J(A) \end{array}$$

is commutative and the induced map $\varinjlim((J(A_n), J(A_n)^+), \nu_n) \to (J(A), J(A)^+)$ is a positive Λ-linear isomorphism. One can replace $(J(-), J(-)^+)$ by the alternate notation $\mathrm{Inv}(-)$ everywhere in the statement of this lemma, since the action of Λ was considered.

Proof. This follows by putting together Lemmas 4.4–4.6. The property given in part (b) of Lemma 4.4 is crucial since it allows to insure positivity of Λ-linear maps by obtaining it only for finitely many elements. □

Lemma 4.8. *Let A be a C^*-algebra which admits an exhaustive sequence (A_n) such that each A_n is a finite direct sum of Kirchberg algebras satisfying the UCT and with finitely generated K-theory. Let B be a C^*-algebra which admits an exhaustive sequence (B_n) such that each B_n is a finite direct sum of Kirchberg algebras. If $\alpha : \mathrm{Inv}(A) \to \mathrm{Inv}(B)$ is a positive Λ-linear map, for any n, there exist $m = m(n)$ and a positive Λ-linear map $\alpha_n : \mathrm{Inv}(A_n) \to \mathrm{Inv}(B_m)$ such that the diagram*

$$\begin{array}{ccc} \mathrm{Inv}(A_n) & \xrightarrow{\mathrm{Inv}(\imath_n)} & \mathrm{Inv}(A) \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha} \\ \mathrm{Inv}(B_m) & \xrightarrow[\mathrm{Inv}(\jmath_m)]{} & \mathrm{Inv}(B) \end{array}$$

is commutative.

Proof. This is just a repetition of the proof of Lemma 4.7. The crucial property of $\mathrm{Inv}(-)$ is that it is "finitely presented" when applied to finite direct sums of Kirchberg algebras with finitely generated K-theory. □

Remark 4.9. If A is a unital Kirchberg algebra, then

$$P_u(A) \cong \{[0]_u, [1]_u\} \sqcup K_0(A),$$

by [2,19]. It is then clear that by using similar arguments one shows that $\mathrm{Inv}_u(-)$ satisfies statements analog to Lemmas 4.7 and 4.8.

5. Classification results

We begin by recalling some terminology and definitions from [5] which will be important in what follows.

Definition 5.1 (*[5]*). Let A be a C*-algebra. A $\underline{K}$-triple $(\mathcal{P}, \mathcal{G}, \delta)$ consists of a finite subset $\mathcal{P}$ of projections,

$$\mathcal{P} \subseteq \bigcup_{m \geqslant 1} \mathrm{Proj}(A \otimes C(\mathbb{T}) \otimes C(W_m) \otimes \mathcal{K}),$$

where the W_m's are the Moore spaces of order m, and a finite subset $\mathcal{G} \subseteq A$, and $\delta > 0$ chosen such that whenever φ is a completely positive contraction which is δ-multiplicative on $\mathcal{G}$, then $e = (\varphi \otimes \mathrm{id})(p)$ is almost a projection in the sense that $\|e^2 - e\| < 1/4$ and in particular

$$\tfrac{1}{2} \notin sp((\varphi \otimes \mathrm{id})(p))$$

for each $p \in \mathcal{P}$, where id is the identity of $C(\mathbb{T}) \otimes C(W_m) \otimes \mathcal{K}$ for suitable m.

