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Homotopy after Tensoring with Uniformly Hyperfinite C^*-Algebras

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Abstract. The paper is devoted to the homotopy classification of C^*-algebras of continuous functions on a finite CW-complex with values in a UHF-algebra. The relevant invariants are based on (connective) K-theory.

Key words. (connective) K-theory, homotopy, C^*-algebras.

0. Introduction

In this paper we give results on the homotopy classification of C^*-algebras of continuous functions with values in uniformly hyperfinite C^*-algebras (UHF-algebras). These results exhibit a phenomenon specific to the homotopy theory of noncommutative C^*-algebras: the homotopy type of certain tensor product C^*-algebras $A \otimes B$ may preserve very little from the homotopy types of A and B. There is no analog of this phenomenon in commutative topology. Indeed, one can recover the homotopy type of X and Y from the homotopy type of the product space $X \times Y$.

Many applications of C^*-algebras in geometry and topology involve C^*-algebras that are 'matricially stable' in a certain sense. While certain C^*-algebras are matricially stable by their nature, some others are stabilized in order to get more flexible objects. We show that this process can radically change the homotopy type of a C^*-algebra. Suppose that X is a nice compact space, say a finite CW-complex, and let U be a UHF-algebra. Our main results are Theorems 1 and 2 below. It turns out that the homotopy type of the C^*-algebra $C(X, U) \cong U \otimes C(X)$ is determined by the *interaction* between the *combinatorial* properties of X and the *arithmetic* properties of the dimension group of U.

To illustrate an extreme case suppose that the reduced K-groups of X are torsion groups and that U is the universal UHF-algebra with $K_0(U) \cong \mathbb{Q}$. By the Künneth formula the C^*-algebras $U \otimes C(X)$ and U have isomorphic K-groups. This algebraic property is no accident but the reflection of a geometric fact. We prove that there is no homotopy invariant that can differentiate between $U \otimes C(X)$ and U. Actually, we show that the two C^*-algebras are homotopy equivalent. More elaborate results are provided by Theorems 1 and 2 below. It turns out that for any finite

CW-complex X the homotopy type of $U \otimes C(X)$ is determined by the Betti numbers of X. We are more precise for spaces X of dimension at most two or simply connected spaces of dimension at most three in which case the K-theory is shown to be a complete homotopy invariant of the tensor product of $C(X)$ by UHF-algebras. These results indicate that the homotopy type of a 'matricially stable' C^*-algebra has a K-theoretic flavor.

THEOREM 1. *Let X, Y be finite connected CW-complexes having isomorphic rational cohomology groups, i.e. $H^q(X, \mathbb{Q}) \cong H^q(Y, \mathbb{Q})$ for all $q \geqslant 0$. Let U be the UHF-algebra with $K_0(U) \cong \mathbb{Q}$. Then $U \otimes C(X)$ is homotopy equivalent to $U \otimes C(Y)$.*

Theorem 1 shows that the isomorphism of the two rational cohomology groups, which is an algebraic property, can be given a geometrical meaning provided one introduces 'noncommutative spaces'.

If the nonzero dimensional cells of X and Y are concentrated in two consecutive dimensions, Theorem 1 can be refined as follows:

THEOREM 2. *Let X, Y, be finite $(n-2)$-connected CW-complexes of dimension at most n, $n \geqslant 2$. Let V_1, V_2 be UHF-algebras such that $V_1 \otimes M_m \cong V_1$ for some integer $m \geqslant 2$. Then $V_1 \otimes C(X)$ is homotopy equivalent to $V_2 \otimes C(Y)$ if and only if $K_*(V_1 \otimes C(X))$ is isomorphic to $K_*(V_2 \otimes C(Y))$ as $\mathbb{Z}/2$-graded ordered groups.*

The main tools we use are the connective KK-groups and the stabilization theorem of [5], and the universal coefficient theorem of [16]. For other applications of connective K-theory to operator algebras see [15, 5, 13, 4, 6]. The techniques developed in [5] are based on a result of G. Segal [17].

