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Classifying C^*-algebras via ordered, mod-p K-theory

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1 Introduction

We introduce an order structure on $K_0(-) \oplus K_0(-;\mathbb{Z}/p)$. This group may also be thought of as $K_0(-;\mathbb{Z} \oplus \mathbb{Z}/p)$. We exhibit new examples of real-rank zero C^*-algebras that are inductive limits of finite dimensional and dimension-drop algebras, have the same ordered, graded K-theory with order unit and yet are not isomorphic. In fact they are not even stably shape equivalent. The order structure on $K_0(-;\mathbb{Z} \oplus \mathbb{Z}/p)$ naturally distinguishes these algebras.

The same invariant is used to give an isomorphism theorem for such real-rank zero inductive limits. As a corollary we obtain an isomorphism theorem for all real-rank zero approximately homogeneous C^*-algebras that arise from systems of bounded dimension growth and torsion-free K_0 group.

At the 1980 Kingston conference, Effros posed the problem of finding suitable invariants for use in studying C^*-algebras that are limits of sequences of homogeneous C^*-algebras. These are now called almost homogeneous (AH) C^*-algebras. The classification of AH algebras is a rapidly developing field and we will not attempt to summarize all this activity. Instead, we will focus on the growth of the invariants used.

Specifically, we consider an AH algebra A that is the direct limit of a system of the form

$$\cdots \to \bigoplus C(X_{n,i}, M_{m_{n,i}}) \to \bigoplus C(X_{n+1,i}, M_{m_{n+1,i}}) \to \cdots ,$$

with the sums assumed finite and the spaces assumed to be finite CW complexes. Technically, these are not all the AH algebras, but we will use the term AH algebra in this paper to refer to this class of inductive limits.

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In the case in which the $X_{n,i}$ are one point spaces, A is just an AF algebra [Br]. It may well be argued that a major source of inspiration for using K-theory as an invariant for C^*-algebras was Elliott's classification [El1] of AF algebras up to isomorphism.

Effros and Kaminker [EfK1] started by studying shape equivalence for some higher-dimensional AH algebras. They defined a shape theory for C^*-algebras and were the first to formally identify the role of semiprojectivity. An apparent barrier soon came into view; we have shown in various papers that very few C^*-algebras, even commutative ones, are semiprojective. For example, among the sphere algebras, $C(S^n)$, only $C(S^1)$ is semiprojective [Lo1], [DL2]. Effros and Kaminker initiated a study of the shape of the limits of circle algebras, and published a paper [EfK2] classifying, up to shape equivalence, inductive limits of Cuntz algebras. That leads to another story, whose telling we leave to others.

Blackadar [Bl2] gave a treatment of shape theory that had nice formal properties. His approach involved semiprojective maps, which are more common than semiprojective C^*-algebras, but relied heavily on universal constructions. The importance of these universal constructions (cf. [D1]) remained somewhat obscure until the advent of E-theory.

Several papers appeared around 1988 that made progress despite the semiprojectivity problem. One was Blackadar's discovery [Bl3] that one possible limit of circle algebras was a UHF algebra. A subsystem could be seen for which the limit was the fixed point algebra under an automorphism of order two but which was not AF by virtue of having nontrivial K_1. This settled another of the questions Effros [Eff] raised in Kingston. It also sparked a discussion at the 1988 Durham Summer Research Institute of whether an inductive limit of circle algebras that had real rank zero and trivial K_1 group was necessarily AF. This was settled, positively, and with rapidity, by Elliott, as we shall see.

Soon after Blackadar's result appeared, Evans and Kishimoto [EK] constructed an inductive limit of sub-homogeneous C^*-algebras that had real rank zero and was not AF by virtue of having non-zero *torsion* K_1-group. Tensoring by a UHF algebra removed this obstruction to being AF, and indeed the result was AF. This solved yet another of the problems in [Eff]. The result of [Bl3] was extended to finite group actions in [BEEK]).

Another paper to emerge in 1988 was by the first named author and Nemethi [DN]. It showed that there was a way around the lack of semiprojectivity. The basic idea of semiprojectivity is that B is semiprojective when a $*$-homomorphism $B \to A$ can be replaced by a $*$-homomorphism $B \to A_n$, where

$$A_n = \bigoplus C(X_{n,i}, M_{m_{n,i}}) \,.$$

(Whether replace is to mean "is close to" or "is homotopic to" depends on the flavor of semiprojectivity used, as does the degree of uniqueness in the replacement.) The approach taken in this paper was to substitute "KK-semiprojectivity" and related concepts, the idea being that one can only ask that one can find an element of $KK(B, A_n)$ that, upon composition with the natural map $A_n \to A$, gives the desired element of $KK(B,A)$. This left the tricky problem of replacing KK-elements, at least the "positive" ones, by $*$-homomorphisms. This was partially solved using connective KK-theory.

By this point, a new structure on K-theory was needed. An order on $K_0(A) \oplus K_1(A)$ exists, namely that coming from the natural identification with $K_0(C(S^1) \otimes A)$. In [DN] are many shape results for which the complete invariant is $K_0(C(S^1) \otimes A)$ as an ordered, scaled, graded group. One of the subclasses thus classified up to shape were the limits of circle algebras.

It was Elliott [El2] who successfully combining the new element of real-rank zero [BP] with the shape ideas emerged with a classification, now up to *isomorphism*, of those inductive limits of circle algebras that happen to have real-rank zero. The invariant was again $K_0(A) \oplus K_1(A)$, with the same order as in the shape results, but a slightly different scale. In this case, the identification was with $KK(C(S^1), A)$ and the positivity was determined by the $*$-homomorphisms.

