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Asymptotic Unitary Equivalence in *KK*-Theory

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Abstract. A description of the Kasparov group $KK(A, B)$ is given in terms of Cuntz pairs of representations and the notion of proper asymptotic unitary equivalence that we introduce here. The use of the word 'proper' reflects the crucial fact that all unitaries implementing the equivalence can be chosen to be compact perturbations of identity. The result has significant applications to the classification theory of nuclear C^*-algebras.

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1. Introduction

Let A, B be C^*-algebras with A separable and B σ-unital. Let $\mathbb{E}(A, B)$ denote the set of (A, B) C^*-bimodules. The versatility of KK-theory is due to some extent to the remarkable fact, emphasized in [Ska91, Remark 10.8], that all 'reasonable' equivalence relations on $\mathbb{E}(A, B)$, ranging from homotopy to cobordism, prove to be the same. They all give rise to the same object, the Kasparov group $KK(A, B)$.

It is a goal of this paper to exhibit yet another equivalence relation leading to $KK(A, B)$, see Theorem 3.8. This equivalence relation is based on the notion of proper asymptotic unitary equivalence, which we define below.

Besides its intrinsic interest, the result is motivated by applications in the classification theory of nuclear C^*-algebras [DE98]. In particular it gives a better understanding of what it means that two $*$-homomorphisms $\varphi, \psi\colon A \to B$ have the same KK-theory class (see Theorems 4.2 and 4.3). It also sheds new light on the realizations of $KK(A, B)$ for purely infinite simple nuclear C^*-algebras, [Kir94], [Phi00], as it reveals a role for asymptotic unitary equivalence in a general KK-theory context. To elaborate on this point, we need the following definition. Let E be a (right) Hilbert B-module. If $\pi, \sigma\colon A \to \mathcal{L}(E)$ are representations, we say that π and σ are properly asymptotically unitarily equivalent and write $\pi \approxeq \sigma$ if there is a continuous path of unitaries $u\colon [0, \infty) \to \mathcal{U}(\mathcal{K}(E) + \mathbb{C}1_E)$, $u = (u_t)_{t\in[0,\infty)}$ such that

- $\lim_{t\to\infty} \|u_t\pi(a)u_t^* - \sigma(a)\| = 0$, for all $a \in A$
- $u_t\pi(a)u_t^* - \sigma(a) \in \mathcal{K}(E)$, for all $t \in [0, \infty)$, and $a \in A$.

The use of the word 'proper' reflects the crucial fact that all unitaries are of the form 'identity + compact'.

The main result is Theorem 3.8, which shows that if $\varphi, \psi\colon A \to M(\mathcal{K}(H)\otimes B)$ is a Cuntz pair of representations [Cun83], that is $\varphi(a) - \psi(a) \in \mathcal{K}(H) \otimes B$ for all $a \in A$, then the class $[\varphi, \psi]$ vanishes in $KK(A, B)$ if and only if $\varphi\oplus\gamma \approxeq \psi\oplus\gamma$ for some representation $\gamma\colon A \to M(\mathcal{K}(H) \otimes B)$. A version of this for $KK_{nuc}(A, B)$ [Ska88] is given in Theorem 3.10. The result is improved for K-homology. Thus if $\varphi, \psi\colon A \to \mathcal{L}(H)$ is a Cuntz pair of faithful nondegenerate representations whose images do not contain nonzero compact operators, then $[\varphi, \psi] = 0$ in $K^0(A)$ if and only if $\varphi \approxeq \psi$. For a better understanding of the difference between asymptotic unitary equivalence and proper asymptotic unitary equivalence, we encourage the reader to contrast this result with Voiculescu's Theorem 3.11.

A key step in the proof of our result was to employ the Paschke duality, its generalizations [Pas81], [Ska88] and other arguments in the same spirit, in order to produce an automorphism of $\varphi(A) + \mathcal{K}(H) \otimes B$ which can be connected to the identity by a path continuous in the uniform topology. Then one uses a theorem of Kadison and Ringrose on automorphisms [KR67], [Ped79], by developing an idea of Lin [Lin97].

Note that our result fits nicely with the following basic example discussed in [BDF73]. Suppose that $e, f \in \mathcal{L}(H)$ are two projections with $e - f \in \mathcal{K}(H)$. The essential codimension of e in f is an integer defined as the Fredholm index of v^*w, where v, w are isometries on H with $vv^* = e$ and $ww^* = f$. This correspondence gives an isomorphism from $KK(\mathbb{C}, \mathbb{C}) = K^0(\mathbb{C})$ to $\mathbb{Z}$. The integer associated to the pair (e, f) vanishes exactly when there is a unitary $u \in \mathcal{K}(H) + \mathbb{C}1$ such that $ueu^* = f$.

2. Preliminaries and Background

2.1. APPROXIMATELY EQUIVALENT REPRESENTATIONS

The reader is referred to [Kas80a] for an introduction to Hilbert C^*-algebra modules. Throughout the paper, A is a separable C^*-algebra, and all Hilbert C^*-algebra modules are assumed to be countably generated over a σ-unital C^*-algebra B. We let H_B denote the Hilbert module $H \otimes B$, where H is an infinite dimensional, separable Hilbert space. We use the term representation for a $*$-homomorphism $A \to \mathcal{L}(E)$, where E is a Hilbert B-module.

DEFINITION 2.1. Fix a σ-unital C^*-algebra B. When $\gamma\colon A \longrightarrow \mathcal{L}(E)$ and $\gamma'\colon A \longrightarrow \mathcal{L}(E')$ are two representations, with E and E' Hilbert B-modules, we say that γ and γ' are approximately unitarily equivalent and write $\gamma \sim \gamma'$, if there exists a sequence of unitaries $u_n \in \mathcal{L}(E', E)$ such that for any $a \in A$

(i) $\lim_{n\to\infty} \|\gamma(a) - u_n\gamma'(a)u_n^*\| = 0$,
(ii) $\gamma(a) - u_n\gamma'(a)u_n^* \in \mathcal{K}(E)$, for all n.

We say that γ and γ' are asymptotically unitarily equivalent, and write $\gamma \sim_{\text{asymp}} \gamma'$, if there exists a unitary valued norm-continuous map $u\colon [0,\infty) \to \mathcal{L}(E', E)$, $u = (u_t)_{t\in[0,\infty)}$, such that the map $t \mapsto \gamma(a) - u_t\gamma'(a)u_t^*$ lies in $C_0[0,\infty) \otimes \mathcal{K}(E)$ for any $a \in A$. In other words, if

(iii) $\lim_{t\to\infty} \|\gamma(a) - u_t\gamma'(a)u_t^*\| = 0$,
(iv) $\gamma(a) - u_t\gamma'(a)u_t^* \in \mathcal{K}(E)$, for all $t \in [0,\infty)$,

for all $a \in A$.

If $\sigma\colon A \to \mathcal{L}(F)$ is a representation, we define $\sigma_\infty\colon A \to \mathcal{L}(F_\infty)$ by $\sigma_\infty = \sigma \oplus \sigma \oplus \cdots$, where $F_\infty = F \oplus F \oplus \cdots$. Let $w_\infty\colon F_\infty \to F \oplus F_\infty$ be defined as $w_\infty(\xi_1, \xi_2, \xi_3, \dots) = \xi_1 \oplus (\xi_2, \xi_3, \dots)$. We need the following lemma of [DE98].

LEMMA 2.2. *Let* $\pi\colon A \to \mathcal{L}(E)$ *and* $\sigma\colon A \to \mathcal{L}(F)$ *be two representations. Then for any isometry* $v\colon F_\infty \to E$, *the unitary* $u = (1_F \oplus v)w_\infty v^* + 1_E - vv^* \in \mathcal{L}(E, F \oplus E)$ *satisfies*

$$\|\sigma(a) \oplus \pi(a) - u\pi(a)u^*\| \leqslant 6\|v\sigma_\infty(a) - \pi(a)v\| + 4\|v\sigma_\infty(a^*) - \pi(a^*)v\|.$$

Moreover, if $v\sigma_\infty(a) - \pi(a)v \in \mathcal{K}(F_\infty, E)$, *for all* $a \in A$, *then* $\sigma(a) \oplus \pi(a) - u\pi(a)u^* \in \mathcal{K}(F \oplus E)$, *for all* $a \in A$.