By Theorem 1, for any finite connected CW-complex X, there is a wedge of spheres Y, such that $U \otimes C(X)$ is homotopy equivalent to $U \otimes C(Y)$. It is an open problem to decide which finite wedges of spheres give homotopy equivalent C^*-algebras after tensoring with U. In particular, I was not able to prove whether or not $U \otimes C(S^1)$ is homotopy equivalent to $U \otimes C(S^3)$. This kind of questions seems to depend on the continuity of certain filtrations on K-theory considered in [13].

Ideas from homotopy theory have led to powerful tools of investigation in the theory of C^*-algebras. Recall that the KK-theory of Kasparov [12] and the E-theory of Connes and Higson [3] are homotopy-invariant by definition. A result of Voiculescu, asserting that a C^*-algebra which is homotopically dominated by a quasidiagonal C^*-algebra is quasidiagonal shows that homotopy is essentially involved in non-commutative phenomena. For more discussion on the role of homotopy in the study of C^*-algebras, see [15, 8].

1. Preliminaries

Given unital C^*-algebras A, B let $\mathrm{Hom}(A, B)$ be the space of all unit-preserving $*$-homomorphisms from A to B endowed with the topology of pointwise normconvergence. If either A or B is nonunital we let $\mathrm{Hom}(A, B)$ denote all the $*$-homomorphisms from A to B. Two homomorphisms $\varphi, \psi \in \mathrm{Hom}(A, B)$ are called homotopic if they belong to the same path component of $\mathrm{Hom}(A, B)$. The corresponding homotopy classes are denoted by $[A, B]$. A is homotopy equivalent to B if there are $\varphi \in \mathrm{Hom}(A, B)$, $\psi \in \mathrm{Hom}(B, A)$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity maps. For a given compact space X with base point we let $C_0(X \backslash \mathrm{pt})$ denote the complex continuous functions on X vanishing at the base point. Let M_m stand for the C^*-algebra of $m \times m$ complex matrices with unit 1_m and let id_m denote its identity map.

A UHF-algebra, U can be described as the C^*-completion of an infinite tensor product algebra of the form

$$M_{p_1} \otimes M_{p_2} \otimes \cdots \otimes M_{p_k} \otimes \cdots,$$

where $p_1, p_2, \ldots, p_k, \ldots$ are prime numbers. Let P_U denote the set of all primes occurring in the above description of U and for $p \in P_U$ let $f_U(p) \in \{1, 2, \ldots \infty\}$ denote the number of occurrences of p in the sequence $p_1, p_2, \ldots, p_k, \ldots$. A result of Glimm, [10], asserts that two UHF-algebras U, V are isomorphic if and only if $P_U = P_V$ and $f_U = f_V$. Alternatively, the classification of UHF-algebras can be given in terms of K-theory, see [9, 7]: U is isomorphic to V if and only if $K_0(U)$ is isomorphic to $K_0(V)$ by an isomorphism that takes $[1_U]$ to $[1_V]$. $K_0(U)$ can be identified with a dense subgroup of the rational numbers and, conversely, any dense subgroup of the rational numbers is isomorphic to the K-theory group of some UHF-algebra. Let $m \geqslant 2$. The following assertions are equivalent:

(a) $U \otimes M_m$ is isomorphic to U,
(b) $mK_0(U) = K_0(U)$,
(c) m is a product of primes $p \in P_U$ with $f_U(p) = \infty$.

The following proposition gives a sufficient condition for two C^*-algebras to become homotopy equivalent after tensoring with a UHF-algebra. It relies on the fact that any two unital endomorphisms of a UHF-algebra are homotopic, see [2, 11].

PROPOSITION 1. *Let U be a UHF-algebra such that $U \otimes M_m \cong U$ for some integer m. Let A, B be C^*-algebras and assume that there are $\varphi \in \mathrm{Hom}(A, M_m \otimes B)$, $\psi \in \mathrm{Hom}(B, M_m \otimes A)$ such that $(\mathrm{id}_m \otimes \psi) \circ \varphi$ is homotopic to the amplification map $u \in \mathrm{Hom}(A, M_m \otimes M_m \otimes A)$, $u(a) = 1_m \otimes 1_m \otimes a$ and $(\mathrm{id}_m \otimes \varphi) \circ \psi$ is homotopic to the amplification map $v \in \mathrm{Hom}(B, M_m \otimes M_m \otimes B)$, $v(b) = 1_m \otimes 1_m \otimes b$.*

Then $U \otimes A$ is homotopy equivalent to $U \otimes B$.

