RESIDUALLY FINITE DIMENSIONAL C*-ALGEBRAS

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A C*-algebra is called residually finite dimensional (RFD for brevity) if it has a separating family of finite dimensional representations. A C*-algebra A is said to be AF embeddable if there is an AF algebra B and a $*$-monomorphisms $\alpha : A \to B$. In this note we discuss the question of AF embeddability of RFD algebras. Since a C*-subalgebra of a nuclear C*-algebra must be exact [Ki], the nonexact RFD algebras (such that the C*-algebra of the free group on two generators) are not AF embeddable.

In this note we show that the cone over any nuclear RFD algebra is AF embeddable (see Theorem 6). Using a result of Spielberg [S] we obtain that the AF embeddability of a nuclear RFD algebra A (with all ideals in the bootstrap category of [RS]) depends only on the homotopy type of A. The question whether all the exact or even nuclear RFD algebras are AF embeddable is open. The main ingredient of the proof is Theorem 5, which shows that if two $*$-homomorphisms from a nuclear RFD algebra A are asymptotically homotopic (in the sense of [CH]), then they are stably approximately unitarily equivalent. The case $A = C(X)$ with X a compact metric space was proved in [D_1]. The case when A is homogeneous is treated in [L_1]. Very general related results appear in [L_2]. We hope that the result given in Theorem 5 will be useful in the classification problem of simple nuclear C*-algebras. Indeed, by a result of Blackadar and Kirchberg [$BK_{1,2}$] any separable nuclear C*-algebra having a separating family of quasidiagonal representations, is an inductive limit of nuclear RFD algebras. The reader is refered to [GoMe], [ExL] and [D_2] for other results on RFD algebras.

Definition 1. *A C*-algebra A is called residually finite dimensional if for any nonzero element $a \in A$ there is a finite dimensional representation π of A such that $\pi(a) \neq 0$.*

For C*-algebras C, D let $CP(C, D)$ denote the set of all linear, contractive, completely positive maps from C to D. The elements of $CP(C, D)$ will be referred to as CP-contractions. If G is a finite subset of C and $\delta > 0$ we say that $\varphi \in CP(C, D)$ is *δ-multiplicative* on G if $\|\varphi(ab) - \varphi(a)\varphi(b)\| < \delta$ for all $a, b \in G$.

The following proposition is a consequence of a result of Kasparov [Ka].

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Proposition 2. [D_2] *Let A be a separable RFD C*-algebra and let $\varphi : A \to B$ be a nuclear $*$-homomorphism to some unital C*-algebra B. Then there is a sequence $\tau_n : A \to M_{r(n)-1}(B)$ of CP-contractions and there is a sequence $\mu_n : A \to M_{r(n)}(B)$ of $*$-homomorphisms with finite dimensional image such that*

$$\lim_{n\to\infty} \|diag(\varphi(a), \tau_n(a)) - \mu_n(a)\| = 0$$

for all $a \in A$. The $*$-homomorphisms μ_n are of the form $\mu_n(a) = u_n(\pi_n(a) \otimes 1_B)u_n^*$ where $\pi_n : A \to M_{r(n)}$ are $*$-representations and $u_n \in M_{r(n)}(B)$ are unitaries. If $\varphi(1_A) = 1_B$, then we may arrange that τ_n and μ_n are unital.

Note that the sequence (τ_n) is necessarily asymptotically multiplicative. That is $\|\tau_n(ab) - \tau_n(a)\tau_n(b)\| \to 0$, for all $a, b \in A$, as $n \to \infty$.

Definition 3. *A C*-algebra A is said to have property (H_w), if for any finite subset $F \subset A$ and any $\epsilon > 0$, there exist $r \in \mathbb{N}$, a CP-contraction $\tau : A \to M_{r-1}(A)$ and a $*$-homomorphism $\mu : A \to M_r(A)$ with finite dimensional image such that*

$$\|diag(\tau(a), a) - \mu(a)\| < \epsilon$$

for all $a \in F$.

Let X be a compact metric space. The C*-algebras of the form $M_n(C(X))$ satisfy a stronger version of property (H_w), where τ is required to be a $*$-homomorphism, see [D2].

