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# On the asymptotic homotopy type of inductive limit $C^*$-algebras

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Let $X$, $Y$ be compact, connected, metrisable spaces with base points $x_0$, $y_0$ and let $\mathscr{K}$ denote the compact operators. It is shown that $C_0(X\backslash x_0)\otimes\mathscr{K}$ is asymptotically homotopic (or shape equivalent) to $C_0(Y\backslash y_0)\otimes\mathscr{K}$ if and only if $X$ and $Y$ have isomorphic $K$-groups. Similar results are obtained for certain inductive limits of nuclear $C^*$-algebras.

Let $\mathscr{A}$ denote the category whose objects are all the separable $C^*$-algebras and whose set of morphisms from $A$ to $B$, denoted [[A, B]], consists of homotopy classes of asymptotic morphisms. The construction of this category is due to Connes and Higson [CH], who defined a bivariant homology theory $E(A, B) = [[SA, SB\otimes\mathscr{K}]]$ and have shown how to define the intersection product for arbitrary extensions of separable $C^*$-algebras. If $A$ is $K$-nuclear then $E$-theory agrees with Kasparov's bivariant $K$-theory [K].

On the other hand the asymptotic homotopy category $\mathscr{A}$ appears to be the "right" framework for the homotopy theory of separable $C^*$-algebras. This point of view is supported by results in [H; CH; D; CH1; CK; D1]. For instance we have shown in [D1] that asymptotic homotopy is equivalent to a strong shape theory and hence is intimately related to the shape theories of [EK1] and [B] which were intended as homotopy theories for noncommutative singular spaces. In particular it turned out that two separable $C^*$-algebras are shape equivalent if and only if they are asymptotically homotopic i.e. isomorphic in $\mathscr{A}$. The isomorphism class in $\mathscr{A}$ of a separable $C^*$-algebra $A$ is called the asymptotic homotopy type of $A$.

In this note we exhibit large classes of (projectionless) stable, nuclear $C^*$-algebras whose asymptotic homotopy type is determined by $K$-theoretical data (Theorem 6). This is done via a suspension isomorphism

$$[[A, B\otimes\mathscr{K}]] \to [[SA, SB\otimes\mathscr{K}]]$$

which, by the main result in [DL] holds whenever $[[\mathrm{id}_A]]$ is invertible in $[[A, A\otimes\mathscr{K}]]$. We show in the paper that among $A$ for which this isomorphism holds true are the inductive limits of direct sums of $C^*$-algebras of the form

$C_0(X\backslash x_0, D)$ where $X$ is a connected polyhedron, $x_0$ a point in $X$ and $D$ is any separable $C^*$-algebra. The technique employed in the proof is based on the approximation of asymptotic morphisms by homotopies of $*$-homomorphisms.

For $C^*$-algebras $A$, $B$ let $\mathrm{Hom}(A, B)$ denote the space of $*$-homomorphisms from $A$ to $B$ equipped with the topology of pointwise convergence. The path components of $\mathrm{Hom}(A, B)$ correspond to the homotopy classes of $*$-homomorphisms denoted by $[A, B]$. Let $\mathscr{K}$ denote the $C^*$-algebra of compact operators acting on an infinite dimensional separable Hilbert space. $[A, B \otimes \mathscr{K}]$ has a natural structure of abelian semigroup with addition induced by the direct sum of $*$-homomorphisms and unit given by the class of the null homomorphism (see Theorem 3.1 in [R]).

**Lemma 1** *Let $A$, $B$ be $C^*$-algebras and let $\eta_0 \in \mathrm{Hom}(A, B \otimes \mathscr{K})$. Suppose that $[\eta_0]$ is an invertible element of the semigroup $[A, B \otimes \mathscr{K}]$. Then the map*

$$F: \mathrm{Hom}(A, B \otimes \mathscr{K}) \to \mathrm{Hom}(A, B \otimes \mathscr{K} \otimes M_2),$$

*$F(\gamma) = \gamma \oplus \eta_0$ is a homotopy equivalence.*

*Proof.* Let $\bar{\eta}_0 \in \mathrm{Hom}(A, B \otimes \mathscr{K})$ such that $\eta_0 \oplus \bar{\eta}_0$ is homotopic to 0. Let $\theta_0$ be an isomorphism of $\mathscr{K} \otimes M_3$ onto $\mathscr{K}$ and set $\theta = \mathrm{id}_B \otimes \theta_0$. Then the map

$$G: \mathrm{Hom}(A, B \otimes \mathscr{K} \otimes M_2) \to \mathrm{Hom}(A, B \otimes \mathscr{K})$$

given by $G(\varphi) = \theta \circ (\varphi \oplus \bar{\eta}_0)$ is a homotopy inverse of $F$. First we compute

$$GF(\gamma) = G(\gamma \oplus \eta_0) = \theta \circ (\gamma \oplus \eta_0 \oplus \bar{\eta}_0).$$