Proof. Let λ_m be an isomorphism of $U \otimes M_m$ onto U and consider the following homomorphisms:

$$\tilde{\varphi} = (\lambda_m \otimes \mathrm{id}_B) \circ (\mathrm{id}_U \otimes \varphi) \in \mathrm{Hom}(U \otimes A, U \otimes B),$$

$$\tilde{\psi} = (\lambda_m \otimes \mathrm{id}_A) \circ (\mathrm{id}_U \otimes \psi) \in \mathrm{Hom}(U \otimes B, U \otimes A).$$

Due to the symmetry of our data, it suffices to prove that $\tilde{\psi} \circ \tilde{\varphi}$ is homotopic to $\mathrm{id}_U \otimes \mathrm{id}_A$.

Let λ_{m^2} denote the isomorphism $\lambda_m \circ (\lambda_m \otimes \mathrm{id}_m)$ of $U \otimes M_m \otimes M_m$ onto U, let θ stand for $(\mathrm{id}_m \otimes \psi) \circ \varphi$, and define

$$\tilde{\theta} = (\lambda_{m^2} \otimes \mathrm{id}_A) \circ (\mathrm{id}_U \otimes \theta) \in \mathrm{Hom}(U \otimes A, U \otimes A)$$

By checking on simple tensor products one sees that

$$\lambda_m \otimes \psi = (\mathrm{id}_U \otimes \psi) \circ (\lambda_m \otimes \mathrm{id}_B) = (\lambda_m \otimes \mathrm{id}_m \otimes \mathrm{id}_A) \circ (\mathrm{id}_U \otimes \mathrm{id}_m \otimes \psi).$$

This easily implies that $\tilde{\theta} = \tilde{\psi} \circ \tilde{\varphi}$. Since, by hypothesis, θ is homotopic to u, all we need to prove is that $\tilde{u}$ is homotopic to $\mathrm{id}_U \otimes \mathrm{id}_A$, where

$$\tilde{u} = (\lambda_{m^2} \otimes \mathrm{id}_A) \circ (\mathrm{id}_U \otimes u).$$

Let $\sigma \in \mathrm{Hom}(U, U \otimes M_m \otimes M_m)$ be given by $\sigma(x) = x \otimes 1_m \otimes 1_m$. One has $\tilde{u} = (\lambda_{m^2} \circ \sigma) \otimes \mathrm{id}_A$. This concludes the proof, since $\lambda_{m^2} \circ \sigma$ is an unital endomorphism of U and therefore is homotopic to id_U (see [2, 11]).

2. Some Facts about Connective *KK*-Theory

In this section we recall some results from [5] which will be used in the proofs of Theorems 1 and 2.

Let X, Y be finite connected CW-complexes. The direct sum with a fixed evaluation map $C(X) \to \mathbb{C}$ induces a map

$$[C(X), M_n \otimes C(Y)] \to [C(X), M_{n+1} \otimes C(Y)].$$

Taking direct limit over n, we define

$$kk(Y, X) = \lim_{n} \, [C(X), M_n \otimes C(Y)].$$

$kk(Y, X)$ is a group with addition induced by the direct sum of the homomorphisms. The usual suspension functor induces an isomorphism

$$kk(Y, X) \cong kk(SY, SX)$$

which is used to extend $kk(Y, X)$ to nonconnected spaces and to define the higher-order groups $\{kk_q(Y, X)\}_{q \in \mathbb{Z}}$

$$kk_q(Y, X) = \lim_{r} \, kk(S^{q+r}Y, S^r X).$$

Then $kk_q(S^0, X) = k_q(X)$ is (reduced) connective K-homology and $kk_q(Y, S^0) = k^{-q}(Y)$ is (reduced) connective K-theory. The groups $kk_q(Y, X)$ have good excision properties in both variables. One can regard kk_* as the natural connective bivariant theory associated with the Kasparov groups $KK_*(C_0(X \backslash \mathrm{pt}), C_0(Y \backslash \mathrm{pt}))$. The composition and the tensor product of homomorphisms induce a rich multiplicative structure on $kk_*(Y, X)$. For instance if $\alpha = [\varphi] \in kk(Y, X)$ is the class of $\varphi \in \mathrm{Hom}(C(X), M_n \otimes C(Y))$ and $\beta = [\psi] \in kk(Z, Y)$ is the class of $\psi \in \mathrm{Hom}(C(Y), M_m \otimes C(Z))$, then