Remarkably this invariant was shown to be complete for the simple AH algebras of real rank zero arising from inductive systems with spectra of dimension at most three [EG]. This was a far reaching generalization of previous work of several authors including notably [Li] and [EGLP]. In several instances an apparently larger collection of C^*-algebras has been shown to be the same as a collection which had already been classified, thus extending that classification result. For example, Lin [Li] showed that when the $X_{n,i}$ are contractible subsets of the plane and the limit A has real rank zero then A is an AF algebra. In the same spirit, the first author [D3] and Gong [G2], extended the classification result of [EG] to simple AH algebras of real rank zero, for which the dimensions of the spaces $X_{n,i}$ are uniformly bounded (or with slow dimension growth). This was done by showing that such C^*-algebras are isomorphic to AH algebras over spaces of dimension at most three which were already classified in [EG].

In the *non-simple* case, it was shown by Gong [G1] that this invariant is not complete for the AH algebras of real rank zero. We construct below examples which shows that the same situation occurs for the class of C^*-algebras Elliott studied in [El2] and which we called AD algebras (of real rank zero). An indirect way for constructing such examples was suggested by Elliott, Gong and Su [G2]. The AD algebras are inductive limits of a limited class of subhomogeneous C^*-algebras. It was shown in [D2], that the class of AD algebras contains a large class of AH algebras.

We now get to the point of the present paper. In view of the above discussion, it is clear that stronger algebraic invariants are needed. What we propose is an order structure on what we think should be called K-theory with coefficients in $\mathbb{Z} \oplus \mathbb{Z}/p$. The mod p K-theory groups for C^*-algebras were studied in [S]. The groups $K_*(-;\mathbb{Z}/p)$ are determined by $K_*(A)$. As opposed to this, the order structure we define is not determined by the order on $K_0(A) \oplus K_1(A)$ and appears to be genuinely new. It is the extra element needed, in many cases, to allow classification of AH (AD) algebras.

We henceforth assume all C^*-algebras to be nuclear and separable.

The classes of C^*-algebras we shall initially study are all those that are obtained, through the processes of direct sum, tensoring by M_n for various n and inductive limits, from some basic C^*-algebras, or building blocks. The building blocks are $\mathbb{C}$, $C(S^1)$ and the unital dimension-drop intervals $\tilde{\mathbb{I}}_p$, described below. We call such C^*-algebras AD *algebras.*

This class of algebras is almost what Elliott studied in [El2]. Indeed, he studied the AD algebras of real rank zero.

By the dimension-drop interval, we mean either the non-unital version $\mathbb{I}_p$

$$\mathbb{I}_p = \{f \in C([0,1], M_p) \mid f(0) = 0 \text{ and } f(1) \in \mathbb{C}\}$$

or the unital version

$$\tilde{\mathbb{I}}_p = \{f \in C([0,1], M_p) \mid f(0), f(1) \in \mathbb{C}\} .$$

We can now give a simplified version of our example by exhibiting two AD C^*-algebras A_p and B_p that are not stably shape equivalent, and hence not homotopic, and yet $K_0(A_p) \oplus K_1(A_p) \cong K_0(B_p) \oplus K_1(B_p)$, even as ordered groups.

In both cases, the inductive limit is of the form

$$\cdots \longrightarrow \tilde{\mathbb{I}}_p \oplus \mathbb{C}^{2n+1} \longrightarrow \tilde{\mathbb{I}}_p \oplus \mathbb{C}^{2n+3} \longrightarrow \cdots .$$

For A_p the connecting maps are

$$(f, x_{-n}, \ldots, x_n) \mapsto (f, f(0), x_{-n}, \ldots, x_n, f(0))$$

while for B_p,

$$(f, x_{-n}, \ldots, x_n) \mapsto (f, f(0), x_{-n}, \ldots, x_n, f(1))$$

It is not hard to see that

$$A_p \cong \{f \in C(X, M_p) \mid f(n) \in \mathbb{C} \text{ for } n \in \mathbb{Z} \text{ and } f(\hat{0}), f(\hat{1}) \in \mathbb{C}\}$$

and

$$B_p \cong \{f \in C(Y, M_p) \mid f(n) \in \mathbb{C} \text{ for } n \in \mathbb{Z} \text{ and } f(\hat{0}), f(\hat{1}) \in \mathbb{C}\}$$

where X and Y are spaces that contain a copy of $\mathbb{Z}$ as an open subset, with complement a copy of $[0,1]$, (with $t \in [0,1]$ denoted by $\hat{t}$) topologized so that

$$\lim_{n \to -\infty} n = \hat{0}, \quad \lim_{n \to +\infty} n = \hat{0}$$

in X while, in Y,

$$\lim_{n \to -\infty} n = \hat{0}, \quad \lim_{n \to +\infty} n = \hat{1} .$$

One way to describe the order Elliott imposes on

$$K_*(D) = K_0(D) \oplus K_1(D)$$

is that the positive elements are those that are in the image of $\hom(C(S^1), D \otimes \mathcal{K})$ under the mapping

$$\varphi \mapsto ([\varphi(1)], [\varphi(e^{2\pi i t})]) .$$

An alternative picture is $K_*(D) = KK(C(S^1), D)$ with

$$K_*(D)_+ = \{[\varphi] \mid \varphi \in \hom(C(S^1), M_n(D)) \text{ for some } n.\}$$

From [El2] we recall

$$K_*(\mathbb{C}) = K_*(\mathbb{C})_{\mathrm{ev}} = \mathbb{Z}, \quad K_*(\mathbb{C})_+ = \mathbb{N}$$

and

$$K_*(\tilde{\mathbb{I}}_p) \cong \mathbb{Z} \oplus \mathbb{Z}/p, \quad K_*(\tilde{I}_p)_{\mathrm{ev}} = \mathbb{Z} \oplus 0 ,$$

$$K_*(\tilde{\mathbb{I}}_p)_+ = \{(a, \bar{b}) \mid a = b = 0 \text{ or } a \geqq 1\} .$$

(We assume b normalized so that $0 \leqq b \leqq p-1$.)