LEMMA 2.3. *Let* $\pi\colon A \to \mathcal{L}(E)$ *and* $\sigma\colon A \to \mathcal{L}(F)$ *be two representations. Suppose that there is a sequence of isometries* $v_i\colon F_\infty \to E$ *such that*

$$v_i\sigma_\infty(a) - \pi(a)v_i \in \mathcal{K}(F_\infty, E), \qquad \|v_i\sigma_\infty(a) - \pi(a)v_i\| \longrightarrow 0, \quad a \in A$$

and $v_i^*v_j = 0$ *for* $i \neq j$. *Then* $\pi \oplus \sigma \sim_{\text{asymp}} \pi$.

Proof. Extend the sequence (v_i) to a continuous family of isometries $v\colon [0,\infty) \to \mathcal{L}(F_\infty, E)$ by defining

$$v_{i+t} = (1-t)^{1/2}v_i + t^{1/2}v_{i+1}, \quad 0 \leqslant t \leqslant 1.$$

Then $t \mapsto v_t\sigma_\infty(a) - \pi(a)v_t$ defines a function in $C_0[0,\infty) \otimes \mathcal{K}(F_\infty, E)$ for all $a \in A$. Now Lemma 2.2 yields a norm continuous family of unitaries $u = (u_t)_{t\in[0,\infty)}$ such that

$$\pi(a) \oplus \sigma(a) - u\pi(a)u^* \in C_0[0,\infty) \otimes \mathcal{K}(E \oplus F). \qquad \square$$

LEMMA 2.4. *Let* $\pi\colon A \to \mathcal{L}(E)$ *and* $\sigma\colon A \to \mathcal{L}(F)$ *be two representations. If* $\sigma \oplus \pi \sim \pi$, *then* $\sigma \oplus \pi_\infty \sim_{\text{asymp}} \pi_\infty$.

Proof. If $\sigma \oplus \pi \sim \pi$, then $\sigma_\infty \oplus \pi_\infty \sim \pi_\infty$. Using that $E_\infty = (E_\infty)_\infty$ we may think of the unitaries given by (i), (ii) of Definition 2.1 as isometries $v_i'\colon F_\infty \oplus E_\infty \to E_\infty$ with $(v_i')^*v_j' = 0$ for $i \neq j$ and

$$v_i'(\sigma_\infty \oplus \pi_\infty)(a) - \pi_\infty(a)v_i' \in \mathcal{K}(F_\infty, E),$$

$$\left\| v_i'(\sigma_\infty \oplus \pi_\infty)(a) - \pi_\infty(a)v_i' \right\| \longrightarrow 0,$$

for $a \in A$. Letting $W\colon F_\infty \to F_\infty \oplus E_\infty$ be the canonical isometry, we note that $[\sigma_\infty \oplus \pi_\infty]W = W\sigma_\infty$, and consequently, that π_∞ and σ satisfy the assumptions of Lemma 2.3 using $v_i = v_i'W$. It follows that $\pi_\infty \oplus \sigma \sim_{\text{asymp}} \pi_\infty$. □

2.2. ABSORBING REPRESENTATIONS

DEFINITION 2.5. A representation $\pi\colon A \to \mathcal{L}(E)$ is called absorbing if $\pi \oplus \sigma \sim \pi$ for any representation $\sigma\colon A \to \mathcal{L}(F)$. Note that a unital representation cannot absorb a nonunital representation. If A is unital, then a representation $\pi\colon A \to \mathcal{L}(E)$ is called unitally absorbing if $\pi \oplus \sigma \sim \pi$ for any unital representation $\sigma\colon A \to \mathcal{L}(F)$.

A recent result of Thomsen ([Tho01, 2]) ensures the existence of absorbing representations from A to $M(\mathcal{K}(H) \otimes B)$ under the modest assumptions that A and B are separable C^*-algebras. If A is unital, then the representation can be chosen unital and unitally absorbing.

DEFINITION 2.6 ([Ska88, Definition 1.1.d]). A completely positive contraction $\pi\colon A \longrightarrow \mathcal{L}(E)$ is said to be strictly nuclear if there exist integers (n_λ) and generalized sequences $\psi_\lambda\colon A \to M_{n_\lambda}(\mathbb{C})$, $\varphi_\lambda\colon M_{n_\lambda}(\mathbb{C}) \to \mathcal{L}(E)$ of completely positive contractions such that

$$\lim_{\lambda \longrightarrow \infty} \varphi_\lambda \psi_\lambda(a) = \pi(a)$$

in the strict topology, for all $a \in A$. Equivalently, for any $v \in \mathcal{K}(E)$ the map $A \to \mathcal{K}(E)$, defined by $a \mapsto v^*\pi(a)v$, is nuclear (see [Ska88]).

Remark 2.7. Any scalar representation $\theta\colon A \to \mathcal{L}(H) \subset \mathcal{L}(H_B)$ is strictly nuclear. If either A or B is nuclear, then any completely positive contraction $\pi\colon A \longrightarrow \mathcal{L}(E)$ is strictly nuclear by [Ska88, 1.7].

DEFINITION 2.8. A representation $\pi\colon A \to \mathcal{L}(E)$ is called nuclearly absorbing if $\pi \oplus \sigma \sim \pi$ for any strictly nuclear representation $\sigma\colon A \to \mathcal{L}(F)$. If A is unital, then a representation $\pi\colon A \to \mathcal{L}(E)$ is called unitally nuclearly absorbing if $\pi \oplus \sigma \sim \pi$ for any unital strictly nuclear representation $\sigma\colon A \to \mathcal{L}(F)$.

If $\pi\colon A \to \mathcal{L}(E)$ is a representation, we denote by $\widetilde{A}$ the C^*-algebra obtained by adjoining an external unit $\widetilde{1}$ to A, and define $\widetilde{\pi}\colon \widetilde{A} \to \mathcal{L}(E)$ by $\widetilde{\pi}(a + \lambda\widetilde{1}) = \pi(a) + \lambda 1_E$. Also let $\widehat{\gamma} = 0 \oplus \gamma\colon A \to M(\mathcal{K}(H \oplus H) \otimes B)$, where the zero summand acts on $H \otimes B$. The following was proved in [DE98].

LEMMA 2.9. *Let A be a separable C^*-algebra and let B be a σ-unital C^*-algebra. If π is a nonunital nuclearly absorbing representation, then $\widetilde{\pi}$ is a unital nuclearly* *absorbing representation. Suppose that A is unital. If $\gamma\colon\ A \to M(\mathcal{K}(H)\otimes B)$ is a unital nuclearly absorbing representation, then $\widehat{\gamma} = 0 \oplus \gamma$ is a nonunital nuclearly absorbing representation.*

Note that one has a version of Lemma 2.9 for absorbing representations.

DEFINITION 2.10. Let A be a separable C^*-algebra. A faithful scalar representation of infinite multiplicity $\theta\colon\ A \longrightarrow M(\mathcal{K}(H)\otimes B)$ is a $*$-homomorphism which factors as

$$A \xrightarrow{\theta'} \mathcal{L}(H) \xrightarrow{-\otimes 1} \mathcal{L}(H)\otimes M(B) \hookrightarrow M(\mathcal{K}(H)\otimes B),$$

where θ' is faithful and of infinite multiplicity, that is of the form $\infty\cdot\gamma$ for some representation γ.

PROPOSITION 2.11. *Let A be a unital separable C^*-algebra. If $\theta\colon\ A \longrightarrow \mathcal{L}(H_B)$ is a unital faithful scalar representation of infinite multiplicity, then θ is unitally nuclearly absorbing.*

Proof. This follows from the arguments of [Kas80a], as pointed out in [Ska88]. Details can be found in [DE98]. □

2.3. AUTOMORPHISMS AND DERIVATIONS

Let A denote a unital separable C^*-algebra, and denote as in [Ped79] by $\mathbf{B}(A)$ the Banach algebra of linear operators on A and by $\mathrm{Aut}(A)$ and $\mathrm{Der}(A)$ the subsets of $\mathbf{B}(A)$ consisting of $*$-automorphisms and $*$-derivations, respectively. Unless otherwise specified, we are going to consider $\mathbf{B}(A)$ and these subsets as equipped with the uniform topology. Thus $\mathrm{Aut}_0(A)$ refers to the connected component of the identity of $\mathrm{Aut}(A)$ in the uniform topology. Note that analytic functional calculus is at our disposal in $\mathbf{B}(A)$, and that $\delta \mapsto \exp\delta$ is a continuous map from $\mathrm{Der}(A)$ to $\mathrm{Aut}(A)$.