$$\alpha\beta = [(\mathrm{id}_n \otimes \psi) \circ \varphi] \in kk(Z, X).$$

The unit of the ring $kk(X, X)$ is given by the class of the identity map of $C(X)$ and will be denoted by $[\mathrm{id}_X]$. The multiplication by the Bott element

$$t \in [C(S^1), M_2 \otimes C(S^3)] = K^1(S^3)$$

gives rise to a $\mathbb{Z}[t]$-module structure on $k^*(X)$. The Bott operation is easily described if one considers rational coefficients. Indeed, $k^q(X) \otimes \mathbb{Q}$ can be identified with $\oplus_{j \geqslant 0} \tilde{H}^{q+2j}(X, \mathbb{Q})$ such that

$$t: k^{q+2}(X) \otimes \mathbb{Q} \to k^q(X) \otimes \mathbb{Q}$$

corresponds to the canonical inclusion

$$\oplus_{j \geqslant 1} \tilde{H}^{q+2j}(X, \mathbb{Q}) \hookrightarrow \oplus_{j \geqslant 0} \tilde{H}^{q+2j}(X, \mathbb{Q}).$$

The composition of homomorphisms gives a natural map

$$\gamma: kk(Y, X) \to \mathrm{Hom}_{\mathbb{Z}[t]}(k^*(X), k^*(Y)).$$

Passing to rational coefficients one has the following proposition.

PROPOSITION 2. *Let X, Y be finite CW-complexes. The map*

$$\gamma_{\mathbb{Q}}: kk(Y, X) \otimes \mathbb{Q} \to \mathrm{Hom}_{\mathbb{Q}[t]}(k^*(X) \otimes \mathbb{Q}, k^*(Y) \otimes \mathbb{Q})$$

is an isomorphism.

Proof. This is implicitly contained in [5, Section 3.5]. A more direct proof follows if $\gamma_{\mathbb{Q}}$ is regarded as a natural transformation of homology theories

$$\gamma_{\mathbb{Q}}: kk_*(Y, X) \otimes \mathbb{Q} \to \mathrm{Hom}^*_{\mathbb{Q}[t]}(k^*(X) \otimes \mathbb{Q}, k^*(Y) \otimes \mathbb{Q})$$

that induces an isomorphism on coefficients, i.e. for $X = S^0$.

It turns out that one can identify

$$kk(Y, X) \otimes \mathbb{Q}$$

with the set of all parity-preserving morphisms

$$(\sigma_0, \sigma_1): \tilde{H}^{\mathrm{even}}(X, \mathbb{Q}) \oplus \tilde{H}^{\mathrm{odd}}(X, \mathbb{Q}) \to \tilde{H}^{\mathrm{even}}(Y, \mathbb{Q}) \oplus \tilde{H}^{\mathrm{odd}}(Y, \mathbb{Q})$$

that are upper triangular, i.e.

$$\sigma_i(\oplus_{j \geqslant 0} \tilde{H}^{q+2j}(X, \mathbb{Q})) \subset \oplus_{j \geqslant 0} \tilde{H}^{q+2j}(Y, \mathbb{Q})$$

for all $q \geqslant 0$, $i = 0, 1$.