Given any C^*-algebra D with unit e we denote by $\delta_j \colon \tilde{\mathbb{I}}_p \to D$, for $j = 0, 1$, the map $\delta_j(f) = f(j)e$, while for $0 < t < 1$ we use $\delta_t \colon \tilde{\mathbb{I}}_p \to M_p(D)$ for the map $\delta_t(f) = e \otimes f(t)$. We can calculate now that

$$\delta_j \colon \tilde{\mathbb{I}}_p \to \mathbb{C}$$

induces

$$(a, \bar{b}) \mapsto a$$

as the map $(\delta_j)_* \colon K_*(\tilde{\mathbb{I}}_p) \to K_*(\mathbb{C})$. However δ_0 and δ_1 correspond to distinct KK-classes.

Our example exploits this inability of K_* to distinguish right from left endpoint. Since K_* commutes with inductive limits, the above calculation is enough to show that $K_*(A_p) \cong K_*(B_p)$ as ordered, graded abelian groups. The additional structure of order unit is also preserved. In fact, one may calculate that $K_*(A_p) = K_*(B_p)$ consists of those elements $(a, (a_j), \bar{b}) \in \mathbb{Z} \oplus \prod_{-\infty}^{\infty} \mathbb{Z} \oplus \mathbb{Z}/p$ for which there is $m \geqq 0$ such that $a_j = a$ for $|j| \geqq m$. An element $(a, (a_n), \bar{b})$ is positive if $a_j \geqq 0$ for all j and either $\bar{b} = 0$ or $a \geqq 1$.

To create an invariant that can distinguish the two ends of $\tilde{\mathbb{I}}_p$ we let $\tilde{\mathbb{I}}_p$ itself play the role that $C(S^1)$ plays for K_*. That is, while the positive elements in $K_*(A)$ are those represented by a $*$-homomorphism $\varphi \colon C(S^1) \to A \otimes M_n$, in our invariant, the positive elements will be defined as those that are represented by a $*$-homomorphism from $\tilde{\mathbb{I}}_p$ to $A \otimes M_n$. This leads to a manageable invariant since we have previously shown that $\mathbb{I}_p$ and $\tilde{\mathbb{I}}_p$ have many of the same properties that are had by $C_0(0,1)$ and $C(S^1)$ (see [Lo2] and [DL1]).

2 The invariant

Throughout the paper we denote by G_p the group $\mathbb{Z} \oplus \mathbb{Z}/p$ where $p \geqq 2$ is not necessarily a prime. As a graded group, our invariant is defined as

$$K_0(A; G_p) = K_0(A) \oplus K_0(A; \mathbb{Z}/p)$$

with the grading

$$K_0(A;G_p)^{\mathrm{ev}} = K_0(A) \oplus 0, \quad K_0(A;G_p)^{\mathrm{odd}} = 0 \oplus K_0(A;\mathbb{Z}/p).$$

To define $K_0(A;\mathbb{Z}/p)$ one may choose any nuclear C^*-algebra P in the bootstrap category of [RS] such that $K_0(P) = 0$ and $K_1(P) = \mathbb{Z}/p$. Then $K_0(A;\mathbb{Z}/p) \cong K_1(A \otimes P) \cong KK(P,A)$ (see [S], [Bl1]). Naturally, we choose $P = \mathbb{I}_p$. We may then use the split-exact sequence in KK arising from $0 \to \mathbb{I}_p \to \tilde{\mathbb{I}}_p \to \mathbb{C} \to 0$ to see that

$$K_0(A;G_p) = KK(\tilde{\mathbb{I}}_p, A).$$

The six-term exact sequence for KK now gives us an exact sequence

$$\begin{array}{ccccc} K_0(I;G_p) & \longrightarrow & K_0(A;G_p) & \longrightarrow & K_0(A/I;G_p) \\ \uparrow & & & & \downarrow \\ K_0(S(A/I);G_p) & \longleftarrow & K_0(SA;G_p) & \longleftarrow & K_0(SI;G_p) \end{array}$$

for any ideal I of A.

The following lemma, which follows from the universal coefficient theorem, shows that the group structure of $K_0(A;G_p)$ is completely determined by the ordinary K-theory. The complication will be in the order.

2.1. Lemma. [S] *There is a natural exact sequence of groups*

$$K_0(A) \xrightarrow{\times p} K_0(A) \to K_0(A;\mathbb{Z}/p) \to K_1(A) \xrightarrow{\times p} K_1(A).$$

2.2. Proposition. *Let A be unital. Given $\alpha \in K_0(A;\mathbb{Z}/p) = KK(\mathbb{I}_p, A)$, there exist n and a $*$-homomorphism $\varphi\colon \tilde{\mathbb{I}}_p \to M_n(A)$ such that $[\varphi|_{\mathbb{I}_p}] = \alpha$.*

Proof. We showed in [DL1] that

$$K_0(A;\mathbb{Z}/p) \cong \varinjlim [\mathbb{I}_p, M_n(A)].$$

Choose any $*$-homomorphism $\mathbb{I}_p \to M_n(A)$ that has KK class α and extend by sending unit to unit. Q.E.D.

Given this proposition, it makes sense to ask what are the possible K_0 classes obtained as $[\varphi(1)]$ for maps $\varphi\colon \tilde{\mathbb{I}}_p \to A \otimes \mathscr{K}$ such that $\varphi|_{\mathbb{I}_p}$ represents a given KK class. We thus define

$$K_0(A;G_p)^+ = \{([\varphi(1)], [\varphi|_{\mathbb{I}_p}] \in K_0(A) \oplus K_0(A;\mathbb{Z}/p)) \mid \varphi \in \hom(\tilde{\mathbb{I}}_p, M_n(A))\}.$$

The above Proposition can be restated as the statement that, for any $\alpha \in K_0(A;\mathbb{Z}/p)$, and assuming $1 \in A$,

$$\{a \in K_0(A) \mid (a,\alpha) \geqq 0\} \neq \emptyset.$$

Another consequence deserves a formal statement.