Definition 5.2 (*[5, Definition 3.9]*). Let $(\mathcal{P}, \mathcal{G}, \delta)$ be a $\underline{K}$-triple, and assume that $\varphi : A \to B$ is a completely positive contraction which is δ-multiplicative on $\mathcal{G}$. We define $\varphi_\sharp(p) = [\chi_0(\varphi \otimes \mathrm{id})(p)]$ where $\chi_0 : [0, 1] \setminus \{1/2\} \to [0, 1]$ is 0 on $[0, 1/2)$ and 1 on $(1/2, 1]$. It is clear that if $(\mathcal{P}, \mathcal{G}, \delta)$ is a $\underline{K}(A)$-triple one also has a natural map $\varphi_\sharp : \mathcal{P} \to P(B \otimes C)$ whenever $\varphi : A \to B$ is a completely positive contraction which is δ-multiplicative on $\mathcal{G}$. Hence, one can define in this way $\varphi_\sharp : \mathcal{P} \to \mathrm{Inv}(B)$ whenever $(\mathcal{P}, \mathcal{G}, \delta)$ is a $\underline{K}(A)$ -triple and $\varphi : A \to B$ is a completely positive contraction which is δ-multiplicative on $\mathcal{G}$. Similarly one defines a map $\varphi_\sharp : \mathcal{P} \to P_u(B)$.

We need the following result.

Theorem 5.3. *Assume that A is a unital Kirchberg algebra which satisfies the UCT. For any finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$, there is a $\underline{K}(A)$-triple $(\mathcal{P}, \mathcal{G}, \delta)$ with the following property. For any unital Kirchberg algebra B and any $(\mathcal{G}, \delta)$-multiplicative completely positive contractions $\varphi, \psi : A \to B$ with $\varphi_\sharp(p) = \psi_\sharp(p) \in \underline{K}(B)$ for all $p \in \mathcal{P}$, there is an unitary $u \in B$ such that $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.*

Proof. This is Theorem 6.20 of [5]. In addition we may work with a set of projections $\mathcal{P} \subset A \otimes C$ rather than $\mathcal{P} \subset A \otimes C \otimes \mathcal{K}$. □

Corollary 5.4. *Assume that A is a finite direct sum of Kirchberg algebras that satisfy the UCT. For any finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$, there is a $\underline{K}(A)$-triple $(\mathcal{P}, \mathcal{G}, \delta)$ with the following property. For any finite direct sum of stable Kirchberg algebras B and any $(\mathcal{G}, \delta)$-multiplicative completely positive contractions $\varphi, \psi : A \to B$ with $\varphi_\sharp(p) = \psi_\sharp(p) \in \mathrm{Inv}(B)$ for all $p \in \mathcal{P}$, there is a unitary $u \in \widetilde{B}$ (the unitalization of B) such that $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.*

Proof. First observe that we may assume that B is simple (otherwise we compose with the projection onto each direct summand). Now, since every Kirchberg algebra has an approximate unit of projections, we may assume that $A = \oplus_{i=1}^k A_i$ where each A_i has a unit denoted by e_i. We choose $\mathcal{P}$ such that $e_i \in \mathcal{P}, 1 \leqslant i \leqslant k$. After a small perturbation we may assume that $(\varphi(e_i))_{i=1}^k$ and $(\psi(e_i))_{i=1}^k$ are finite sequences consisting each of mutually orthogonal projections. Since $\varphi(e_i)$ and $\psi(e_i)$ have the same class in $P(B \otimes \mathcal{O}_2)$ they are simultaneously zero or nonzero. Since they have the same class in $\underline{K}(B)$, in particular they have the same class in $K_0(B)$. But B is purely infinite and simple. By results of Cuntz [2], it follows that $\varphi(e_i)$ is equivalent to $\psi(e_i)$, $1 \leqslant i \leqslant k$. Hence, after conjugating φ with a suitable partial isometry $v \in B$, we may assume that $\varphi(e_i) = \psi(e_i), 1 \leqslant i \leqslant k$. Since B is stable, this partial isometry extends to a unitary $u \in \widetilde{B}$. Next, we apply Theorem 5.3 for the restriction of the two given maps to $A_i \to \varphi(e_i)B\varphi(e_i), 1 \leqslant i \leqslant k$. Thus we obtain a unitary $w \in eBe$, where $e = \varphi(e_1) + \cdots + \varphi(e_k)$, with $\|w\varphi(a)w^* - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$. Finally we choose $u = 1 - e + w \in U(\widetilde{B})$. □