In contrast with this, recall that

$$KK(C_0(X \setminus \mathrm{pt}), C_0(Y \setminus \mathrm{pt})) \otimes \mathbb{Q}$$
$$\cong \mathrm{Hom}(K_*(C_0(X \setminus \mathrm{pt})) \otimes \mathbb{Q}, K_*(C_0(Y \setminus \mathrm{pt})) \otimes \mathbb{Q})$$

and therefore $KK(C_0(X \setminus \mathrm{pt}), C_0(Y \setminus \mathrm{pt})) \otimes \mathbb{Q}$ can be identified with the set of *all* parity-preserving morphisms

$$(\tau_0, \tau_1): \tilde{H}^{\mathrm{even}}(X, \mathbb{Q}) \oplus \tilde{H}^{\mathrm{odd}}(X, \mathbb{Q}) \to \tilde{H}^{\mathrm{even}}(Y, \mathbb{Q}) \oplus \tilde{H}^{\mathrm{odd}}(Y, \mathbb{Q}).$$

Let $\mathcal{K}$ denote the compact operators on an infinite-dimensional separable Hilbert space. Then $kk(Y, X)$ can be described alternatively as $[C_0(X \setminus \mathrm{pt}) \otimes \mathcal{K}, C_0(Y \setminus \mathrm{pt}) \otimes \mathcal{K}]$. It follows that there is a natural transformation

$$\chi: kk(Y, X) \to KK(C_0(X \setminus \mathrm{pt}), C_0(Y \setminus \mathrm{pt}))$$

which is compatible with the above identifications. Thus, modulo torsion, we have a good image of how far $KK(C_0(X \setminus \mathrm{pt}), C_0(Y \setminus \mathrm{pt}))$ can stay from the homotopy classes of actual homomorphisms from $C_0(X \setminus \mathrm{pt}) \otimes \mathcal{K}$ to $C_0(Y \setminus \mathrm{pt}) \otimes \mathcal{K}$.

The following result of [5] will be used in the proof of Theorem 2.

PROPOSITION 3. *Assume that X, Y are finite $(n-2)$-connected CW-complexes of dimension at most n, $n \geqslant 2$. Then the canonical map*

$$\chi: kk(Y, X) \to KK(C_0(X \setminus \mathrm{pt}), C_0(Y \setminus \mathrm{pt}))$$

is an isomorphism.

We also need the following result of [5] which extends certain stability properties of vector bundles to $*$-homomorphisms.

THEOREM 3. *Let X, Y be finite connected CW-complexes. Then*

$$kk(Y, X) \cong [C(X), M_m \otimes C(Y)]$$

for any $m > 3\dim(Y)/2$.

3. The Proof of Theorem 1

PROPOSITION 4. *Let U be a UHF-algebra such that $U \otimes M_m$ is isomorphic to U for some integer $m \geqslant 2$. Let X, Y be finite connected CW-complexes and assume that there are*

$$\alpha \in kk(Y, X), \qquad \beta \in kk(X, Y)$$

such that

$$\alpha\beta = r[\mathrm{id}_X], \qquad \beta\alpha = r[\mathrm{id}_Y]$$

for some integer r dividing some power of m. Then $U \otimes C(X)$ is homotopy equivalent to $U \otimes C(Y)$.

Proof. By replacing m by m^s and α by $t\alpha$ for suitable s, t, we may assume that $\alpha\beta = m^2[\mathrm{id}_X], \beta\alpha = m^2[\mathrm{id}_Y]$ and $m > 3\dim(Y)/2$. Using Theorem 3, we find $\varphi \in \mathrm{Hom}(C(X), M_m \otimes C(Y))$ and $\psi \in \mathrm{Hom}(C(Y), M_m \otimes C(X))$ such that $[\varphi] = \alpha$ and $[\psi] = \beta$. Let u, v be the homomorphisms defined in the statement of Proposition 1 with $A = C(X)$ and $B = C(Y)$. By the very definition of addition in the kk-groups, we have $[u] = m^2[\mathrm{id}_X]$ and $[v] = m^2[\mathrm{id}_Y]$. Therefore, we get

$$[(\mathrm{id}_m \otimes \psi) \circ \varphi] = [\varphi][\psi] = m^2[\mathrm{id}_X] = [u],$$
$$[(\mathrm{id}_m \otimes \varphi) \circ \psi] = [\psi][\varphi] = m^2[\mathrm{id}_X] = [v].$$

Using once more Theorem 3, we find that $(\mathrm{id}_m \otimes \psi) \circ \varphi$ is homotopic to u and $(\mathrm{id}_m \otimes \varphi) \circ \psi$ is homotopic to v. Having these homotopies, the statement follows from Proposition 1.