2.3. Lemma. *Suppose $1 \in A$. Every element of $K_0(A; G_p)$ can be expressed as $[\varphi] - n[\delta_0]$ for some natural number n and some $*$-homomorphism $\varphi\colon \tilde{\mathbb{I}}_p \to M_n(A)$.*

2.4. Lemma. *Suppose $1 \in A$. Given $\varphi_0, \varphi_1\colon \tilde{\mathbb{I}}_p \to M_n(A)$ $*$-homomorphisms, $[\varphi_0] = [\varphi_1]$ in $K_0(A; G_p)$ if and only if, there is a $*$-homomorphism η of the form $\eta = \delta_0 \oplus \cdots \oplus \delta_0 \oplus 0 \oplus \cdots \oplus 0$ such that $\varphi_0 \oplus \eta$ is homotopic to $\varphi_1 \oplus \eta$ as $*$-homomorphisms $\tilde{\mathbb{I}}_p \to M_K(A)$.*

Proof. This follows from [DL1] in a similar fashion as Lemma 2.3. Q.E.D.

2.5. Proposition. *If $A = \varinjlim A_n$ then*

$$K_0(A; G_p) = \varinjlim K_0(A_n; G_p)$$

as ordered, graded groups.

Proof. It is easy to reduce to case where A and the A_n have a common unit. The semiprojectivity for $\tilde{\mathbb{I}}_p$ implies any map

$$\varphi\colon \tilde{\mathbb{I}}_p \to M_k(A) = \varinjlim M_k(A_n)$$

is homotopic, hence equal in KK, to the composition of a map $\tilde{\mathbb{I}}_p \to M_k(A_n)$ with the natural map $M_k(A_n) \to M_k(A)$. Since also $\delta_0\colon \tilde{\mathbb{I}}_p \to A_n$ followed by the natural map $A_n \to A$ is again $\delta_0\colon \tilde{\mathbb{I}}_p \to A$, we have shown that the induced maps $K_0(A_n; G_p) \to K_0(A; G_p)$ combine to define a surjection of $\varinjlim K_0(A_n; G_p)$ onto $K_0(A; G_p)$. We have also shown that every positive element is the image of a positive element. Semiprojectivity also shows that if $\varphi_0, \varphi_1\colon \tilde{\mathbb{I}}_p \to M_k(A_n)$ become homotopic after composition with $M_k(A_n) \to M_k(A)$, they are in fact homotopic after only composing with the natural map $M_k(A_n) \to M_k(A_l)$ for some $l \geqq n$. Injectivity follows, by using Lemma 2.4. Q.E.D.

2.6. Proposition. *The graded ordered group $K_0(-; G_p)$ is a shape invariant.*

Proof. We know from [D1] that A and B are shape equivalent if and only if there are asymptotic morphisms

$$(\alpha_t)\colon A \to B, \quad (\beta_t)\colon B \to A$$

that compose in either direction to be homotopic to the appropriate identity map. These determine, via the Connes-Higson E-theory, elements in $KK(A, B)$ and $KK(B, A)$, which, being inverse, determine $\alpha_* = \beta_*^{-1}\colon KK(\tilde{\mathbb{I}}_p, A) \to KK(\tilde{\mathbb{I}}_p, B)$. A consequence of the semiprojectivity of $\tilde{\mathbb{I}}_p$ is that the natural map

$$[\tilde{\mathbb{I}}_p, D] \to [[\tilde{\mathbb{I}}_p, D]]$$

is always an isomorphism. ([[-,-]] denoting homotopy classes of asymptotic morphisms.) Consider a positive element of $KK(\tilde{\mathbb{I}}_p, A)$, represented by

$\varphi\colon \tilde{\mathbb{I}}_p \to A \otimes M_n$. This is sent by α_* to the KK class of the asymptotic morphism $((\alpha_t \otimes 1)\varphi)\colon \tilde{\mathbb{I}}_p \to A \otimes M_n$. But this is homotopic to a $*$-homomorphism, and so represents a positive KK element. Thus α_*, and similarly α_*^{-1}, is positive. Q.E.D.

One can regard the ordered group $K_0(A; G_p)$ as a homotopy invariant functor on the category of C^*-algebras. By Proposition 2.5 this functor is continuous. Therefore $K_0(A, G_p)$ factorizes though the shape functor of [Bl2]. This argument gives an alternative proof to Proposition 2.6.

3 Examples

Our goal is to calculate $K_0(A_p; G_p)$ and $K_0(B_p; G_p)$ and show that the order distinguishes these C^*-algebras. With a little more work we will do the same for related C^*-algebras of real rank zero.

3.1. Lemma.

$$K_0(\mathbb{C}; G_p) = G_p = \mathbb{Z} \oplus \mathbb{Z}/p, \quad K_0(\mathbb{C}; G_p)^{\mathrm{ev}} = \mathbb{Z} \oplus 0\,,$$

$$K_0(\mathbb{C}; G_p)^+ = \{(a, \bar{x}) \mid a \geqq x\}$$

with the identification $[\delta_j] = (1, j)$ *for* $j = 0, 1$.