The following is our "uniqueness" result

Proposition 5.5. *Assume that $A \in \mathcal{L}_{\mathrm{uct}}$. For any finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$, there is a $\underline{K}(A)$-triple $(\mathcal{P}, \mathcal{G}, \delta)$ with the following property. For any stable $B \in \mathcal{L}$ and any $(\mathcal{G}, \delta)$-multiplicative completely positive contractions $\varphi, \psi : A \to B$ with $\varphi_\sharp(p) = \psi_\sharp(p) \in \mathrm{Inv}(B)$ for all $p \in \mathcal{P}$, there is a unitary $u \in \widetilde{B}$ such that $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$. If B is not assumed to be stable, the conclusion remains valid if we assume that $\varphi_\sharp(p) \in P_u(B)$ for all $p \in \mathcal{P} \cap A$.*

Proof. Let (A_n) be an exhaustive sequence for A with $A_n \in \mathcal{B}_{\mathrm{uct}}$. It suffices to prove the statement for the restrictions of φ and ψ to A_n. Thus we reduced the proof to the case when $A \in \mathcal{B}_{\mathrm{uct}}$.

Using Lemma 2.2 we find an exhausting sequence (B_n) for B with $B_n \in \mathcal{B}$ and a sequence of completely positive contractions $\pi_n : B \to B_n$ which is asymptotically multiplicative and such that

$$\lim_{n\to\infty} \|\jmath_n \pi_n(b) - b\| = 0, \quad \forall\, b \in B. \tag{15}$$

Define for every $n \in \mathbb{N}, \varphi_n, \psi_n : A \to B_n$ by $\varphi_n := \pi_n\varphi$, $\psi_n := \pi_n\psi$. Then (15) implies that for $\forall\, a \in A$, we have

$$\lim_{n\to\infty} \|\jmath_n \varphi_n(a) - \varphi(a)\| = 0.$$

A similar property is satisfied by ψ_n. Replacing φ with φ_n and similarly, ψ with ψ_n, for large n, we reduce the proof of the proposition to Corollary 5.4. □

Lemma 5.6. Let $A \in \mathcal{B}_{\text{fg}-\text{uct}}$ *and let* B *be an finite direct sum of Kirchberg algebras. Then any positive and* Λ*-linear morphism* $\alpha : \mathrm{Inv}(A) \to \mathrm{Inv}(B)$ *lifts to a* $*$*-homomorphism* $\varphi : A \to B$.

Proof. We may suppose that B is a Kirchberg algebra. Write $A = \bigoplus_{i=1}^k A_i$ with $A_i \in \mathcal{E}_{\text{fg}-\text{uct}}$. Then, clearly

$$\mathrm{Inv}(A) = \bigoplus_{i=1}^{k} \mathrm{Inv}(A_i)$$

and correspondingly, α_i is the restriction of α to $\mathrm{Inv}(A_i)$. All we have to do is to lift each α_i to a $*$-homomorphism $\varphi_i : A_i \to B$. Then, since B is purely infinite and simple, using [2] we may assume, after conjugating each φ_i with a suitable partial isometry in B, that $\varphi_i(1_{A_i})\varphi_j(1_{A_j}) = 0$ whenever $i \neq j$. Finally, we shall define $\varphi : A \to B$ as the direct sum of the φ_i's. Now, as explained earlier

$$J(A_i)^+ = \{(0,0)\} \cup \{\infty \oplus \underline{K}(A_i)\} \subset P(A_i \otimes \mathcal{O}_2) \oplus \underline{K}(A_i)$$

and

$$J(B)^+ = \{(0,0)\} \cup \{\infty \oplus \underline{K}(B)\} \subset P(B \otimes \mathcal{O}_2) \oplus \underline{K}(B).$$

The map α_i has components $\alpha_i^{(0)} : P(A_i \otimes \mathcal{O}_2) \to P(B \otimes \mathcal{O}_2)$ and $\alpha_i^{(1)} : \underline{K}(A_i) \to \underline{K}(B)$,

$$\alpha_i = \begin{pmatrix} \alpha_i^{(0)} & 0 \\ 0 & \alpha_i^{(1)} \end{pmatrix}.$$