PROPOSITION 5. *Let X, Y be finite connected CW-complexes and assume that $H^q(X, \mathbb{Q})$ is isomorphic to $H^q(Y, \mathbb{Q})$ for all $q \geqslant 0$. Then there are $\alpha \in kk(Y, X)$ and $\beta \in kk(X, Y)$ such that $\alpha\beta = m^2[\mathrm{id}_X]$ and $\beta\alpha = m^2[\mathrm{id}_Y]$ for some nonzero integer m.*

Proof. By the discussion following Proposition 2, there is an isomorphism $i_0 \in \mathrm{Hom}_{\mathbb{Q}[t]}(k^*(X) \otimes \mathbb{Q}, k^*(Y) \otimes \mathbb{Q})$. Let j_0 be its inverse map. Using Proposition 2, we find

$$\alpha_0 \in kk(Y, X) \otimes \mathbb{Q} \quad \text{and} \quad \beta \in kk(X, Y) \otimes \mathbb{Q}$$

such that

$$\gamma_{\mathbb{Q}}(\alpha_0) = i_0 \quad \text{and} \quad \gamma_{\mathbb{Q}}(\beta_0) = j_0.$$

The natural map $\eta\colon kk \to kk \otimes \mathbb{Q}$ given by $\eta(x) = x \otimes 1$ is not onto in general. However, there is some nonzero integer s such that $s\alpha_0$ and $s\beta_0$ lift to elements $\alpha_1 \in kk(Y, X)$ and $\beta_1 \in kk(X, Y)$, respectively. It follows that $\alpha_1\beta_1 - s^2[\mathrm{id}_X]$ and $\beta_1\alpha_1 - s^2[\mathrm{id}_Y]$ belong to the kernel of η and therefore they are torsion elements. This means that there is some nonzero integer r such that $r(\alpha_1\beta_1 - s^2[\mathrm{id}_X]) = 0$ and $r(\beta_1\alpha_1 - s^2[\mathrm{id}_Y]) = 0$. Finally, we put $\alpha = r\alpha_1, \beta = r\beta_1$ and $m = rs$.

The end of the proof of Theorem 1.

Given X, Y let α, β, m be as provided by Proposition 5. Replacing, if necessary, m by $2m$ and α by 4α, we may assume that $m \geqslant 2$. No other control on m is necessary for if U is the universal UHF-algebra, then $U \otimes M_m$ is isomorphic to U and we can apply Proposition 4 to conclude that $U \otimes C(X)$ is homotopy equivalent to $U \otimes C(Y)$.

4. The Proof of Theorem 2

PROPOSITION 6. *Let V be a UHF-algebra such that $V \otimes M_m$ is isomorphic to V for some integer $m \geqslant 2$. Let X, Y be finite $(n-2)$-connected CW-complexes of dimension at most n, $n \geqslant 2$. Suppose that there are $\alpha \in KK(C_0(X\backslash \mathrm{pt}), C_0(Y\backslash \mathrm{pt}))$ and $\beta \in KK(C_0(Y\backslash \mathrm{pt}), C_0(X\backslash \mathrm{pt}))$ such that $\alpha\beta = r[\mathrm{id}_X]$ and $\beta\alpha = r[\mathrm{id}_Y]$ for some r dividing a power of m. Then $V \otimes C(X)$ is homotopy equivalent to $V \otimes C(Y)$.*

Proof. By Proposition 3, the map

$$\chi\colon kk(Y, X) \to KK(C_0(X\backslash \mathrm{pt}), C_0(Y\backslash \mathrm{pt}))$$

is an isomorphism. Since χ preserves the multiplicative structure, the result follows by Proposition 4.

LEMMA. *Let H, H' be finitely generated Abelian groups and let G be a nonzero subgroup of $\mathbb{Q}$ such that $G \otimes H$ is isomorphic to $G \otimes H'$. Then there are groups S, T, T' and a nonzero integer r such that*

$$H \cong S \oplus T, \qquad H' \cong S \oplus T',$$
$$rT = 0, \qquad rT' = 0, \qquad rG = G.$$