We can define inner automorphisms and derivations by

$$\mathrm{Ad}(u)(a) = uau^*, \qquad \mathrm{ad}(ih)(a) = i[h,a] = i(ha - ah),$$

for $u, h \in A$ which are unitary and selfadjoint, respectively. We get $\exp\mathrm{ad}(ih) = \mathrm{Ad}(\exp ih)$ in this setting. We call elements of $\mathrm{Aut}(A)$ or $\mathrm{Der}(A)$ asymptotically inner when they are the pointwise limits of uniformly continuous one-parameter families of inner automorphisms or derivations. Thus an asymptotically inner automorphism α is one for which there is a continuous path of unitaries $(u_t)_{t\in[0,\infty)}$ such that

$$\lim_{t\to\infty} \|\mathrm{Ad}(u_t)(a) - \alpha(a)\| = 0, \quad a \in A.$$

Similarly, a derivation δ is asymptotically inner if there is a norm continuous bounded family of selfadjoint elements $(h_t)_{t\in[0,\infty)}$ such that

$$\lim_{t\to\infty} \|\mathrm{ad}(ih_t)(a) - \delta(a)\| = 0, \quad a \in A. \tag{1}$$

LEMMA 2.12. *If A is a unital separable C^*-algebra, then any derivation $\delta \in \mathrm{Der}(A)$ is asymptotically inner. In fact, one may choose h_t in* (1) *with $\|h_t\| \leqslant \|\delta\|$.*

Proof. By [Ped79, 8.6.12] there is a sequence h_n of selfadjoint elements of A such that $\lim_{n\to\infty} \|\mathrm{ad}(ih_n)(a) - \delta(a)\| = 0$ for all $a \in A$. We may choose h_n such that $\|h_n\| \leqslant \|\delta\|$. By linear interpolation, we find a norm continuous bounded family of selfadjoint elements $(h_t)_{t\in[0,\infty)}$ satisfying (1). □

The proof of the following elementary lemma is left to the reader.

LEMMA 2.13. *The map $\exp\colon \mathrm{Der}(A) \to \mathrm{Aut}(A)$ is continuous in the point-norm topology. That is, if $\lim_{n\to\infty} \|\delta_n(a) - \delta(a)\| = 0$ for all $a \in A$, then $\lim_{n\to\infty} \|\exp(\delta_n)(a) - \exp(\delta)(a)\| = 0$, for all $a \in A$.*

LEMMA 2.14. *If A is a unital separable C^*-algebra, then any automorphism in $\mathrm{Aut}_0(A)$ is asymptotically inner. The one-parameter family can be found within $\mathrm{Aut}_0(A) \cap \mathrm{Inn}(A)$.*

Proof. If $\alpha \in \mathrm{Aut}_0(A)$, then by [Ped79, 8.7.8] α is a product of finitely many automorphisms of the form $\exp\delta$, $\delta \in \mathrm{Der}(A)$. Thus it suffices to assume that in fact $\alpha = \exp\delta$. Using Lemma 2.13, the result now follows from exponentiation of the path constructed in Lemma 2.12, noting that $\exp(\mathrm{Der}(A)) \subseteq \mathrm{Aut}_0(A)$. □

PROPOSITION 2.15. *Let A be a unital separable C^*-algebra. If $(\alpha_t)_{t\in[0,\infty)}$ is a uniformly continuous family in $\mathrm{Aut}(A)$ with $\alpha_0 = \mathrm{id}_A$, then there exists a continuous family $(v_t)_{t\in[0,\infty)}$ of unitaries in A with $v_0 = 1$ such that*

$$\lim_{t\longrightarrow\infty} \|\alpha_t(a) - \mathrm{Ad}(v_t)(a)\| = 0, \quad a \in A.$$

Proof. Applying uniform continuity separately on each interval $[n, n+1]$ we see that we can divide $[0,\infty)$ into pieces by $0 = t_0 < t_1 < \cdots < t_j < t_{j+1} < \cdots$ with $t_j \longrightarrow \infty$ such that when we set

$$\mu^{(j)} = \sup_{t\in[t_{j-1},t_j]} \left\|\alpha_t - \alpha_{t_j}\right\|,$$

$\mu^{(j)} < 2$ and $\mu^{(j)} \longrightarrow 0$ for $j \longrightarrow \infty$.

For any $j \geqslant 1$, let $\beta^{(j)} = \alpha_{t_j} \circ \alpha^{-1}_{t_{j-1}}$ and note that $\left\|\mathrm{id} - \beta^{(j)}\right\| \leqslant \mu^{(j)}$. By [Ped79, 8.7.7], when we set $\delta^{(j)} = \mathrm{Log}(\beta^{(j)})$, in $\mathbf{B}(A)$, we get a sequence of $*$-derivations. Furthermore, $\left\|\delta^{(j)}\right\| \longrightarrow \|\mathrm{Log}(\mathrm{id})\| = 0$. Using Lemma 2.12 we choose selfadjoint families $(h_t^{(j)})_{t\in[0,\infty)}$ such that $\left\|h_t^{(j)}\right\| \leqslant \left\|\delta^{(j)}\right\|$ for all t, j, $h_0^{(j)} = 0$ and

$$\left\|\delta^{(j)}(a) - \mathrm{ad}(ih_t^{(j)})(a)\right\| \longrightarrow 0, \tag{2}$$

for all a. Note that $\|e^{ih_t^{(j)}} - 1\| \leqslant e^{\|h_t^{(j)}\|} - 1 \leqslant e^{\|\delta^{(j)}\|} - 1$.

Fix a dense sequence $(a_i)_{i=0}^{\infty}$ of the unit ball of A. When $k \geqslant j \geqslant 1$ we set

$$\mathcal{F}_k^{(j)} = \{\beta^{(j-1)} \circ \cdots \circ \beta^{(1)}(a_i) \mid i \leqslant k\},$$

interpreted in the obvious way when $j = 1$. Exponentiating each family $h_t^{(j)}$ and then reparametrizing using a homeomorphism between $[t_j, \infty)$ and $[0, \infty)$, we get continuous families of unitaries $v_t^{(j)}$ with the properties

$$v_t^{(j)} = 1 \qquad t \in [0, t_j]$$
$$\left\| 1 - v_t^{(j)} \right\| \leqslant e^{\|\delta^{(j)}\|} - 1, \quad t \in [t_j, t_{j+1}]$$
$$\left\| \beta^{(j)}(a) - \mathrm{Ad}(v_t^{(j)})(a) \right\| \leqslant 2^{-j-k}, \quad t \in [t_k, t_{k+1}], a \in \mathcal{F}_k^{(j)}, k > j.$$

This is arranged using (2), noting that the condition on $[t_j, t_{j+1}]$ in each case holds true on any element of the form $\exp(ih_t^{(j)})$.