Proof. By Lemma 2.1, $K_0(\mathbb{C}; G_p)$ is, as a graded group, isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/p$. A specific map from $K_0(\mathbb{C}; G_p)$ to $\mathbb{Z} \oplus \mathbb{Z}/p$ can be defined as follows. Every $*$-homomorphism $\varphi\colon \tilde{\mathbb{I}}_p \to M_n$ can be decomposed into c_0 copies of δ_0, c_1 copies of δ_1 and m copies of various δ_t for $t \in (0,1)$. As φ varies continuously, c_0 can vary as δ_t converges to p copies of δ_0, and similarly c_1 may vary, but both c_0 and c_1 remain well defined mod p. Sending the homotopy class of φ to $(c_0 + c_1 + mp, \bar{c}_1) \in \mathbb{Z} \oplus \mathbb{Z}/p$ is a map that is additive, relative to direct sums. Therefore this extends to a group homomorphism $K_0(\mathbb{C}; G_p) \to \mathbb{Z} \oplus \mathbb{Z}/p$. It is onto because

$$[\delta_j] \to (1, \bar{j}), \quad j = 0, 1\,,$$

and so must be an isomorphism and, one may verify, graded. Also, since c_0 and m are non-negative, we have

$$K_*(\mathbb{C}; G_p)^+ \subseteq \{(a, \bar{x}) \mid a \geqq x\}\,,$$

while the other inclusion follows from the fact that $[\delta_j]$ is positive. Q.E.D.

By $\overline{\mathrm{id}}\colon \tilde{\mathbb{I}}_p \to \tilde{\mathbb{I}}_p$ we mean $\overline{\mathrm{id}}(f)(t) = f(1-t)$.

3.2. Lemma.

$$K_0(\tilde{\mathbb{I}}_p; G_p) = \mathbb{Z} \oplus \mathbb{Z}/p \oplus \mathbb{Z}/p, \quad K_0(\tilde{I}_p; G_p)^{\mathrm{ev}} = \mathbb{Z} \oplus 0 \oplus 0\,,$$

$$K_0(\tilde{\mathbb{I}}_p; G_p)^+ = \{(a, \bar{b}, \bar{c}) \mid a \geqq b \text{ and } a \geqq c\}$$

with the identification $[\delta_j] = (1,j,j)$, *for* $j = 0, 1$, $[\mathrm{id}] = (1,0,1)$ *and* $[\overline{\mathrm{id}}] = (1,1,0)$.

Proof. It follows from Lemma 2.1 that $K_0(SM_p; G_p) = 0$. Recall that $\delta_j\colon \tilde{\mathbb{I}}_p \to \mathbb{C}$ are evaluation maps. Since a copy of SM_p sits inside $\tilde{\mathbb{I}}_p$, with quotient $\mathbb{C} \oplus \mathbb{C}$, the quotient map being $\delta_0 \oplus \delta_1\colon \tilde{\mathbb{I}}_p \to \mathbb{C} \oplus \mathbb{C}$, we have by the six-term exact sequence an injection

$$(\delta_0)_* \oplus (\delta_1)_*\colon K_0(\tilde{\mathbb{I}}_p; G_p) \to (\mathbb{Z} \oplus \mathbb{Z}/p) \oplus (\mathbb{Z} \oplus \mathbb{Z}/p)\,.$$

The image is clearly contained in the subgroup $\{(a, \bar{b}, a, \bar{c})\}$. Since $(\delta_j)_*$ is positive, the image of the positive cone is contained in

$$\{(a, \bar{b}, a, \bar{c}) \mid a \geqq b \text{ and } a \geqq c\}\,.$$

That these containments are actually equalities follows from the calculation of the image of some specific elements under $(\delta_0)_* \oplus (\delta_1)_*$:

$$[\delta_j] \mapsto (1,j,1,j), \quad [\mathrm{id}] \mapsto (1,0,1,1), \quad [\overline{\mathrm{id}}] \mapsto (1,1,1,0)\,. \qquad \text{Q.E.D.}$$

Using the above identifications it is now easy to calculate $\alpha_*\colon K_*(A; G_p) \to K_*(B; G_p)$ for various maps α between building blocks. Using matrix notation for the group homomorphisms, we find:

$$\alpha = \delta_0\colon \tilde{\mathbb{I}}_p \to \mathbb{C} \Longrightarrow \alpha_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$\alpha = \delta_1\colon \tilde{\mathbb{I}}_p \to \mathbb{C} \Longrightarrow \alpha_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\alpha\colon \mathbb{C} \to \tilde{\mathbb{I}}_p,\ \alpha(\lambda) = \lambda I \Longrightarrow \alpha_* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

$$\alpha = \overline{\mathrm{id}}\colon \tilde{\mathbb{I}}_p \to \tilde{\mathbb{I}}_p \Longrightarrow \alpha_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$\alpha = \delta_t\colon \tilde{\mathbb{I}}_p \to M_p(\tilde{\mathbb{I}}_p),\ 0 < t < 1, \Longrightarrow \alpha_* = \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We now return to the examples A_p and B_p described in section one. The group $K_0(-; G_p)$ is, in both cases, the limit of a system

$$\cdots \to (\mathbb{Z} \oplus \mathbb{Z}/p \oplus \mathbb{Z}/p) \oplus (\mathbb{Z} \oplus \mathbb{Z}/p)^{2n+1} \to \cdots,$$

where a typical element

$$g = (a, \bar{b}, \bar{c}) \oplus (a_{-n}, \bar{x}_{-n}, \ldots, a_n, \bar{x}_n)$$

is positive if $a \geqq b, c$ and $a_j \geqq x_j$ for all j. For A_p, the connecting maps act by sending g to

$$(a, \bar{b}, \bar{c}) \oplus (a, \bar{b}, a_{-n}, \bar{x}_{-n}, \ldots, a_n, \bar{x}_n, a, \bar{b})$$

while, for B_p, the same element is sent to

$$(a, \bar{b}, \bar{c}) \oplus (a, \bar{b}, a_{-n}, \bar{x}_{-n}, \ldots, a_n, \bar{x}_n, a, \bar{c})$$