Therefore the condition $\alpha_i(J(A_i)^+) \subseteq J(B)^+$ shows that $\alpha_i^{(1)} = 0$ whenever $\alpha_i^{(0)}(\infty) = 0$. If $\alpha_i^{(0)}(\infty) = 0$, then $\alpha_i^{(0)} = 0$ and we set $\varphi_i = 0$, $1 \leqslant i \leqslant n$. If $\alpha_i^{(0)}(\infty) = \infty$, we apply the UMCT of [7] to lift $\alpha_i^{(1)}$ to an element β_i of $KK(A_i, B)$. Since $K_*(A_i)$ is finitely generated, A_i satisfies the *UCT* and B has a countable approximate unit of projections, by using the continuity of $KK(A_i, -)$ we may assume that B is unital. Now, using [21, Theorem 8.3.3], we lift β_i to a full $*$-homomorphism $\varphi_i' : A_i \to B \otimes \mathcal{K}$. But, notice that every projection in a matrix algebra over B is equivalent to a projection in $B \subset B \otimes \mathcal{K}$ (since $B \otimes \mathcal{O}_\infty \cong B$ and B has an approximate unit of projections). Hence, there is a partial isometry $v_i \in B \otimes \mathcal{K}$ such that $v_i^* v_i = \varphi_i'(1_{A_i})$ and $v_i v_i^* \leqslant 1_B$. Define $\varphi_i : A_i \to B$ by $\varphi_i := v_i \varphi_i' v_i^*$. Then, clearly, $[\beta_i]$ lifts to φ_i and hence $\alpha_i^{(1)}$ lifts to φ_i. It is clear that $\varphi_i : A_i \to B$ is injective (φ_i is nonzero and A_i is unital) and hence φ_i also lifts $\alpha_i^{(0)}$. $\square$

The following is our "existence" result:

Proposition 5.7. *Assume that $A \in \mathcal{L}_{\mathrm{uct}}$. For any $\underline{\mathrm{K}}(A)$-triple $(\mathcal{P}, \mathcal{G}, \delta)$, any $B \in \mathcal{L}$ and any positive Λ-linear maps $\alpha : \mathrm{Inv}(A) \to \mathrm{Inv}(B)$, there is a $(\mathcal{G}, \delta)$-multiplicative completely positive contraction $\varphi : A \to B$ such that $\varphi_\sharp(p) = \alpha([p])$ for all $p \in \mathcal{P}$.*

Proof. Note that $\mathcal{E}_{\mathrm{uct}} \subseteq \mathcal{L}_{\mathrm{fg-uct}}$ (see e.g. [21, Proposition 8.4.13]) hence $\mathcal{L}_{\mathrm{uct}} = \mathcal{L}_{\mathrm{fg-uct}}$. Let (A_n) be an exhaustive sequence for A, such that $A_n \in \mathcal{B}_{\mathrm{fg-uct}}$ for each n. Passing to a subsequence, we may assume that (A_n) satisfies (1). By Lemma 2.2, there is a sequence of completely positive contractions $\mu_n : A \to A_n$ which is asymptotically multiplicative and such that

$$\lim_{n\to\infty} \| \imath_n \mu_n(a) - a \| = 0, \quad \forall a \in A. \tag{16}$$

Applying Lemma 4.8, we obtain a commutative diagram:

$$\begin{array}{ccc} \mathrm{Inv}(A_n) & \xrightarrow{\mathrm{Inv}(\imath_n)} & \mathrm{Inv}(A) \\ {\scriptstyle \alpha_n}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathrm{Inv}(B_m) & \xrightarrow[\mathrm{Inv}(\jmath_m)]{} & \mathrm{Inv}(B) \end{array}$$

where $\alpha_n : \mathrm{Inv}(A_n) \to \mathrm{Inv}(B_m)$ is positive and Λ-linear, for some $m = m(n)$. Next we lift α_n to a $*$-homomorphism $\varphi_{n,m}' : A_n \to B_m$ using Lemma 5.6. Finally, we set $\varphi_{n,m} := \jmath_m \varphi_{n,m}' \mu_n : A \to B$. Using (16), one verifies that if n is large enough, $\varphi := \varphi_{n,m}$ will satisfy the conclusion of the theorem. $\square$