When $t \in [t_j, t_{j+1}]$ and $i \leqslant j$ we may use our assumption on $v_t^{(j-1)}$ on the interval $[t_k, t_{k+1}]$ with $k = j$ to see that

$$\begin{aligned}
&\left\| \alpha_{t_{j-1}}(a_i) - \mathrm{Ad}(v_t^{(j-1)} \cdots v_t^{(1)})(a_i) \right\| \\
&\quad \leqslant \left\| \beta^{(j-1)}[\beta^{(j-2)} \circ \cdots \circ \beta^{(1)}(a_i)] - \mathrm{Ad}(v_t^{(j-1)})[\beta^{(j-2)} \circ \cdots \circ \beta^{(1)}(a_i)] \right\| + \\
&\quad\quad + \left\| \beta^{(j-2)} \circ \cdots \circ \beta^{(1)}(a_i) - \mathrm{Ad}(v_t^{(j-2)} \cdots v_t^{(1)})(a_i) \right\| \\
&\quad \leqslant 2^{-2j+1} + \left\| \alpha_{t_{j-2}}(a_i) - \mathrm{Ad}(v_t^{(j-2)} \cdots v_t^{(1)})(a_i) \right\|.
\end{aligned}$$

We can iterate this argument to prove that under these circumstances,

$$\left\| \alpha_{t_{j-1}}(a_i) - \mathrm{Ad}(v_t^{(j-1)} \cdots v_t^{(1)})(a_i) \right\| \leqslant 2^{-2j+1} + \cdots + 2^{-j-1} < 2^{-j}. \tag{3}$$

We may now define

$$u_t = \lim_{j \longrightarrow \infty} v_t^{(j)} \cdots v_t^{(1)}$$

since for every t, the sequence is eventually constant, as $v_1^{(j)} = 1$ on the interval $[0, t_j]$. This is then obviously a continuous family of unitaries. Now fix i and consider a_i. When $j \geqslant i$ and $t \in [t_j, t_{j+1}]$ we have by (3), the definition of $\mu^{(j)}$, and the condition of $v_t^{(j)}$ on $[t_j, t_{j+1}]$ that

$$\begin{aligned}
&\|\alpha_t(a_i) - \mathrm{Ad}(u_t)(a_i)\| \\
&\quad = \left\| \alpha_t(a_i) - \mathrm{Ad}(v_t^{(j)} \cdots v_t^{(1)})(a_i) \right\| \\
&\quad \leqslant \mu^{(j)} + 2\mu^{(j+1)} + \left\| \alpha_{t_{j-1}}(a_i) - \mathrm{Ad}(v_t^{(j)} \cdots v_t^{(1)})(a_i) \right\| \\
&\quad \leqslant \mu^{(j)} + 2\mu^{(j+1)} + 2\left\| v_t^{(j)} - 1 \right\| + \\
&\quad\quad + \left\| \alpha_{t_{j-1}}(a_i) - \mathrm{Ad}(v_t^{(j-1)} \cdots v_t^{(1)})(a_i) \right\| \\
&\quad \leqslant \mu^{(j)} + 2\mu^{(j+1)} + 2(e^{\|\delta^{(j)}\|} - 1) + 2^{-j},
\end{aligned}$$

showing that the norm does converge to zero as $t \longrightarrow \infty$. $\square$

3. The Main Result

In all of Section 3, we only work with infinite-dimensional, separable Hilbert spaces, so all Hilbert spaces in this paper are isomorphic. However, we introduce the following notation to aid the reader in distinguishing between different instances of them. We start with a separable Hilbert space H_1 and define

$$H_m = \overbrace{H_1 \oplus \cdots \oplus H_1}^{m}.$$

for any $m \in \mathbb{N}$. There are now canonical identifications between, say, $\mathbf{M}_2(\mathcal{K}(H_1) \otimes B)$ and $\mathcal{K}(H_2) \otimes B$, and we shall employ them tacitly in the following. However, sometimes we choose not to apply the (noncanonical) isomorphisms between, for example, $\mathcal{K}(H_1) \otimes B$ and $\mathcal{K}(H_2) \otimes B$, as we feel this helps to clarify our constructions.

We work with the multiplier algebras $M(\mathcal{K}(H_m) \otimes B)$, as well as the corona algebras

$$Q(\mathcal{K}(H_m) \otimes B) = M(\mathcal{K}(H_m) \otimes B)/\mathcal{K}(H_m) \otimes B.$$

The quotient map from $M(\mathcal{K}(H_m) \otimes B)$ to $Q(\mathcal{K}(H_m) \otimes B)$ is denoted by π_m, and whenever there is need for distinction, we write 1_m and 0_m for the identity and zero elements of these algebras.

3.1. *KK*-THEORY

We depend on Kasparov and Skandalis' bifunctors KK and KK_{nuc} in the paper, and refer the reader to [Kas80a] and [Ska88] for their definition. We recall that throughout the paper A is a separable C^*-algebra and B is a σ-unital C^*-algebra. All Hilbert modules E are countably generated. If $H_B = H \otimes B$, then $\mathcal{L}(H_B) \cong M(\mathcal{K}(H) \otimes B)$. We need both the Fredholm picture and the Cuntz picture of $KK(A, B)$. In the Fredholm picture (see [Hig87, 2.1]), $KK(A, B)$ is described in terms of triples $(\varphi_0, \varphi_1, u)$, which we call KK-cycles or just cycles, where $\varphi_i \colon A \longrightarrow \mathcal{L}(E_i)$ are $*$-homomorphisms and $u \in \mathcal{L}(E_0, E_1)$ satisfies

$$u\varphi_0(a) - \varphi_1(a)u \in \mathcal{K}(E_0, E_1), \tag{4}$$

$$\varphi_0(a)(u^*u - 1) \in \mathcal{K}(E_0), \varphi_1(a)(uu^* - 1) \in \mathcal{K}(E_1). \tag{5}$$

The set of all cycles as above is denoted by $\mathbb{E}(A, B)$. A cycle is degenerate if

$$u\varphi_0(a) - \varphi_1(a)u = 0, \qquad \varphi_0(a)(u^*u - 1) = 0, \qquad \varphi_1(a)(uu^* - 1) = 0,$$

for $a \in A$.
An operatorial homotopy through KK-cycles is a homotopy $(\varphi_0, \varphi_1, u_t)$, where the map $t \mapsto u_t$ is norm continuous. A theorem of Kasparov [Kas80b] shows that $KK(A, B)$ is isomorphic to the quotient of $\mathbb{E}(A, B)$ by the equivalence relation

generated by addition of degenerate cycles, unitary equivalence and operatorial homotopy.

The Cuntz picture is described in terms of pairs of representations $(\varphi, \psi)\colon A \to M(\mathcal{K}(H) \otimes B)$, where

$$\varphi(a) - \psi(a) \in \mathcal{K}(H) \otimes B, \quad a \in A.$$

Such a pair will be called a Cuntz pair. They form a set denoted by $\mathbb{E}_h(A, B)$. A homotopy of Cuntz pairs consists of a Cuntz pair $(\Phi, \Psi)\colon A \to M(\mathcal{K}(H) \otimes B[0, 1])$. The quotient of $\mathbb{E}_h(A, B)$ by homotopy equivalence is a group $KK_h(A, B)$ isomorphic to $KK(A, B)$. The isomorphism

$$KK_h(A, B) \longrightarrow KK(A, B) \tag{6}$$

maps $[\varphi, \psi]$ to $[\varphi, \psi, 1]$. A quick way to see that the map (6) is surjective is to show that any cycle is equivalent to a cycle where u is a unitary. Then since $(\varphi_0, \varphi_1, u)$ is unitarily equivalent to $(u\varphi_0 u^*, \varphi_1, 1)$, we see that (6) maps $[u\varphi_0 u^*, \varphi_1]$ to $[\varphi_0, \varphi_1, u]$. The injectivity of (6) is proved by applying a similar method to an operatorial homotopy.

One defines KK_{nuc} by restricting attention to the case where φ_i are strictly nuclear $*$-homomorphisms. Hence $KK = KK_{\mathrm{nuc}}$ when A or B is nuclear. Similarly $KK_{\mathrm{nuc},h}(A, B)$ is defined by working only with Cuntz pairs (and homotopies) consisting of strictly nuclear representations. And again one has an isomorphism

$$KK_{\mathrm{nuc},h}(A, B) \longrightarrow KK_{\mathrm{nuc}}(A, B).$$

This isomorphism is proved using the description of $KK_{\mathrm{nuc}}(A, B)$ based on operatorial homotopy given by [Ska88, 2.6].

Note that any $*$-homomorphism $\psi\colon A \longrightarrow B$ induces an element $[\psi] \in KK(A, B)$ via the cycle $(\psi, 0, 0)$. Similarly any nuclear $*$-homomorphism $\psi\colon A \longrightarrow B$ induces an element $[\psi] \in KK_{\mathrm{nuc}}(A, B)$.