Therefore, we may identify both groups as subgroups of

$$\mathbb{Z} \oplus \mathbb{Z}/p \oplus \mathbb{Z}/p \oplus \prod_{j=-\infty}^{+\infty} \mathbb{Z} \oplus \prod_{j=-\infty}^{+\infty} \mathbb{Z}/p \, .$$

Indeed, $K_0(A_p; G_p)$ consists of those elements $((a, \bar{b}, \bar{c}), (a_j), (\bar{x}_j))$ for which there is $m \geqq 0$ such that $a_j = a$ and $\bar{x}_j = \bar{b}$ for $|j| \geqq m$. An element $((a, \bar{b}, \bar{c}), (a_j), (\bar{x}_j))$ is positive if

$$a \geqq b, c \quad \text{and} \quad \forall j, \; a_j \geqq x_j \, .$$

(As always, inequalities are taken after normalizing $\bar{r}$ so that $0 \leqq r \leqq p - 1$.)

On the other hand, $K_0(B_p; G_p)$ consists of those elements $((a, \bar{b}, \bar{c}), (a_j), (\bar{x}_j))$ for which there is $m \geqq 0$ such that $a_j = a$ for $|j| \geqq m$, $\bar{x}_j = \bar{b}$ for $j \leqq -m$ and $\bar{x}_j = \bar{c}$ for $j \geqq m$. An element $((a, \bar{b}, \bar{c}), (a_j), (\bar{x}_j))$ is positive if

$$a \geqq b, c \quad \text{and} \quad \forall j, \; a_j \geqq x_j \, .$$

In both cases, an element $((a, \bar{b}, \bar{c}), (a_j), (\bar{x}_j))$ is even if and only if $\bar{b} = \bar{c} = \bar{x}_j = 0$. This even subgroup is a familiar ordered group, $c_1(\mathbb{Z})$.

Now suppose that Φ: $K_0(A_p; G_p) \to K_0(B_p; G_p)$ is an isomorphism of ordered, graded groups. The only order-preserving automorphisms of $c_1(\mathbb{Z})$ are found by reindexing the indices. As there is an evident automorphism (graded and order-preserving) on $K_0(A_p; G_p)$ found by simultaneously reindexing the a_j and the x_j we may as well assume that Φ acts as the identity on the even subgroup.

Consider the elements z and h_k of $K_0(A_p; G_p)$,

$$z = ((0, 0, 1), (0), (0)) \, ,$$

$$h_k = ((1, 0, 0), (1 - \delta_{jk}), (0)) \geqq 0 \, .$$

Since Φ preserves the grading, $\Phi(z)$ is of the form

$$\Phi(z) = ((0, \bar{e}, \bar{f}), (0), (\bar{z}_j)) \, .$$

Since Φ acts as identity on the even subgroup, $\Phi(h_k) = h_k$. Each element $h_k + z$ is positive. Thus, the element

$$\Phi(h_k + z) = ((1, \bar{e}, \bar{f}), (1 - \delta_{jk}), (\bar{z}_j))$$

must be positive. This implies that $z_k = 0$. Varying k we see that $\bar{z}_j = 0$ for all j. Then $\bar{e}$ and $\bar{f}$, being limits of the $\bar{z}_j$, are also zero. Therefore, $\Phi(z) = 0$, contradicting the injectivity of Φ. By this contradiction we have proven that the ordered group $K_0(A_p; G_p)$ is not isomorphic to $K_0(B_p; G_p)$.

With the final observation that $K_0(D \otimes \mathcal{K}; G_p)^+ \cong K_0(D; G_p)^+$, we have proven that $A_p \otimes \mathcal{K}$ and $B_p \otimes \mathcal{K}$ are not shape equivalent, hence not homotopic and not isomorphic.

The fact that this phenomenon can occur for real rank zero AD algebras was suggested by Elliott, Gong and Su, [G2]. Their idea was to construct an even dimensional analogue of an example of Gong [G1]. That will involve inductive limits of homogeneous C^*-algebras with 3-dimensional spectra and torsion free K_0-groups. Then one may apply Theorem 2.4 in [D2] to conclude that such an algebra is isomorphic to an AD algebra. We adopt a more direct approach that illustrates the essential nature of our invariant.

3.3. Theorem. *There exists, for every $p = 2, 3, 4, \ldots$, unital* AD *algebras C_p and D_p such that* $\mathrm{RR}(C_p) = \mathrm{RR}(D_p) = 0$ *and $K_*(C_p) \cong K_*(D_p)$ as graded, ordered groups with order unit, and yet $C_p \otimes \mathcal{K}$ and $D_p \otimes \mathcal{K}$ are not isomorphic or shape equivalent.*

Proof. To modify A_p and B_p to have real rank zero, we need to replace the dimension drop intervals by large matrix algebras over them. This allows us to include many point-evaluations at interior points. The C^*-algebras C_p and D_p we construct below will have real rank zero by Theorem 6.2(ii) in [El2].