Thus $G \circ F$ is homotopic to the map $\gamma \to \theta \circ (\gamma \oplus 0 \oplus 0)$ which in its turn is homotopic to the identity map of $\mathrm{Hom}(A, B \otimes \mathscr{K})$, as in the proof of Theorem 3.1 a) in [*R*] or Lemma 1.3.11 in [JT], where a slightly weaker result is stated. Next we compute

$$FG(\varphi) = F(\theta \circ (\varphi \oplus \bar{\eta}_0)) = (\theta \circ (\varphi \oplus \bar{\eta}_0)) \oplus \eta_0$$
$$= (\theta \oplus \mathrm{id}_{B \otimes \mathscr{K}}) \circ (\varphi \oplus \bar{\eta}_0 \oplus \eta_0).$$

It follows that $F \circ G$ is homotopic to the map

$$\varphi \mapsto (\theta \oplus \mathrm{id}_{B \otimes \mathscr{K}}) \circ (\varphi \oplus 0 \oplus 0) = \theta(\varphi \oplus 0) \oplus 0$$

which is homotopic to the identity map of $\mathrm{Hom}(A, B \otimes \mathscr{K} \otimes M_2)$ by the same argument as above. □

**Corollary 2** *Let $A$, $B$, $\eta_0$ and $F$ be as in Lemma 1. For any base point $\gamma_0 \in \mathrm{Hom}(A, B \otimes \mathscr{K})$, $F$ induces an isomorphism of fundamental groups*

$$F_*: \pi_1(\mathrm{Hom}(A, B \otimes \mathscr{K}), \gamma_0) \to \pi_1(\mathrm{Hom}(A, B \otimes \mathscr{K} \otimes M_2), \gamma_0 \oplus \eta_0).$$

*Proof.* If $f: X \to Y$ is a homotopy equivalence then for any $x_0 \in X$, $f_*: \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism of groups. □

As with $*$-homomorphisms, the homotopy classes of asymptotic morphisms $[[A, B, \otimes \mathscr{K}]]$ form an abelian semigroup. The main technical result of this note is the following.

**Theorem 3** *Let $A$ be a $C^*$-algebra that is the inductive limit of a sequence $(A_n)$ of separable $C^*$-algebras. Suppose that $[\mathrm{id}_{A_n}]$ is invertible in $[A_n, A_n \otimes \mathscr{K}]$ for each n. Then $[[\mathrm{id}_A]]$ is invertible in $[[A, A \otimes \mathscr{K}]]$.*

*Proof.* For any $C^*$-algebra $D$ there are natural isomorphisms $[D, D \otimes \mathscr{K}] \cong [D \otimes \mathscr{K}, D \otimes \mathscr{K}]$ and $[[D, D \otimes \mathscr{K}]] \cong [[D \otimes \mathscr{K}, D \otimes \mathscr{K}]]$ induced by tensorization with $\mathscr{K}$. Therefore we may assume that each $A_n$ is stable. We are going to produce an asymptotic morphism $\varphi_t: A \to A$ such that $[[\varphi_t]] + [[\mathrm{id}_A]] = 0$ in $[[A, A]]$. By hypothesis there are $f_n \in \mathrm{Hom}(A_n, A_n)$ such that $[f_n] + [\mathrm{id}_{A_n}] = 0$ in $[A_n, A_n]$. The idea of the proof is to assemble the $*$-homomorphisms $f_n$ along with properly chosen connecting homotopies into a strong shape map $(A_n) \to (A_n)$. Via the homotopy inductive limit functor of [D1] this strong shape map gives rise to an inverse of $[[\mathrm{id}_A]]$ in $[[A, A]]$. Corollary 2 will be used to eliminate certain topological obstructions that may appear in the process.

For stable $C^*$-algebras $B$, $C$, $D$ the multiplication $[B, C] \times [C, D] \to [B, D]$ is bilinear. Thus if $[\mathrm{id}_B]$ is invertible in $[B, B]$ then $[B, C]$ is a group. Let $p_{n+1n}: A_n \to A_{n+1}$ denote the connecting maps in the inductive system $(A_n)$. We compute

$$[p_{n+1n} f_n] + [p_{n+1n}] = [p_{n+1n}]([f_n] + [\mathrm{id}_{A_n}]) = 0$$

$$[f_{n+1} p_{n+1n}] + [p_{n+1n}] = ([f_{n+1}] + [\mathrm{id}_{A_{n+1}}])[p_{n+1n}] = 0$$

We find that

$$[p_{n+1n} f_n] = [f_{n+1} p_{n+1n}]$$

since $[A_n, A_{n+1}]$ is a group. Therefore for any $n$ there is a homotopy $h_n \in \mathrm{Hom}(A_n, A_{n+1}[0, 1])$, $h_n = (h_n^\tau)_{\tau \in [0,1]}$ such that $h_n^0 = p_{n+1n} f_n$ and $h_n^1 = f_{n+1} p_{n+1n}$.