3.2. PROPER ASYMPTOTIC UNITARY EQUIVALENCE

LEMMA 3.1. *If $\varphi, \psi\colon A \longrightarrow M(\mathcal{K}(H) \otimes B)$ is a Cuntz pair of representations, and $(u_t)_{t\in[1,\infty)}$ is a continuous path of unitaries in $M(\mathcal{K}(H) \otimes B)$ satisfying*

$$u_t\varphi(a)u_t^* - \psi(a) \in \mathcal{K}(H) \otimes B,$$

for all $t \in [1, \infty)$, then $[\varphi, \psi] = [\varphi, u_1\varphi u_1^*]$ in $KK_h(A, B)$.

Proof. If $\Phi_0, \Phi_1\colon A \to M(\mathcal{K}(H) \otimes C[0, 1]) = C_{b,\mathrm{strictly}}([0, 1], \mathcal{L}(H))$ are defined by

$$\Phi_0(a)(s) = \varphi(a), \qquad \Phi_1(a)(s) = \begin{cases} \mathrm{Ad}(u_{1/s}) \circ \varphi(a), & s > 0 \\ \psi(a), & s = 0 \end{cases}$$

then (Φ_0, Φ_1) is a homotopy of Cuntz pairs from (φ, ψ) to $(\varphi, u_1\varphi u_1^*)$. □

DEFINITION 3.2. If $\pi, \sigma\colon A \to \mathcal{L}(E)$ are representations, we say that π and σ are properly asymptotically unitarily equivalent and write $\pi \approxeq \sigma$ if there is a continuous path of unitaries $u\colon [0,\infty) \to \mathcal{U}(\mathcal{K}(E) + \mathbb{C}1_E)$, $u = (u_t)_{t\in[0,\infty)}$ such that for all $a \in A$

(i) $\lim_{t\to\infty} \|u_t\pi(a)u_t^* - \sigma(a)\| = 0$,
(ii) $u_t\pi(a)u_t^* - \sigma(a) \in \mathcal{K}(E)$, for all $t \in [0,\infty)$.

LEMMA 3.3. *If $\varphi \approxeq \psi$, then $\varphi(a) - \psi(a) \in \mathcal{K}(H) \otimes B$ for all $a \in A$ and $[\varphi, \psi, 1] = 0$ in $KK(A, B)$.*

Proof. Let $(u_t)_{t\in[0,\infty)}$ be as in Definition 3.2. The first part of the lemma is obvious since $u_t \in \mathcal{K}(H)\otimes B + \mathbb{C}1$. Then $[\varphi, \psi, 1] = [\varphi, u_1\varphi u_1^*, 1]$ by the lemma above. Since $(\varphi, u_1\varphi u_1^*, 1)$ is unitarily equivalent to $(\varphi, \varphi, u_1^*)$ and $u_1^* \in \mathcal{K}(H) \otimes B + \mathbb{C}1$, we obtain that $[\varphi, \psi, 1] = [\varphi, \varphi, 1] = 0$. □

LEMMA 3.4. *Let $\varphi, \psi, \gamma, \sigma\colon A \to M(\mathcal{K}(H) \otimes B)$ be unital representations. If $\varphi \oplus \gamma \approxeq \psi \oplus \gamma$, and $\gamma \sim_{\mathrm{asymp}} \sigma$, then $\varphi \oplus \sigma \approxeq \psi \oplus \sigma$.*

Proof. By assumption, there is a continuous path of unitaries $u\colon [0,\infty) \to \mathcal{U}(\mathcal{K}(H_2) \otimes B + \mathbb{C}1_2)$ such that

$$u_t(\varphi(a) \oplus \gamma(a))u_t^* - \psi(a) \oplus \gamma(a) \in \mathcal{K}(H_2) \otimes B,$$
$$\lim_{t\to\infty} \|u_t(\varphi(a) \oplus \gamma(a))u_t^* - \psi(a) \oplus \gamma(a)\| = 0 \tag{7}$$

for all $a \in \mathcal{F}_n$. Since $\gamma \sim_{\mathrm{asymp}} \sigma$, there is a continuous path of unitaries $(v_t)_{t\in[0,\infty)}$ such that

$$v_t\gamma(a)v_t^* - \sigma(a) \in \mathcal{K}(H) \otimes B, \qquad \lim_{t\to\infty} \|v_t\gamma(a)v_t^* - \sigma(a)\| = 0 \tag{8}$$

for all $a \in \mathcal{F}_n$. Clearly $w_t = (1 \oplus v_t)u_t(1 \oplus v_t^*)$ is a unitary in $\mathcal{K}(H_2) \otimes B + \mathbb{C}1_2$. If $a \in A$, then from (7) and (8)

$$\begin{aligned}
&\|w_t(\varphi(a) \oplus \sigma(a))w_t^* - \psi(a) \oplus \sigma(a)\| \\
&\quad = \|u_t(\varphi(a) \oplus v_t^*\sigma(a)v_t)u_t^* - \psi(a) \oplus v_t^*\sigma(a)v_t\| \\
&\quad \leqslant 2\|v_t^*\sigma(a)v_t - \gamma(a)\| + \|u_t(\varphi(a) \oplus \gamma(a))u_t^* - \varphi(a) \oplus \gamma(a)\| \to 0.
\end{aligned}$$

By passing to the generalized Calkin algebra in the above inequalities we also obtain

$$w_t(\varphi(a) \oplus \sigma(a))w_t^* - \psi(a) \oplus \sigma(a) \in \mathcal{K}(H_2) \otimes B, \quad a \in A.$$

□

If $\Phi\colon A \to M(\mathcal{K}(H) \otimes B)$ is a representation, we define a C^*-algebra D_Φ by

$$D_\Phi = \{b \in M(\mathcal{K}(H) \otimes B) \mid [b, \Phi(A)] \subset \mathcal{K}(H) \otimes B\}.$$

We use the dot as a shorthand to indicate composition by the quotient map $\pi\colon M(\mathcal{K}(H) \otimes B) \longrightarrow Q(\mathcal{K}(H) \otimes B)$. Thus, $\dot{\Phi} = \pi \circ \Phi$ maps from A to

$Q(\mathcal{K}(H)\otimes B)$. Furthermore, if $X \subseteq Q(\mathcal{K}(H)\otimes B)$, we denote by X^c the commutator of X in $Q(\mathcal{K}(H)\otimes B)$. One checks directly that

$$0 \longrightarrow \mathcal{K}(H)\otimes B \overset{j}{\hookrightarrow} D_\Phi \overset{\pi}{\longrightarrow} \dot{\Phi}(A)^c \longrightarrow 0 \tag{9}$$

is a short exact sequence of C^*-algebras.

LEMMA 3.5. *Let A be a unital separable C^*-algebra and let B be a σ-unital C^*-algebra. Let $\Phi\colon A \to M(\mathcal{K}(H)\otimes B)$ be a unital representation and let w_i be unitaries in D_Φ. Suppose that $[\Phi, \Phi, w_1] = [\Phi, \Phi, w_2]$ in $KK(A, B)$. Then there is a unital representation $\gamma\colon A \to M(\mathcal{K}(H)\otimes B)$ such that*

$$(\Phi\oplus\gamma, \Phi\oplus\gamma, w_1\oplus 1) \sim_{\mathrm{oh}} (\Phi\oplus\gamma, \Phi\oplus\gamma, w_2\oplus 1) \tag{10}$$

If Φ is strictly nuclear and $[\Phi, \Phi, w_1] = [\Phi, \Phi, w_2]$ in $KK_{\mathrm{nuc}}(A, B)$, then γ can be chosen to be strictly nuclear.

Proof. By [Kas80b, §6, Theorem 1], there is a degenerate cycle (γ_0, γ_1, v) with $\gamma_i\colon A \to \mathcal{L}(E_i)$ and $v \in \mathcal{L}(E_0, E_1)$ such that

$$(\Phi\oplus\gamma_0, \Phi\oplus\gamma_1, w_1\oplus v) \sim_{\mathrm{oh}} (\Phi\oplus\gamma_0, \Phi\oplus\gamma_1, w_1\oplus v). \tag{11}$$

Note that if Φ is strictly nuclear, then γ can be chosen to be strictly nuclear by [Ska88, Proposition 2.2(c)]. Let W_t be the continuous path from $w_1\oplus v$ to $w_2\oplus v$ which implements (11).