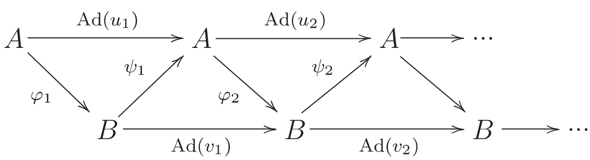
Theorem 5.8. *Let A, B be stable C^*-algebras which admit exhaustive sequences consisting of finite direct sums of Kirchberg algebras satisfying the UCT. Then A is isomorphic to B if and only if $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$ as Λ-modules. Moreover, if $A \in \mathcal{L}_{\mathrm{uct}}$ and*

$B \in \mathcal{L}$, then the canonical map

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}_{\Lambda}(\mathrm{Inv}(A), \mathrm{Inv}(B))$$

is surjective. Two $*$-homomorphisms from A to B induce the same map from $\mathrm{Inv}(A)$ *to* $\mathrm{Inv}(B)$ *if and only if they are approximately unitarily equivalent.*

Proof. We will only prove the first part, as the second part is similar. Fix a positive Λ-linear isomorphism $\alpha : \mathrm{Inv}(A) \to \mathrm{Inv}(B)$. We may apply the "existence" result Proposition 5.7 to get completely positive contractions $\varphi_i : A \to B$ and $\psi_i : B \to A$ which are increasingly multiplicative on larger and larger sets, and induce α and α^{-1}, respectively, on larger and larger subsets of $\mathrm{Inv}(A)$ and $\mathrm{Inv}(B)$. Arranging this appropriately, we may conclude by our "uniqueness" result Proposition 5.5 that there are unitaries u_n and v_n making



an approximate intertwining in the sense of Elliott [9]. Hence

$$A \cong \varinjlim(A, \mathrm{Ad}(u_n)) \cong \varinjlim(B, \mathrm{Ad}(v_n)) \cong B. \qquad \square$$

Theorem 5.9. *Let A, B be stable C*-algebras associated to continuous fields of Kirchberg algebras satisfying the UCT over zero-dimensional metrizable locally compact spaces. Then A is isomorphic to B if and only if* $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$.

Proof. This follows from Theorems 3.6 and 5.8. $\square$

Corollary 5.10. *Let A, B be a separable nuclear C*-algebras with zero-dimensional Hausdorff primitive spectra. Assume that all simple quotients of A and B satisfy the UCT. The following assertions are equivalent*:

(a) $A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$;
(b) $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$;
(c) *A is KK_X-equivalent to B.*

Proof. Equivalence (a) $\Leftrightarrow$ (c) is due to Kirchberg and holds in much more generality [12]. The novelty here is (a) $\Leftrightarrow$ (b) for which we give a direct proof and hence obtain an algebraic criterion for when (c) happens to hold. Let $X := \mathrm{Prim}(A)$. By Fell's theorem [10], A is isomorphic to a the C*-algebra associated to a continuous field of C*-algebras $\mathcal{A} = ((A(x))_{x \in X}, \Gamma)$ with each $A(x)$ simple. Then $A \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ is a C*-algebra satisfying the assumptions of Theorem 5.9 and $B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ has a similar property, see [15, Corollary 2.8]. After this preliminary discussion note that (a) $\Leftrightarrow$ (b) follows from Theorem 5.9. □

Corollary 5.11. *Let A be a stable C^*-algebra which admits an exhaustive sequence consisting of finite direct sums of Kirchberg algebras satisfying the UCT. Then $A \cong \varinjlim A_n$, where each A_n is a finite direct sum of Kirchberg algebras satisfying the UCT and having finitely generated K-theory. In particular this applies to the stabilization of the C^*-algebra associated to a continuous field of Kirchberg algebras satisfying the UCT, over a metrizable zero-dimensional locally compact space.*