In the terminology of [D1] the sequences $(f_n)$ and $(h_n)$ form a strong map of inductive systems $(f_n, h_n): (A_n) \to (A_n)$. There is a natural notion of homotopy for such maps and there is a homotopy inductive limit functor $L$ from the homotopy classes of strong maps to the homotopy classes of asymptotic morphisms (see section 1 and 2 in [D1]). It is obvious from the definition that the functor $L$ is compatible with direct sums i.e.

$$L[[(f'_n, h'_n) \oplus (f''_n, h''_n)]] = L[[f'_n, h'_n]] + L[[f''_n, h''_n]] .$$

Thus all we have to prove is that the strong map $(A_n) \to (A_n \otimes M_2)$ consisting of $*$-homomorphisms

$$f_n \oplus \mathrm{id}_{A_n}: A_n \to A_n \otimes M_2$$

and homotopies

$$h_n^\tau \oplus p_{n+1n}: A_n \to A_{n+1} \otimes M_2$$

is homotopic to the null strong map (0, 0). Indeed this will imply

$$\begin{aligned} L[[f_n, h_n]] + [[\mathrm{id}_A]] &= L[[f_n, h_n]] + L[[\mathrm{id}_{A_n}, p_{n+1n}]] \\ &= L[[f_n \oplus \mathrm{id}_{A_n}, h_n \oplus p_{n+1n}]] \\ &= L[[0, 0]] = 0. \end{aligned}$$

To conclude the proof we produce a homotopy of strong maps from $(f_n, h_n) \oplus (\mathrm{id}_{A_n}, p_{n+1n})$ to $(0, 0)$. This homotopy denoted by $(v_n, \mu_n)$ is a strong map $(A_n) \to (A_n \otimes M_2[0,1])$ consisting of $*$-homomorphisms

$$v_n: A_n \to A_n \otimes M_2[0,1], \quad v_n = (v_n^s)_{s\in[0,1]}$$

and two-homotopies

$$\mu_n: A_n \to A_{n+1} \otimes M_2[0,1] \times [0,1], \quad \mu_n = (\mu_n^{\tau,s})_{\tau,s\in[0,1]}$$

such that for all $\tau, s \in [0,1]$ and all $n$:

$$\mu_n^{s,0} = (p_{n+1n} \otimes \mathrm{id}_{M_2}) v_n^s$$

$$\mu_n^{s,1} = v_{n+1}^s p_{n+1n}$$

$$\mu_n^{0,\tau} = h_n^\tau \oplus p_{n+1n}$$

$$\mu_n^{1,\tau} = 0 .$$

This is done as follows. For any $n$ we take $(v_n^s)_{s\in[0,1]}$ to be any continuous path in $\mathrm{Hom}(A_n, A_n \otimes M_2)$ from $f_n \oplus \mathrm{id}_{A_n}$ to 0. We regard $\mu_n$ as a map $\mu_n: [0,1] \times [0,1] \to \mathrm{Hom}(A_n, A_{n+1} \otimes M_2)$ whose values on the boundary of the unit square are prescribed by the above equations. One can fill the square by a continuous function $\mu_n$ if and only if the loop in $\mathrm{Hom}(A_n, A_{n+1} \otimes M_2)$ given by the boundary conditions corresponds to the zero element of $\pi_1(\mathrm{Hom}(A_n, A_{n+1} \otimes M_2), \mu_n^{0,0})$. Using Corollary 2 for

$$F: (\mathrm{Hom}(A_n, A_{n+1}), p_{n+1n} f_n) \to (\mathrm{Hom}(A_n, A_{n+1} \otimes M_2), p_{n+1n} f_n \oplus p_{n+1n}),$$

$F(\gamma) = \gamma \oplus p_{n+1n}$, we replace $h_n^\tau$ (if necessary) by another path with the same endpoints such that the corresponding obstruction vanishes and we can fill the square. This completes the proof. □

Let $A$, $B$ be separable $C^*$-algebras. By Theorem 4.3 in [DL] if $[[\mathrm{id}_A]]$ is invertible in $[[A, A \otimes \mathscr{K}]]$ then $[[A, B \otimes \mathscr{K}]] \cong E(A, B)$. In conjunction with Theorem 3 this gives the following.