Since (γ_0, γ_1, v) is degenerate, we have for $a \in A$ that

$$v\gamma_0(a) - \gamma_1(a)v = 0, \qquad \gamma_0(a)(1 - v^*v) = 0, \qquad \gamma_1(a)(1 - vv^*) = 0,$$

Let $p_i = \gamma_i(1)$. After replacing E_i by p_iE_i, v by p_1vp_0 and W_t by $(1\oplus p_1)W_t(1\oplus p_0)$ we may assume that (11) holds with γ_i unital and v a unitary.

Next we observe that $(\Phi\oplus\gamma_0, \Phi\oplus\gamma_0, (1\oplus v^*)W_t)$ is a KK-cycle for all t, and we see from (11) that this implements an operatorial homotopy

$$(\Phi\oplus\gamma_0, \Phi\oplus\gamma_0, w_1\oplus 1) \sim_{\mathrm{oh}} (\Phi\oplus\gamma_0, \Phi\oplus\gamma_0, w_2\oplus 1). \tag{12}$$

To complete the proof we need to arrange that $E_i = H_B$. Let $\theta\colon A \to \mathcal{L}(H_B)$ be a unital faithful scalar representation of infinite multiplicity. Let $(\theta, \theta, 1)$ be the corresponding trivial triple. From (12), we have that

$$\begin{aligned}&(\Phi\oplus\gamma_0\oplus\theta, \Phi\oplus\gamma_0\oplus\theta, w_1\oplus 1\oplus 1)\\ &\quad\sim_{\mathrm{oh}} (\Phi\oplus\gamma_0\oplus\theta, \Phi\oplus\gamma_0\oplus\theta, w_2\oplus 1\oplus 1).\end{aligned} \tag{13}$$

Let $z\colon E_0\oplus H_B \to H_B$ be a unitary given by Kasparov's stabilization theorem. Setting $\gamma = z(\gamma_0\oplus\theta)z^*$ we see that (10) is a consequence of (13). Note that γ is strictly nuclear if γ_0 is so. □

PROPOSITION 3.6. *Let A be a separable C^*-algebra and let B be a σ-unital C^*-algebra. Let $\varphi, \psi\colon A \to M(\mathcal{K}(H) \otimes B)$ be representations such that $\varphi(a) - \psi(a) \in \mathcal{K}(H) \otimes B$ for all $a \in A$. Then if $[\varphi, \psi, 1] = 0$ in $KK(A, B)$, there is a representation $\gamma\colon A \to M(\mathcal{K}(H) \otimes B)$ such that $\varphi \oplus \gamma \approxeq \psi \oplus \gamma$.*

When

(a) *A has a unit and φ, ψ are unital,*

we may arrange that γ is unital. When

(b) *φ and ψ are strictly nuclear, and $[\varphi, \psi, 1] = 0$ in $KK_{\mathrm{nuc}}(A, B)$*

we may arrange that γ is strictly nuclear. When both (a) and (b) hold, we may arrange that γ is unital and strictly nuclear.

Proof. PART 1. We first prove the existence of γ under the unitality condition (a).

STEP 1A. Let $\sigma = \varphi_\infty \oplus \psi_\infty$ and set $\Phi' = \varphi \oplus \sigma$ and $\Psi' = \psi \oplus \sigma$. We note that then $\Psi' = \mathrm{Ad}(u') \circ \Phi'$ for some unitary $u' \in \mathcal{L}(H_B)$. We have

$$[\Phi', \Phi', u'] = [\Phi', \Psi', 1] = [\varphi, \psi, 1] = 0 = [\Phi', \Phi', 1]. \tag{14}$$

in $KK(A, B)$, so by Lemma 3.5 there is a unital representation $\gamma'\colon A \to \mathcal{L}(H_B)$ such that

$$(\Phi' \oplus \gamma', \Phi' \oplus \gamma', u' \oplus 1) \sim_{\mathrm{oh}} (\Phi' \oplus \gamma', \Phi' \oplus \gamma', 1 \oplus 1).$$

Without loss of generality we can assume that $\dot{\gamma}'\colon A \longrightarrow \mathcal{L}(H_B)/\mathcal{K}(H_B)$ is injective.

STEP 1B. Define $\Phi = \Phi' \oplus \gamma'$, $\Psi = \Psi' \oplus \gamma'$, $u = u' \oplus 1$. Then $\Psi = \mathrm{Ad}(u) \circ \Phi$, and $(\Phi, \Phi, u) \sim_{\mathrm{oh}} (\Phi, \Phi, 1)$. Using this operatorial homotopy, we show that u is homotopic in $\mathcal{U}(D_\Phi)$ to a unitary in $\mathcal{K}(H) \otimes B + \mathbb{C}1$. Indeed, if $\omega_s \in M(\mathcal{K}(H) \otimes B)$, $s \in [0, 1]$ is a norm continuous path of operators, implementing the operatorial homotopy, with $\omega_0 = u$, $\omega_1 = 1$ then

$$[\Phi(a), \omega_s] \in \mathcal{K}(H) \otimes B,$$
$$\Phi(a)(\omega_s\omega_s^* - 1),\ \Phi(a)(\omega_s^*\omega_s - 1) \in \mathcal{K}(H) \otimes B,$$

for all $s \in [0, 1]$. Since $\Phi(1) = 1$, we see that $\dot{\omega}_s$ is a continuous path of unitaries in $\dot{\Phi}(A)^c$ joining $\dot{u}$ with 1. Since $\mathcal{U}(D_\Phi) \to \mathcal{U}(\dot{\Phi}(A)^c)$ is a fibration with fiber $\mathcal{U}(\mathcal{K}(H \otimes B) + \mathbb{C}1)$, we obtain that u is homotopic to an element of $\mathcal{U}(\mathcal{K}(H \otimes B) + \mathbb{C}1)$.

STEP 1C. We prove that there is a continuous path of unitaries $(v_t)_{t \in [0,\infty)}$ with

$$v_t \in E_\Phi \stackrel{\mathrm{def}}{=} \Phi(A) + \mathcal{K}(H) \otimes B,$$

such that

$$\lim_{t \to \infty} \|v_t \Phi(a) v_t^* - \Psi(a)\| = 0, \qquad a \in A. \tag{15}$$

We have seen that here is a unitary $w \in \mathcal{U}(\mathcal{K}(H) \otimes B + \mathbb{C}1)$, which is homotopic to u in the unitary group of D_Φ. Since $\mathrm{Ad}(u)$ is homotopic to $\mathrm{Ad}(w)$ in the space $\mathrm{Aut}(E_\Phi)$ endowed with the uniform topology, it follows that $\mathrm{Ad}(w^*u) \in \mathrm{Aut}_0(E_\Phi)$. Let $(\alpha_s)_{s\in[0,1]}$ be a uniformly continuous path in $\mathrm{Aut}(E_\Phi)$ connecting id to $\alpha = \mathrm{Ad}(w^*u)$. Let D be the unital C^*-subalgebra of E_Φ generated by all subalgebras of the form

$$\alpha_{s_1}^{j_1} \circ \alpha_{s_2}^{j_2} \circ \cdots \circ \alpha_{s_n}^{j_n} \circ \Phi(A),$$

where $n \geqslant 1$, $j_k \in \mathbb{Z}$, and $s_k \in [0,1] \cap \mathbb{Q}$. Then D is separable, since A is separable, and $\alpha_t(D) = D$ for all $t \in [0,1]$. This proves that $\mathrm{Ad}(w^*u) \in \mathrm{Aut}_0(D)$. It follows by Proposition 2.14 that $\mathrm{Ad}(w^*u)$ is asymptotically inner in $\mathrm{Aut}(D)$. Thus there is a continuous path of unitaries $(v_t)_{t\in[0,\infty)} \subset wD \subset E_\Phi$ with $v_0 = w$ and

$$\lim_{t\to\infty} \|v_t d v_t^* - u d u^*\| = 0, \qquad d \in D.$$

Therefore

$$\lim_{t\to\infty} \|v_t \Phi(a) v_t^* - \Psi(a)\| = \lim_{t\to\infty} \|v_t \Phi(a) v_t^* - u\Phi(a)u^*\| = 0 \tag{16}$$

for all $a \in A$. This proves (15).