Proposition 4. *Let A be a separable unital C*-algebra. Then the following assertions are equivalent.*
(i) A is a nuclear RFD algebra.
(ii) A has the property (H_w).

Proof. (i) $\Rightarrow$ (ii) This follows from Proposition 2, applied for $B = A$ and $\varphi = id_A$.
(ii) $\Rightarrow$ (i) Let F be a finite subset of A and let $\epsilon > 0$. Let τ, μ be as in the definition of property (H_w). If $e := e_{rr} \otimes 1_A$, then $\|a - e\mu(a)e\| < \epsilon$ for all $a \in F$. It follows that A is nuclear since id_A is pointwise-norm limit of CP-contractions with finite dimensional image. Moreover we see that $\|\mu(a)\| \geq \|a\| - \epsilon$ hence $\mu(a) \neq 0$ if $\epsilon < \|a\|/2 \neq 0$. This proves that A is RFD. □

The proof of the next result is very similar to the proof of [D_1, Lemma 1.4], with the crucial remark that one can replace property (H) by the much less restrictive property (H_w).

Theorem 5. *Let A be a nuclear, separable, RFD algebra. Let $\varphi, \psi : A \to B$ be two $*$-homomorphisms to a unital C*-algebra B. Suppose that φ is asymptotically homotopic to ψ. Then for any $\epsilon > 0$ and $F \subset A$ a finite set, there exist $k \in \mathbb{N}$, a $*$-homomorphism $\eta : A \to M_k(B)$ with finite dimensional image and a unitary $u \in U_{k+1}(\mathbb{C}1_B)$ such that*

$$\|u\, diag(\varphi(a), \eta(a))\, u^* - diag(\psi(a), \eta(a))\| < \epsilon$$

for all $a \in F$.

Proof. Without any loss of generality, we may assume that A is unital and that $\varphi(1) = \psi(1) = 1$. This is arranged by replacing A by $\widetilde{A}$ and φ, ψ and the homotopy by their unital extensions. For given $F \subset A$ and $\epsilon > 0$, let $r \in \mathbb{N}$, τ and μ be as in Definition 3. Then $D = \mu(A)$ is a finite dimensional C*-subalgebra of $M_r(A)$. By elementary perturbation theory (see [Br]), there is a finite subset G of D and there is $\delta > 0$ such that whenever E is a unital C*-algebra and $\Psi \in CP(D, E)$ is δ-multiplicative on G, there exists a $*$-homomorphism $\Psi' : D \to E$ satisfying $\|\Psi'(d) - \Psi(d)\| < \epsilon$ for all $d \in \mu(F)$. With G, r and δ as above it is not hard to see that there exist $\widehat{\delta} > 0$ and a finite set $\widehat{F} \subset A$ such that if $\theta \in CP(A, B)$ is $\widehat{\delta}$-multiplicative on $\widehat{F}$ then $\theta \otimes id_r : M_r(A) \to M_r(B)$ is δ-multiplicative on G.

By assumption, there is an asymptotic homotopy $(\Phi_t) : A \to B[0, 1]$ such that $\Phi_t^{(0)} = \varphi$ and $\Phi_t^{(1)} = \psi$ for all $t \in \mathbb{R}$. Since A is nuclear, by using the Choi-Effros Theorem, we may arrange that (Φ_t) is a CP-asymptotic morphism. This means that $\Phi_t \in CP(A, B[0, 1])$ for each $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$ large enough such that Φ_t is $\widehat{\delta}$-multiplicative on $\widehat{F}$. Having t fixed, by uniform continuity there is $n \in \mathbb{N}$ such that $\|\Phi_t^{(s)}(a) - \Phi_t^{(s')}(a)\| < \epsilon$ for all $a \in F$ and $|s - s'| < 1/n$. Define the sequence $\varphi_j = \Phi_t^{(j/n)} \in CP(A, B)$, $0 \le j \le n$. Note that $\varphi = \varphi_0$, $\psi = \varphi_n$ and

$$\lambda := \max_{a \in F} \max_{0 \le j \le n-1} \|\varphi_{j+1}(a) - \varphi_j(a)\| < \epsilon.$$

For $s \in \mathbb{N}$ set $\varphi_{s,j} = \varphi_j \otimes id_s : M_s(A) \to M_s(B)$. By construction each φ_j is $\widehat{\delta}$-multiplicative on $\widehat{F}$, hence $\varphi_{r,j}$ is δ-multiplicative on G. Because of the way G and δ were chosen, for each j, there is a $*$-homomorphism $\chi_j : D \to M_r(B)$ such that $\|\varphi_{r,j}(d) - \psi_j(d)\| < \epsilon$ for all $d \in \mu(F)$ and $j = 0, \ldots, n$.