Define $\psi_n = \mathrm{id} \oplus \delta_{t_n} : \tilde{\mathbb{I}}_p \to M_{p+1}(\tilde{\mathbb{I}}_p)$ where t_n is a sequence dense in $(0,1)$. Now set $q = p+1$ and set

$$\varphi_n = \psi_n \otimes \mathrm{id} : M_{q^{n+1}}(\tilde{\mathbb{I}}_p) \to M_{q^{n+2}}(\tilde{\mathbb{I}}_p) .$$

Both C_p and D_p are inductive limits of the form

$$\cdots \longrightarrow M_{q^{n+1}}(\tilde{\mathbb{I}}_p) \oplus M_{q^n} \oplus \cdots \oplus M_q \oplus \mathbb{C} \oplus M_q \oplus \cdots \oplus M_{q^n} \longrightarrow$$

$$M_{q^{n+2}}(\tilde{\mathbb{I}}_p) \oplus M_{q^{n+1}} \oplus M_{q^n} \oplus \cdots \oplus M_q \oplus \mathbb{C} \oplus M_q \oplus \cdots \oplus M_{q^n} \oplus M_{q^{n+1}} .$$

For C_p the connecting maps are

$$(f, x_{-n}, \ldots, x_n) \to (\varphi_n(f), f(0), x_{-n}, \ldots, x_n, f(0))$$

while for D_p,

$$(f, x_{-n}, \ldots, x_n) \to (\varphi_n(f), f(0), x_{-n}, \ldots, x_n, f(1))$$

These inductive systems satisfy the condition (ii) of Theorem 6.2 in [El2], hence both C_p and D_p have real rank zero. One may check that $K_*(C_p) = K_*(D_p)$ consists of those elements $(a, (a_j), \bar{b}) \in \mathbb{Z}[1/q] \oplus \prod_{-\infty}^{\infty} \mathbb{Z} \oplus \mathbb{Z}/p$ for which there is $m \geqq 0$ such that $a_j = aq^{|j|}$ for $|j| \geqq m$. An element $((a_j), \bar{b})$ is positive if $a_j \geqq 0$ for all j and

$$b = 0 \quad \text{or} \quad a > 0$$

The ordered, graded, $K_0(-;G_p)$ groups are, as before, the limit of a system

$$\cdots \to (\mathbb{Z} \oplus \mathbb{Z}/p \oplus \mathbb{Z}/p) \oplus (\mathbb{Z} \oplus \mathbb{Z}/p)^{2n+1} \to \cdots ,$$

where a typical element

$$g = (a, \bar{b}, \bar{c}) \oplus (a_{-n}, \bar{x}_{-n}, \dots, a_n, \bar{x}_n)$$

is sent by the connecting maps to

$$(qa, \bar{b}, \bar{c}) \oplus (a, \bar{b}, a_{-n}, \bar{x}_{-n}, \dots, a_n, \bar{x}_n, a, \bar{b})$$

in the case of C_p, while, for D_p, it is sent to

$$(qa, \bar{b}, \bar{c}) \oplus (a, \bar{b}, a_{-n}, \bar{x}_{-n}, \dots, a_n, \bar{x}_n, a, \bar{c})$$

Therefore, we may identify both $K_0(-;G_p)$ groups as subgroups of

$$\mathbb{Z}[1/q] \oplus \mathbb{Z}/p \oplus \mathbb{Z}/p \oplus \prod_{j=-\infty}^{+\infty} \mathbb{Z} \oplus \prod_{j=-\infty}^{+\infty} \mathbb{Z}/p .$$

Indeed, $K_0(C_p;G_p)$ consists of those elements $((a,\bar{b},\bar{c}),(a_j),(\bar{x}_j))$ for which there is $m \geqq 0$ such that $a_j = aq^{|j|}$ and $\bar{x}_j = \bar{b}$ for $|j| \geqq m$. An element $((a,\bar{b},\bar{c}),(a_j),(\bar{x}_j))$ is positive if

$$(b = c = 0 \text{ or } a > 0) \quad \text{and} \quad \forall j,\ a_j \geqq x_j .$$

On the other hand, $K_0(D_p;G_p)$ consists of those elements $((a,\bar{b},\bar{c}),(a_j),(\bar{x}_j))$ for which there is $m \geqq 0$ such that $a_j = aq^{|j|}$ for $|j| \geqq m$, $\bar{x}_j = \bar{b}$ for $j \leqq -m$ and $\bar{x}_j = \bar{c}$ for $j \geqq m$. The positivity is determined by the same conditions.

In both cases the even part is a copy of the ordered group

$$\{(a,(a_j)) \in \mathbb{Z}[1/q] \oplus \textstyle\prod \mathbb{Z} \mid a_j = aq^{|j|} \text{ for large enough } |j|\} .$$

It is not hard to see that any order-preserving automorphism of this group is of the form $\Psi(a,(a_j)) = (q^r a, (a_{\sigma(j)}))$ where $r \in \mathbb{Z}$ and $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ is a bijection such that $\sigma(j) \in \{-j-r, j+r\}$ for all but finitely many j. Any such automorphism extends in an obvious way to a graded automorphism of the ordered group $K_0(C_p;G_p)$. Therefore any possible isomorphism $\Phi\colon K_0(C_p;G_p) \to K_0(D_p;G_p)$ can be modified to one that acts identically on the even subgroup.

The rest of the argument works with obvious minor changes. One uses the elements

$$z = ((0,0,1),(0),(0)), \quad h_k = ((1,0,0),((1-\delta_{jk})q^{|j|}),(0)) . \qquad \text{Q.E.D.}$$

4 An isomorphism theorem

Recall that the scale $\Sigma(A)$ of a C^*-algebra A is defined as the image of $[\mathbb{C},A]$ into $K_0(A) = KK(\mathbb{C},A)$. For $\alpha \in KK(A,B)$ the Kasparov product with α induces maps

$$\alpha_*: K_*(A) = KK(C(S^1),A) \to KK(C(S^1),B) = K_*(B)$$
$$\alpha_*: K_0(A;G_p) = KK(\tilde{\mathbb{I}}_p,A) \to KK(\tilde{\mathbb{I}}_p,B) = K_0(B;G_p)$$

Let $\mathcal{K}$ denote the compact operators acting on a separable infinite dimensional Hilbert space.