Proof. By an easy reduction, we may assume that both H and H' are finite. The case $G \cong \mathbb{Z}$ is trivial. If G is not isomorphic to $\mathbb{Z}$ write G as an inductive limit, $G = \lim(G_i, \varphi_i)$, where each G_i is isomorphic to $\mathbb{Z}$ and $\varphi_i\colon G_i \to G_{i+1}$ is the multiplication with some prime $p_i \geqslant 2$. Let P denote the set consisting of all p_i that occur infinitely many times in the sequence $p_1, p_2, \ldots$ and notice that $pG \cong G$ for any p in P. Let d be the order of H and decompose $d = st$ such that t is a product of (possibly distinct) primes in P, and no member of P divides s. Consider the similar decomposition $d' = s't'$ for H', where d' is the order of H'. Since s is relatively prime to t one has an internal direct sum decomposition $H = S \oplus T$, where S (respectively, T) consists of all elements of H of order dividing s (respectively, t). Similarly one has $H' = S' \oplus T'$. Since

$$tT = 0, \qquad t'T' = 0 \quad \text{and} \quad tG \cong G, \qquad t'G \cong G,$$

we get $T \otimes G \cong 0$ and $T' \otimes G \cong 0$, therefore

$$H \otimes G \cong S \otimes G \quad \text{and} \quad H' \otimes G \cong S' \otimes G.$$

We conclude the proof by showing that $S \otimes G \cong S$. Indeed since $S \otimes G_i \cong S$, $S \otimes G$ is isomorphic to the inductive limit $\lim(S, \varphi_i)$, where the connecting maps $\varphi_i\colon S \to S$ are given by $\varphi_i(x) = p_i x$. Since no prime in P divides s, there is some j such that p_i does not divide s whenever $i \geqslant j$. It follows that φ_i is an isomorphism for $i \geqslant j$ and, therefore, $S \otimes G \cong \lim(S, \varphi) \cong S$. Similarly, one gets $S' \otimes G \cong S'$. Since $H \otimes G$ is isomorphic to $H' \otimes G$ by hypothesis, we get $S \cong S'$. Finally take $r = tt'$.

An Abelian group having all elements of order r will be called below *a group of exponent r*.

The end of the proof of Theorem 2.

Let X, Y, V_1, V_2, m be as in the statement of Theorem 2.
Assume that $K_*(V_1 \otimes C(X))$ is isomorphic to $K_*(V_2 \otimes C(Y))$ as ordered (scaled) groups. A positivity argument like that of Proposition 5.1.6 in [5], shows that $K_0(V_1)$ is isomorphic to $K_0(V_2)$ as ordered scaled groups. This implies that V_1 is isomorphic to V_2, [9]. Thus we may assume that both V_1 and V_2 are equal to some UHF-algebra V. By the Künneth formula

$$K_0(V) \otimes K_0(C(X)) \cong K_0(V) \otimes K_0(C(Y)),$$
$$K_0(V) \otimes K_1(C(X)) \cong K_0(V) \otimes K_1(C(Y)).$$

Suppose that n is even. The proof for n odd is entirely similar—just interchange K_0 with K_1. Since X is $(n-2)$-connected of dimension at most n, it follows that X is homotopy equivalent to a CW-complex with all cells in dimension $n-1$ and n. Since n is even this implies that $K_1(C(X))$ is a free group and of course the same holds true for $K_1(C(Y))$. By applying the above Lemma with

$$G = K_0(V), \qquad H = K_*(C(X)), \qquad H' = K_*(C(Y))$$

we get

$$K_0(C(X)) \cong S \oplus T, \qquad K_0(C(Y)) \cong S \oplus T',$$
$$K_1(C(X)) \cong K_1(C(Y)),$$

where T, T' are groups of exponent r and $rK_0(V) = K_0(V)$. Note that this implies $V \otimes M_r \cong V$. Let Z, W be $(n-2)$-connected CW-complexes of dimension at most n such that

$$K_0(C(Z)) \cong S, \qquad K_1(C(Z)) \cong K_1(C(X)),$$
$$K_0(C(W)) \cong \mathbb{Z} \oplus T, \qquad K_1(C(W)) = 0.$$

Such spaces are easily constructed by attaching n-cells to an wedge of $(n-1)$-spheres. Since Z plays a symmetric role with respect to X and Y, the proof will be complete once we show that $V \otimes C(X)$ is homotopy equivalent to $V \otimes C(Z)$. This is accomplished in two steps.