STEP 1D. Let $\gamma = \sigma \oplus \gamma'$. Since $\Phi = \varphi \oplus \gamma$ and $\Psi = \psi \oplus \gamma$, in order to complete the proof that γ has the desired properties in the presence of (a) it suffices to show that there is a continuous path of unitaries $u_t \in \mathcal{K}(H) \otimes B + \mathbb{C}1$ such that

$$\lim_{t\to\infty} \|u_t \Phi(a) u_t^* - \Psi(a)\| = 0, \qquad a \in A. \tag{17}$$

By definition of E_Φ, there are $x_t \in A$ and $y_t \in \mathcal{K}(H) \otimes B$ such that $v_t = \Phi(x_t) + y_t$. Since we have arranged that $\dot{\Phi}$ is a unital monomorphism, (x_t) must be a continuous path of unitaries. Moreover

$$\lim_{t\to\infty} \left\| x_t a x_t^* - a \right\| = 0, \quad a \in A \tag{18}$$

as a consequence of (15), since $\dot{\Phi} = \dot{\Psi}$ is norm-preserving. Define

$$u_t = v_t \Phi(x_t)^* = 1 + y_t \Phi(x_t)^*.$$

Then, from (15) and (18),

$$\begin{aligned} \|u_t \Phi(a) u_t^* - \Psi(a)\| &\leqslant \|v_t \Phi(a) v_t^* - \Psi(a)\| + \|v_t \Phi(x_t^* a x_t - a) v_t^*\| \\ &\quad \to 0, \quad \text{as}\, t \to \infty, \quad a \in A. \end{aligned}$$

This proves (17) and concludes the proof of the unital case of the proposition.

PART 2. We now briefly indicate how to choose γ strictly nuclear in the presence of both (a) and (b). Note that in STEP 1A if φ, ψ are strictly nuclear, then so is σ. Also, in this case one gets that then (14) holds in $KK_{\mathrm{nuc}}(A, B)$. We then may choose γ' strictly nuclear in STEP 1C by appealing to the nuclear version of Lemma 3.5 above. The remaining steps are carried out verbatim.

PART 3. The general claims are straightforward consequences of the unital cases. Indeed, after replacing A by $\widetilde{A}$ and φ, φ, γ by $\widetilde{\varphi}, \widetilde{\varphi}, \widetilde{\gamma}$, we may assume that the C^*-algebra A and all these maps are unital. When (b) holds, we still have $[\widetilde{\varphi}, \widetilde{\varphi}, 1] = 0$ in $KK_{\text{nuc}}(\widetilde{A}, \widetilde{B})$ and that $\widetilde{\gamma}$ is unital and strictly nuclear. □

Notes 3.7. The idea of invoking results about the automorphisms of C^*-algebras of the form E_Φ originates with Lin ([Lin97]) in the case $\Phi = d_\iota$ with d_ι defined as in Section 4 below.

Because of Lemma 3.3, the proposition above gives a new characterization of equivalence of cycles in the Cuntz picture of KK-theory.

THEOREM 3.8. *Let A be a (unital) separable C^*-algebra and let B be a separable C^*-algebra. Let $\varphi, \psi\colon A \to M(\mathcal{K}(H) \otimes B)$ be (unital) representations such that $\varphi(a) - \psi(a) \in \mathcal{K}(H) \otimes B$ for all $a \in A$. Then the following are equivalent.*

(i) $[\varphi, \psi, 1] = 0$ in $KK(A, B)$
(ii) $[\varphi, \psi] = 0$ in $KK_h(A, B)$
(iii) *There exists a (unital) representation $\sigma\colon A \to M(\mathcal{K}(H) \otimes B)$ with $\varphi \oplus \sigma \approxeq \psi \oplus \sigma$*
(iv) *For any (unital) absorbing representation $\gamma\colon A \to M(\mathcal{K}(H) \otimes B)$, $\varphi \oplus \gamma_\infty \approxeq \psi \oplus \gamma_\infty$*

Proof. We saw that (i)$\Longleftrightarrow$(ii) in Section 3.1. We have proved (ii)$\Longrightarrow$(iii) in the proposition above, and (iii)$\Longrightarrow$(ii) follows by Lemma 3.3. When (iii) holds, we have $\varphi \oplus \sigma \oplus \gamma_\infty \approxeq \psi \oplus \sigma \oplus \gamma_\infty$. In view of Lemma 3.4, this implies $\varphi \oplus \gamma_\infty \approxeq \psi \oplus \gamma_\infty$ since $\sigma \oplus \gamma_\infty \sim_{\text{asymp}} \gamma_\infty$ by Proposition 2.4 and the absorption property of γ. This proves (iii)$\Longrightarrow$(iv). Finally, to prove (iv)$\Longrightarrow$(iii), we note that an absorbing representation does exist by [Tho01, 2.4,2.7]. □

Remark 3.9. It is clear from the proof of Theorem 3.8, that the implications (i)$\Longleftrightarrow$(ii)$\Longrightarrow$(iii) hold for any σ-unital C^*-algebra B, while (iii)$\Longrightarrow$(i) holds whenever absorbing representations $\gamma\colon A \to M(\mathcal{K}(H) \otimes B)$ do exist, such as in the case when B is separable.

The similar characterization in the setting of KK_{nuc} is valid for any σ-unital B.

THEOREM 3.10. *Let A be a (unital) separable C^*-algebra and let B be a σ-unital C^*-algebra. Let $\varphi, \psi\colon A \to M(\mathcal{K}(H) \otimes B)$ be (unital) strictly nuclear representations such that $\varphi(a) - \psi(a) \in \mathcal{K}(H) \otimes B$ for all $a \in A$. Then the following are equivalent.*

(i) $[\varphi, \psi, 1] = 0$ in $KK_{\text{nuc}}(A, B)$

(ii) $[\varphi, \psi] = 0$ in $KK_{\mathrm{nuc},h}(A, B)$

(iii) *There exists a (unital) strictly nuclear representation* $\sigma\colon A \to M(\mathcal{K}(H) \otimes B)$ *with* $\varphi \oplus \sigma \approxeq \psi \oplus \sigma$

(iv) *For any (unital) nuclearly absorbing and strictly nuclear representation* $\gamma\colon A \to M(\mathcal{K}(H) \otimes B)$, $\varphi \oplus \gamma_\infty \approxeq \psi \oplus \gamma_\infty$

Proof. Arguments parallel to those in the proof above show that (i)–(iii) are equivalent. For (iii)$\Longrightarrow$(iv) we note that γ absorbs σ by definition. To prove (iv) $\Longrightarrow$(iii), we use the fact that a strictly nuclear and nuclearly absorbing representation does exist by Proposition 2.11. □

3.3. *K*-HOMOLOGY

We say that a $*$-representation $\varphi\colon A \longrightarrow \mathcal{L}(H)$ is admissible if it is faithful and nondegenerate and satisfies $\varphi(A) \cap \mathcal{K}(H) = \{0\}$. Equivalently, φ is nondegenerate and $\dot\varphi\colon A \to Q(H)$ is injective. The main result in [Voi76] states that any pair of admissible representations satisfies $\varphi \sim \psi$. In fact, more is proved:

THEOREM 3.11 [Voiculescu]. *If* φ *and* ψ *are admissible representations, we have* $\varphi \sim_{\mathrm{asymp}} \psi$.

Proof. The results of [Voi76] are stated only for unital A. To deal with the general case, note that if A is unital, then any admissible representation is necessarily unital. Moreover, if A is nonunital and φ is admissible, then $\widetilde{\varphi}\colon \widetilde{A} \to \mathcal{L}(H)$ is admissible. Therefore we may assume that A and both of φ and ψ are unital.

The proof of [Voi76, Theorem 1.3] gives a sequence of isometries $v_n \in \mathcal{L}(H)$ which satisfy

$$v_n\varphi_\infty(a) - \psi(a)v_n \in \mathcal{K}(H), \quad \|v_n\varphi_\infty(a) - \psi(a)v_n\| \longrightarrow 0, \quad a \in A.$$

Lemma 2.3 shows that $\varphi \oplus \psi \sim_{\mathrm{asymp}} \psi$, and the proof is complete by symmetry. □

THEOREM 3.12. *Let* A *be a separable* C^**-algebra and let* $\varphi, \psi\colon A \longrightarrow \mathcal{L}(H)$ *be a Cuntz pair of admissible representations. Then* $[\varphi, \psi] = 0$ *in* $KK(A, \mathbb{C}) = K^0(A)$ *if and only if* $\varphi \approxeq \psi$.