Proof. Since $A \in \mathcal{L}_{\text{uct}} = \mathcal{L}_{\text{fg}-\text{uct}}$, it follows that there is an exhausting sequence (A_n) for A, with $A_n \in \mathcal{B}_{\text{fg}-\text{uct}}$. By Lemma 4.7, after passing to a subsequence of (A_n) if necessary, we can arrange that $\text{Inv}(A) \cong \varinjlim(\text{Inv}(A_n), v_n)$, for some positive Λ-linear maps $v_n : \text{Inv}(A_n) \to \text{Inv}(A_{n+1})$ $(n \in \mathbb{N})$. Now, using Lemma 5.6, we lift each v_n to a $*$-homomorphism $\Phi_n : A_n \to A_{n+1}$. Define $B' := \varinjlim(A_n, \Phi_n)$ and $B := B' \otimes \mathcal{K}$. Since $B = \varinjlim A_n \otimes M_n$ and $A_n \otimes M_n \in \mathcal{B}_{\text{fg}}$ (since $A_n \in \mathcal{B}_{\text{fg}}$) for each n, it follows obviously that $B \in \mathcal{L}_{\text{uct}}$. Hence A, B are stable C^*-algebras in $\mathcal{L}_{\text{uct}}$ and

$$\text{Inv}(B) = \text{Inv}(B' \otimes \mathcal{K}) \cong \text{Inv}(B') \cong \varinjlim(\text{Inv}(A_n), \text{Inv}(\Phi_n))$$
$$= \varinjlim(\text{Inv}(A_n), v_n) \cong \text{Inv}(A).$$

Then Theorem 5.8 implies that $A \cong B \cong \varinjlim(A_n, \Phi_n)$, which ends the proof, since each $A_n \in \mathcal{B}_{\text{fg}-\text{uct}}$. □

Corollary 5.12. *Let A, B be stable C^*-algebras which admit exhaustive sequences consisting of finite direct sums of Kirchberg algebras satisfying the UCT. Then, the following assertions are equivalent*:

(a) *A and B are $*$-isomorphic*;
(b) *A and B are shape equivalent*;
(c) *A and B are homotopy equivalent.*

Proof. Since implications (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) are trivially true, to prove the corollary it is enough to prove that (b) $\Rightarrow$ (a) is true. $\text{Inv}(-)$ is a homotopy invariant continuous functor. Therefore if A and B are shape equivalent, then we clearly have $\text{Inv}(A) \cong \text{Inv}(B)$, which by Theorem 5.8 implies that A and B are $*$-isomorphic. □

Remark 5.13. Theorems 5.8 and 5.9 remain true for nonstable C*-algebras, provided that one replaces $\text{Inv}(-)$ by $\text{Inv}_u(-)$ in their statements. The proof is essentially the same, except for small changes as in [19]. Consequently, the stability assumptions can be dropped from Corollaries 5.11 and 5.12. For Corollary 5.11, which is probably the more interesting statement of the two, one can also verify our claim, at least in the unital case, as follows. Assume that $A \in \mathcal{L}_{\text{uct}}$ is unital. Then, by the stable case, we can write $A \otimes \mathcal{K}$ as the closure of an increasing sequence (A_n) of C*-algebras in $\mathcal{B}_{\mathrm{fg-uct}}$. Letting $p = 1_A \otimes e_{11}$, we may assume that $\|p - q\| < 1$ for some projection $q \in A_1$. Let $v \in A \otimes \mathcal{K}$ be a partial isometry with $v^*v = p$ and $vv^* = q$. Thus $\mathrm{Ad}(v) : p(A \otimes \mathcal{K})p \to q(A \otimes \mathcal{K})q$ is an isomorphism. Then

$$A \cong p(A \otimes \mathcal{K})p \cong q(A \otimes \mathcal{K})q = \overline{\cup_n q A_n q}$$

and $qA_nq \in \mathcal{B}_{\mathrm{fg-uct}}$.

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