Define L, $L' : A \to M_{nr}(B)$ by

$$L = diag(\varphi_{r-1,0}\tau, \varphi_0,\ \varphi_{r-1,1}\tau, \varphi_1,\ \ldots, \varphi_{r-1,n-1}\tau, \varphi_{n-1})$$

$$L' = diag(\varphi_0, \varphi_{r-1,0}\tau,\ \varphi_1, \varphi_{r-1,1}\tau,\ \ldots, \varphi_{n-1}, \varphi_{r-1,n-1}\tau)$$

Note that L is unitarily equivalent to L'. Thus there is a permutation unitary $u \in U_{nr+1}(B)$ such that

$$u\, diag(L', \varphi_n)u^* = diag(\varphi_n, L). \tag{1}$$

Since $\|\varphi_{j+1}(a) - \varphi_j(a)\| \le \lambda$ for all $a \in F$

$$\|diag(\varphi_0(a), L(a)) - diag(L'(a), \varphi_n(a))\| = \max_j \|\varphi_{j+1}(a) - \varphi_j(a)\| = \lambda. \tag{2}$$

Using (1) and (2) we obtain

$$(3) \qquad \|u\, diag(\varphi_0(a), L(a))u^* - diag(\varphi_n(a), L(a))\| \le \lambda$$

for all $a \in F$.

On the other hand

$$(4) \quad \|L(a) - diag(\varphi_{r,0}\mu(a), \dots, \varphi_{r,n-1}\mu(a))\|$$
$$= \|diag(\varphi_{r,0}(\tau(a) \oplus a - \mu(a)), \dots, \varphi_{r,n-1}(\tau(a) \oplus a - \mu(a)))\|$$
$$\le \|\tau(a) \oplus a - \mu(a)\| < \epsilon$$

for all $a \in F$, since $\varphi_{r,j}$ are norm decreasing. Note that $\|\varphi_{r,j}\mu(a) - \chi_j\mu(a)\| < \epsilon$ for all $a \in F$ since $\|\varphi_{r,j}(d) - \chi_j(d)\| < \epsilon$ for all $d \in \mu(F)$. This implies

$$(5) \qquad \|diag(\varphi_{r,0}\mu(a), \dots, \varphi_{r,n-1}\mu(a)) - diag(\chi_0\mu(a), \dots, \chi_{n-1}\mu(a))\| < \epsilon$$

for all $a \in F$. The $*$-homomorphism defined by $\eta = diag(\chi_0\mu, \dots, \chi_{n-1}\mu)$ has finite dimensional image. Using (4) and (5) we obtain

$$(6) \qquad \|L(a) - \eta(a)\| < 2\epsilon$$

for all $a \in F$. Combining (3) and (6) we find

$$\|u\, diag(\varphi_0(a), \eta(a))u^* - diag(\varphi_n(a), \eta(a))\| < 4\epsilon + \lambda < 5\epsilon$$

for all $a \in F$. □

We say that a C*-algebra A is homotopically dominated by a C*-algebra B if there are $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\psi\varphi$ is homotopic to id_A.

The proof of the next result is very similar to the proof of [D$_2$, Lemma 4], with the crucial remark that property (H) can be replaced by the less restrictive property (H_w).