Proof. One implication is covered by Lemma 3.3. To prove the other, arguing as in the proof of Theorem 3.11, we may assume that A, φ and ψ are unital. By Theorem 3.11 we get a continuous family of unitaries $(u_t)_{t\in[0,\infty)}$ with

$$u_t\varphi(a)u_t^* - \psi(a) \in C_0([0, \infty)) \otimes \mathcal{K}(H). \tag{19}$$

We assume that $[\varphi, \psi] = 0$ and conclude from Lemma 3.1 and (19) that also $[\varphi, u_1\varphi u_1^*] = 0$ in $KK_h(A, B)$. Since $(\varphi, \varphi, u_1^*)$ is unitarily equivalent to $(\varphi, u_1\varphi u_1^*, 1)$, we conclude that $[\varphi, \varphi, u_1] = [\varphi, \varphi, u_1^*] = 0$ in $KK(A, B)$. By Paschke duality, $\dot u_1$ is homotopic to 1 in the unitary group of the commutant of $\dot\varphi(A)$ in the Calkin algebra. Indeed, since $KK(A, \mathbb{C}) \cong K_1(\dot\varphi(A)^c)$ we have that

$[\dot{u}_1] = 0$ in $K_1(\dot{\varphi}(A)^c)$ and the latter group is isomorphic to the quotient of the unitary group of $\dot{\varphi}(A)^c$ by the path component of the identity by [Pas81, Lemma 3].

Then as in STEP 1B of the proof of Proposition 3.6, using the principal fibration

$$\mathcal{U}(\mathcal{K}(H) + \mathbb{C}1) \to \mathcal{U}(D_\varphi) \to \mathcal{U}(\dot{\varphi}(A)^c)$$

we conclude that u_1 is homotopic to 1 in D_φ. Note that any element v of such a homotopy satisfies $v\varphi(a)v^* - \psi(a) \in \mathcal{K}(H)$. We may (and shall) hence assume that in fact $u_0 = 1$ in (19).

We define $(\alpha_t)_{t\in[0,\infty)}$ in $\mathrm{Aut}_0(E_\varphi)$ by $\alpha_t = \mathrm{Ad}(u_t)$, noting that this is a uniformly continuous family there. Hence Proposition 2.15 applies to give us a continuous family $(v_t)_{t\in[0,\infty)}$ such that

$$\lim_{t\longrightarrow\infty} \|\alpha_t(x) - \mathrm{Ad}(v_t)(x)\| = 0,$$

for any $x \in E_\varphi$. Combining this with (19) we get that v_t satisfies

$$\lim_{t\longrightarrow\infty} \left\| v_t\varphi(a)v_t^* - \psi(a)\right\| = 0,$$

for any $a \in A$. Since $\dot{\varphi}$ is faithful, we can replace $(v_t)_{t\in[0,\infty)}$ by a family of unitaries in $\mathcal{K}(H) + \mathbb{C}1$ as in STEP 1D of the proof of Proposition 3.6. □

4. Applications

Lin [Lin97] showed that if $\iota\colon A \to B$ is a unital embedding, if A is nuclear, and either A or B is simple, then the map $d_\iota\colon A \longrightarrow M(\mathcal{K}(H) \otimes B)$ defined by $d_\iota(a) = 1 \otimes \iota(a)$ is absorbing. This was generalized in [DE98] as follows.

THEOREM 4.1 *Let A be a unital separable C^*-algebra and let $\iota\colon A \hookrightarrow B$ be a unital full embedding, that is a unital embedding with the property that the linear span of $B\iota(a)B$ is dense in B for all nonzero $a \in A$. Then d_ι is nuclearly absorbing.*

The following two theorems are immediate applications of Theorem 3.10. Their proofs are given in full in [DE98], however we indicate here the basic idea behind them. Theorem 4.2 generalizes a result of Lin [Lin97], where A was assumed to be nuclear and to satisfy the universal coefficient theorem of Rosenberg and Schochet, and either A or B were simple.

THEOREM 4.2. *Let A be a unital, separable C^*-algebra and let B be a unital C^*-algebra. Assume that there exists a unital full embedding $\iota : A \to B$. Suppose that $\varphi, \psi : A \longrightarrow B$ are two nuclear unital $*$-homomorphisms with $[\varphi] = [\psi]$ in $KK_{\mathrm{nuc}}(A, B)$. Then there is a sequence of unitaries $u_n \in \mathcal{U}_{n+1}(B)$ such that*

$$\lim_{n\to\infty} \|u_n(\varphi(a) \oplus n\cdot\iota(a))u_n^* - \psi(a) \oplus n\cdot\iota(a)\| = 0, \quad a \in A.$$

Proof. We give only the idea of the proof. Since φ and ψ assume only compact values, we have

$$[\varphi, \psi, 1] = [\varphi, \psi, 0] = [\varphi, 0, 0] - [\psi, 0, 0] = 0$$

in $KK_{\mathrm{nuc}}(A, B)$. The representation d_ι is nuclearly absorbing by Theorem 4.1. By Theorem 3.10 we have $\varphi \oplus \sigma \approxeq \psi \oplus \sigma$, hence $\varphi \oplus \sigma \oplus d_\iota \approxeq \psi \oplus \sigma \oplus d_\iota$, for some unital strictly nuclear representation $\sigma\colon A \to M(\mathcal{K}(H) \otimes B)$. By Lemma 3.4 this implies

$$\varphi \oplus d_\iota \approxeq \psi \oplus d_\iota \tag{20}$$

since $\sigma \oplus d_\iota \sim_{\mathrm{asymp}} \sigma \oplus (d_\iota)_\infty \sim_{\mathrm{asymp}} (d_\iota)_\infty \sim_{\mathrm{asymp}} d_\iota$ by using Lemma 2.4 and the fact that d_ι is nuclearly absorbing. Compressing (20) by $p_n = (n+1) \cdot \iota(1)$ yields the result in the statement, since p_n is an approximate unit of projections of $\mathcal{K}(H) \otimes B$ which commutes with d_ι. □

Let A be a separable C^*-algebra and let (π_n) be a sequence of finite dimensional representations of A. We say that (π_n) is a separating sequence if for every $a \in A$ there is n such that $\pi_n(a) \neq 0$. If this is the case, we say that A is residually finite dimensional. The following result generalizes [Dad95, Theorem A] from commutative algebras to residually finite dimensional algebras.

THEOREM 4.3. *Let A be a separable unital residually finite dimensional C^*-algebra and let B be a unital C^*-algebra. Let (π_n) be a separating sequence of finite dimensional representations of A such that each π_n is repeated infinitely many times. Define $\gamma_n = \pi_1 \oplus \cdots \oplus \pi_n$, with $\gamma_n\colon A \to \mathbf{M}_{r_n}(\mathbb{C}) \subseteq \mathbf{M}_{r_n}(\mathbb{C}1_B)$. Let $\varphi, \psi\colon A \to B$ be two unital nuclear $*$-homomorphisms with $[\varphi] = [\psi]$ in $KK_{\mathrm{nuc}}(A, B)$. Then there is a sequence of unitaries $u_n \in \mathcal{U}_{r_n+1}(B)$ such that*

$$\lim_{n\to\infty} \|u_n(\varphi(a) \oplus \gamma_n(a))u_n^* - \psi(a) \oplus \gamma_n(a)\| = 0, \quad a \in A.$$

Proof. The proof is similar to the proof of the previous theorem. One replaces d_ι by $\gamma = \pi_1 \oplus \pi_2 \oplus \cdots$ regarded as a faithful scalar representation of infinite multiplicity into $M(\mathcal{K}(H) \otimes B)$, and 'compress' $\varphi \oplus \gamma \approxeq \psi \oplus \gamma$ by $p_n = 1 \oplus \gamma_n(1)$. □

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