4.1. Theorem. *Suppose A and B are AD algebras of real rank zero. If there exists an invertible element $\alpha \in KK(A,B)$ such that, for all $p \geqq 2$,*

$$\alpha_*: K_*(A) \to K_*(B), \quad \alpha_*: K_0(A;G_p) \to K_0(B;G_p)$$

and their inverses, are all positive, then $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. If in addition α_* induces a bijection $\Sigma(A) \to \Sigma(B)$, then $A \cong B$.

Proof. Our strategy is basically the same as Elliott's. In fact, after we have realized α and $\beta = \alpha^{-1}$ in an intertwining diagram of $*$-homomorphisms that commute in KK, we will appeal to step four of the proof of [El2; Theorem 7.1].

Since $\Sigma(A \otimes \mathcal{K}) = K_0(A)^+$ it suffices to consider the case when the two C^*-algebras have isomorphic scales. Write A and B as inductive limits of basic blocks A_k and B_k. Let μ_k and ν_k denote the maps $A_k \to A$ and $B_k \to B$. Consider $A_1 = \bigoplus M_{n_j}(D_j)$ where each D_j equals $C(S^1)$ or $\tilde{\mathbb{I}}_{p_j}$. (Dropping $\mathbb{C}$ as a building block is no restriction.) We denote by $[-]$ the KK class of a $*$-homomorphism. Let γ_j denote the KK-class of the $*$-homomorphism

$$D_j \cong D_j \otimes e_{11} \hookrightarrow M_{n_j}(D_j) \hookrightarrow A_1 \to A\,.$$

We know by our assumptions that $\alpha_*(\gamma_j)$ is a positive element in either $KK(C(S^1),B)$ or $KK(\tilde{\mathbb{I}}_p,B)$, so there is a $*$-homomorphism $\psi_j: D_j \to B \otimes \mathcal{K}$ whose KK-class equals this. Using the fact that D_j is unital we are able to find now a $*$-homomorphism $\varphi_1: A_1 \to B \otimes \mathcal{K}$ such that, as a KK-diagram,

$$\begin{tikzcd} A_1 \arrow[r, "\mu_1"] \arrow[dr, "\varphi_1"'] & A \arrow[d, "\alpha"] \\ & B \otimes \mathcal{K} \end{tikzcd}$$

commutes, with all arrows except α actual $*$-homomorphisms. Let e denote the unit of A_1. Since α_* preserves the scales, $[\varphi_1(e)] = \alpha_*[\mu_1(e)] = [f]$ for some projection $f \in B$. Since the AD-algebras have stable rank one, it follows that $\varphi_1(e)$ is Murray-von Neumann equivalent to f. Therefore after conjugating with a partial isometry in $B \otimes \mathcal{K}$, we may assume that φ_1 maps into B. Using the semiprojectivity of A_1 we can find a $*$-homomorphism α_1 so that, after reindexing, we have a commutative diagram in KK

$$\begin{tikzcd} A_1 \arrow[r, "\mu_1"] \arrow[d, "\alpha_1"'] & A \arrow[d, "\alpha"] \\ B_1 \arrow[r, "\nu_1"'] & B \end{tikzcd}$$

Now, working with β and B_1 in place of α and A_1, we can find a $*$-homomorphism β_1 so that, after reindexing once, we have a diagram

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "\alpha_1"'] & A_2 \arrow[r] & A \arrow[d, shift left, "\alpha"] \\ B_1 \arrow[ur, "\beta_1"] \arrow[rr] & & B \arrow[u, shift left, "\beta"] \end{tikzcd}$$

that commutes except that the top-left triangle commutes only after composing with the map to A. Since $K_*(B_1)$ is finitely generated, $KK(B_1,A) = \varinjlim KK(B_1,A_j)$ by [RS], hence we have equality in fact after composing with some connecting map $A_2 \to A_m$. So re-indexing a second time we know the above diagram commutes in KK, with only α and β not actual $*$-homomorphisms.

By Elliott's step four, we conclude $A \cong B$. Q.E.D.

4.2. *Remarks* (a) Suppose A and B are unital AD algebras of real rank zero. Then the last condition of Theorem 4.1 can be replaced by $\alpha_*[1_A] = [1_B]$.

(b) Our isomorphism result is not a true classification result as it involves an invariant, $KK(A,B)$, of the pair A and B. However, it does imply Elliott's classification theorem for simple AD algebras of real rank zero. This goes as follows. Let A and B denote simple AD algebras of real rank zero. If $a \in K_0(A)^+$ is nonzero, then $(a,x) \in K_0(A;G_p)^+$ for any $x \in K_0(A;\mathbb{Z}/p)$ and $p \geqq 2$. Also $(a,y) \in K_*(A)^+$ for all $y \in K_1(A)$. Using the universal coefficient theorem of [RS], it follows from Theorem 4.1 that A is isomorphic to B if and only if $K_0(A) \cong K_0(B)$ as scaled, ordered groups and $K_1(A) \cong K_1(B)$ as abstract groups.

4.3. Corollary *The isomorphism Theorem* 4.1 *is true for C^*-algebras of real rank zero which are inductive limits of C^*-algebras of the form*

$$A_n = \bigoplus_{i=1}^{m} M_{k(i)}(X_i)$$

where X_i are finite CW complexes with $K^0(X_i)$ torsion free and the dimensions of the spectra of A_n form a bounded sequence.

Proof. Let A be a C^*-algebra as in the statement of the Corollary. By Theorem 2.4 in [D2], A is isomorphic to an AD algebra. Q.E.D.

The invariant studied in this paper has an odd-dimensional analogue. We can present these invariants in a compact way by defining $K_*(A;G_p)^+$ to be the image of $\hom(\tilde{\mathbb{I}}_p, A \otimes C(S^1) \otimes \mathbb{K})$ in $KK(\tilde{\mathbb{I}}_p, A \otimes C(S^1)) \cong K_*(A) \oplus K_*(A, \mathbb{Z}/p)$.

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