Theorem 6. *Let A be a separable, nuclear RFD algebra. Suppose that A is homotopically dominated by an AF algebra. Then A is AF embeddable.*

Proof. By assumption id_A is homotopic to a $*$-homomorphism $\psi : A \to A$ such that $\psi(A)$ is AF. After adjoining units we may assume that A is unital and $\psi(1) = 1$. Let F_n be an increasing sequence of finite subsets of A whose union is dense in A. By Theorem 5 there is a sequence $\eta_n : A \to M_{k(n)-1}(A)$ of $*$-homomorphisms with finite dimensional image and a sequence of unitaries $u_n \in M_{k(n)}(A)$ such that

$$\|diag(a, \eta_n(a)) - u_n\, diag(\psi(a), \eta_n(a))\, u_n^*\| < 1/n$$

for all $a \in F_n$. Define $\phi_n, \gamma_n : A \to M_{k(n)}(A)$ by $\phi_n(a) = u_n\, diag(\psi(a), \eta_n(a))\, u_n^*$, and $\gamma_n(a) = diag(a, \eta_n(a))$. Then

$$\lim_{n\to\infty} \|\gamma_n(a) - \phi_n(a)\| = 0 \tag{7}$$

for all $a \in A$. Note that $\phi_n(A)$ is an AF algebra.

Next we construct an AF algebra B and an embedding $A \to B$. Let $r(1) = 1$ and $r(n+1) = r(n)k(n)$ for $n \geq 1$. Let $A_n := M_{r(n)}(A)$, $n \geq 1$, and define $*$-monomorphisms $\Gamma_n : A_n \to A_{n+1}$ by $\Gamma_n = id_{r(n)} \otimes \gamma_n$ and $\Gamma_{n,i} : A_i \to A_n$, $i < n$, $\Gamma_{n,i} := \Gamma_{n-1} \circ \cdots \circ \Gamma_i$. Let B be the inductive limit of the system (A_n, Γ_n). Then $A = A_1$ is clearly a C*-subalgebra of B. It remains to prove that B is AF. To this purpose it is enough to show that for any $\epsilon > 0$, any $i \geq 1$ and for any finite subset F of A_i there is $n > i$ and there is an AF subalgebra E of A_{n+1} such that $dist(\Gamma_{n+1,i}(a), E) < \epsilon$ for all $a \in F$ (see [Br]). Let F and ϵ be as above. Using (7) we find $n > i$ such that

$$\|id_{r(i)} \otimes \gamma_n(a) - id_{r(i)} \otimes \phi_n(a))\| < \epsilon \tag{8}$$

for all $a \in F$. Note that $\Gamma_{n,i} : A_i \to A_n$ is unitarily equivalent to a $*$-homomorphism of the form $a \mapsto diag(a, \theta(a))$, where θ is a $*$-homomorphism with finite dimensional image. It follows that there is a unitary $u \in A_{n+1}$ such that if we identify A_{n+1} with $M_{r(i)} \otimes M_{r(n+1)/r(i)}(A)$ then

$$\Gamma_{n+1,i} = u\, diag(id_{r(i)} \otimes \gamma_n, \theta_{n,i})u^*, \tag{9}$$

where $\theta_{n,i}$ is $*$-homomorphism with finite dimensional image. Using (8) and (9) we see that

$$dist(\Gamma_{n,i}(a), u(M_{r(i)}(\phi_n(A)) \oplus \theta_{n,i}(A))u^*) < \epsilon$$

for all $a \in F$. This concludes the proof since $u(M_{r(i)}(\phi_n(A)) \oplus \theta_{n,i}(A))u^*$ is an AF subalgebra of A_{n+1}. □

For a C*-algebra A, the cone over A is the C*-algebra $C_0(0,1] \otimes A$ and the suspension of A is $C_0(0,1) \otimes A$.

Corollary 7. *Let A be a separable, nuclear RFD algebra. Then the cone over A, CA (and hence the suspension of A, SA) is AF embeddable.*

Proof. This follows from Theorem 6 applied for the C*-algebra $C_0(0,1] \otimes A$ which is homotopy equivalent to $\{0\}$. □

Corollary 8. *Let A be a separable, nuclear RFD algebra having all its ideals KK-equivalent to C*-algebras in the bootstrap category of* [RS]. *Suppose that A is homotopically dominated by a separable AF-embeddable C*-algebra. Then A is AF embeddable.*

Proof. This is a consequence of Corollary 7 and [S, Theorem 3.9]. □

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