The 1st step: $V \otimes C(X)$ is homotopy equivalent to $V \otimes C(Z \vee W)$. This follows from Proposition 6 since $C_0(X \setminus \mathrm{pt})$ and $C_0(Z \vee W \setminus \mathrm{pt})$ have the same K-theory groups and, therefore, they are KK-equivalent by [16].

The 2nd step: $V \otimes C(Z \vee W)$ is homotopy equivalent to $V \otimes C(Z)$. We have seen that $V \otimes M_{mr} \cong V$. Taking advantage of Proposition 6, it is enough to find

$$\alpha \in KK(C_0(Z \vee W \setminus \mathrm{pt}), C_0(Z \setminus \mathrm{pt})),$$
$$\beta \in KK(C_0(Z \setminus \mathrm{pt}), C_0(Z \vee W \setminus \mathrm{pt})),$$

such that

$$\alpha\beta = r^2m^2[\mathrm{id}_{Z \vee W}] \quad \text{and} \quad \beta\alpha = r^2m^2[\mathrm{id}_Z].$$

To this purpose consider the following groups

$$G_{00} = KK(C_0(Z \setminus \mathrm{pt}), C_0(Z \setminus \mathrm{pt})), \qquad G_{01} = KK(C_0(Z \setminus \mathrm{pt}), C_0(W \setminus \mathrm{pt})),$$
$$G_{10} = KK(C_0(W \setminus \mathrm{pt}), C_0(Z \setminus \mathrm{pt})), \qquad G_{11} = KK(C_0(W \setminus \mathrm{pt}), C_0(W \setminus \mathrm{pt})).$$

Note that G_{01}, G_{10} and G_{11} are groups of exponent r since $K_*(C_0(W \setminus \mathrm{pt})) \cong T$ is a group of exponent r. This is easily seen by using the universal coefficient theorem of [16] which determines the Kasparov groups in terms of K-theory. Since

$$C_0(Z \vee W \setminus \mathrm{pt}) \cong C_0(Z \setminus \mathrm{pt}) \oplus C_0(W \setminus \mathrm{pt}),$$

we have the following decompositions

$$KK(C_0(Z \vee W\backslash\mathrm{pt}), C_0(Z\backslash\mathrm{pt})) \cong G_{00} \oplus G_{10},$$
$$KK(C_0(Z\backslash\mathrm{pt}), C_0(Z \vee W\backslash\mathrm{pt})) \cong G_{00} \oplus G_{01},$$
$$KK(C_0(Z \vee W\backslash\mathrm{pt}), C_0(Z \vee W\backslash\mathrm{pt})) \cong G_{00} \oplus G_{01} \oplus G_{10} \oplus G_{11}.$$

Define

$$\alpha_0 = [\mathrm{id}_Z] \in KK(C_0(Z \vee W\backslash\mathrm{pt}), C_0(Z\backslash\mathrm{pt})),$$
$$\beta_0 = [\mathrm{id}_Z] \in KK(C_0(Z\backslash\mathrm{pt}), C_0(Z \vee W\backslash\mathrm{pt})).$$

We have

$$\alpha_0\beta_0 = [\mathrm{id}_Z] \in G_{00} \subset KK(C_0(Z \vee W\backslash\mathrm{pt}), C_0(Z \vee W\backslash\mathrm{pt})),$$
$$\beta_0\alpha_0 = [\mathrm{id}_Z] \in KK(C_0(Z\backslash\mathrm{pt}), C_0(Z\backslash\mathrm{pt})).$$

Therefore

$$\alpha_0\beta_0 - [\mathrm{id}_{Z\vee W}] \in G_{10} \oplus G_{01} \oplus G_{11}.$$

Finally, let $\alpha = mr\alpha_0, \beta = mr\beta_0$. It is clear that

$$\alpha\beta - m^2r^2[\mathrm{id}_{Z\vee W}] = m^2r^2(\alpha_0\beta_0 - [\mathrm{id}_{Z\vee W}]) = 0,$$
$$\beta\alpha - m^2r^2[\mathrm{id}_Z] = 0,$$

since we have seen that G_{10}, G_{01}, and G_{11} are groups of exponent r.

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