**Theorem 4** *Let $A$ be the inductive limit of a sequence $(A_n)$ of separable $C^*$-algebras such that $[\mathrm{id}_{A_n}]$ is invertible in $[A_n, A_n \otimes \mathscr{K}]$ for each $n$. Then for any separable $C^*$-algebra $B$ the suspension map*

$$[[A, B \otimes \mathscr{K}]] \to [[SA, SB \otimes \mathscr{K}]] = E(A, B)$$

*is an isomorphism.*

**Corollary 5** *Let $X$ be a compact, connected, metrisable space and let $x_0 \in X$. For any separable $C^*$-algebra $B$*

$$[[C_0(X \backslash x_0), B \otimes \mathscr{K}]] \cong KK(C_0(X \backslash x_0), B) .$$

*Proof.* By Theorem 10.1 p. 284 in [ES] $(X, x_0)$ can be written as the projective limit of a sequence of polyhedra $(X_n, x_n)$. An inspection of the proof shows that if $X$ is connected then all $X_n$ can be chosen connected. If $Y$ is a connected polyhedron then $[C_0(Y \backslash y_0), C_0(Y \backslash y_0) \otimes \mathscr{K}]$ is a group by Proposition 3.1.3 in [DN]. Therefore we may apply Theorem 4 with $A = C_0(X \backslash x_0)$ and $A_n = C_0(X_n \backslash x_n)$. For nuclear $A$, $E(A, B)$ is isomorphic to $KK(A, B)$ [CH]. □

For spaces $X$ having the homotopy type of a finite, connected $CW$-complex, Corollary 5 was proven in [DL]. It is clear that Corollary 5 does not hold true for the two-point space $X = \{0, 1\}$. This shows that it is necessary to assume that $X$ is connected.

**Theorem 6** *Let $A$, $B$ be $C^*$-algebras that are inductive limits of direct sums of $C^*$-algebras of the form $C_0(X \backslash x_0, D)$ for connected polyhedra $X$, $x_0 \in X$ and separable nuclear $C^*$-algebras $D$. The following are equivalent*
(i) $A$ is $KK$-equivalent to $B$.
(ii) $A \otimes \mathscr{K}$ is asymptotically homotopic to $B \otimes \mathscr{K}$.
(iii) $A \otimes \mathscr{K}$ is shape equivalent to $B \otimes \mathscr{K}$.
*If $A$ and $B$ belong to the category of "nice" nuclear $C^*$-algebras introduced in* [RS] *then the above conditions are equivalent to*
(iv) $K_*(A) \cong K_*(B)$ as $\mathbb{Z}/2$-graded groups.

*Proof.* (ii) $\Leftrightarrow$ (iii) by Theorem 3.9 in [D1].

For "nice" $A$, $B$ (i) $\Leftrightarrow$ (iv) by [RS].

(i) $\Leftrightarrow$ (ii). Since $A$, $B$ are nuclear $C^*$-algebras, $KK(A, B)$ is isomorphic to $E(A, B)$ by an isomorphism that preserves the multiplicative structure. Therefore $A$ is $KK$-equivalent to $B$ if and only if $SA \otimes \mathscr{K}$ is asymptotically homotopic to $SB \otimes \mathscr{K}$. Since $A \otimes \mathscr{K}$ and $B \otimes \mathscr{K}$ satisfy the hypotheses of Theorem 4, this happens if and only if $A \otimes \mathscr{K}$ is asymptotically equivalent to $B \otimes \mathscr{K}$. □

*Remark 7* Let $X$, $Y$ be compact, connected, metrisable space. Then $C_0(X \backslash x_0)$ is shape equivalent to $C_0(Y \backslash y_0)$ if and only if $(X, x_0)$ is shape equivalent to $(Y, y_0)$ [MS, EK1, B]. Tensoring with the compact operators we get a completely different situation. Indeed, by Theorem 6, $C_0(X \backslash x_0) \otimes \mathscr{K}$ is shape equivalent to $C_0(Y \backslash y_0) \otimes \mathscr{K}$ if and only if $K^*(X) \cong K^*(Y)$ as $\mathbb{Z}/2$-graded groups. Recall that a functor that preserves the inductive limits is called continuous. It was shown in [D1] that any homotopic, continuous functor on the category of separable $C^*$-algebras factors through the category $\mathscr{A}$. Hence if $X$ and $Y$ have isomorphic $K$-groups then such a functor cannot distinguish $C_0(X \backslash x_0) \otimes \mathscr{K}$ from $C_0(Y \backslash y_0) \otimes \mathscr{K}$. However these $C^*$-algebras need not be homotopy equivalent. Indeed their homotopy type is essentially determined by the connective $K$-theory groups of their spectra rather than by the $K$-groups. (see [D2; D3]). In particular this shows that there are no continuous extensions of connective $K$-theory to the category of separable $C^*$-algebras.

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