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K-Homology, Asymptotic Representations, and Unsuspended E-Theory

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Connes and Higson defined a bivariant homology theory $E(A, B)$ for separable C^*-algebras. The elements of $E(A, B)$ are taken to be homotopy classes of asymptotic momorphisms from $SA \otimes \mathscr{K}$ to $SB \otimes \mathscr{K}$. In symbols $E(A, B) \cong [[SA, SB \otimes \mathscr{K}]]$. If A is K-nuclear then E-theory agrees with Kasparov's bivariant K-theory. We show that, in many cases, one need not take suspensions to calculate the E-theory group $E(A, B)$. For many A, we show

$$E(A, B) \cong [[A, B \otimes \mathscr{K}]]$$

for all B. Among the A for which this is true are $C_0(X \backslash \{pt\})$ for X with the homotopy type of a finite, connected CW complex. This gives a concrete realization of K-homology, related to the Brown–Douglas–Fillmore description. For example, $K_0(X)$ arises as asymptotic representations,

$$K_0(X) \cong [[C_0(X), \mathscr{K}]].$$

Other A for which our isomorphic holds include the nonunital dimension-drop intervals. In this case, there is no distinction between $*$-homomorphisms and asymptotic morphisms so we have succeeded in classifying all $*$-homomorphisms from a dimension-drop interval to a stable C^*-algebra. This was subsequently used by Elliott (*J. Reine Angew. Math.* **443** (1993), 179–219) in the classification of certain C^*-algebras. The dimension-drop interval may also be used to describe $K_*(X; \mathbf{Z}/n)$ in terms of paths of asymptotic representations of $C_0(X)$.

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1. Asymptotic Representations

The most obvious generalization of a continuous function between compact spaces is a $*$-homomorphism between unital C^*-algebras. Recently, interest among operator algebraists has turned to more general types of mappings between C^*-algebras, such as the asymptotic momorphisms used by Connes and Highson [5] to define E-theory. A natural question is what topological objects correspond to these mappings.

The continuous functions $Y \to X$ between compact spaces correspond to unital $*$-homomorphisms $C(X) \to C(Y)$. We will assume X and Y are metrizable so the C^*-algebras $C(X)$ and $C(Y)$ will be separable. It can be shown [4, 10] that a unital asymptotic morphism $C(X) \to C(Y)$ corresponds to a path of continuous functions $f_t \colon Y \to Z$, parameterized by $t \in [1, \infty)$, where Z is an appropriate space containing X. Moreover, as t tends to infinity, the image of f_t will approach X in a suitable sense. As long as X is an ANR, f_t can be deformed to a path of functions from Y to X. The commutative case is very interesting and nontrivial [4, 6, 7, 10].

There are, however, topological reasons to be interested in asymptotic momorphisms from $C(X)$ to other, noncommutative C^*-algebras. Here, the distinction between asymptotic momorphisms and $*$-homomorphisms becomes even more critical. For example, suppose X is locally compact and path-connected, $x_0 \in X$, and let $\mathcal{K}$ denote the C^*-algebra of compact operators on a separable Hilbert space. The $*$-homomorphisms $C_0(X \setminus \{x_0\}) \to \mathcal{K}$ are all homotopic. Not only is this false for asymptotic momorphisms, but we will show that the homotopy classes correspond to K-homology classes.

In addition, the injective asymptotic momorphisms from $C(X)$ to any C^*-algebra correspond to deformations of the algebra $C(X)$ and so are candidates for quantum spaces. We will not explore this connection here, but refer the reader to [5, 13, 18, 20].

Let X denote a locally compact space and $C_0(X)$ the algebra of continuous complex valued functions on X vanishing at infinity. By an *asymptotic representation* of $C_0(X)$ we mean a collection, parameterized by $t \in [1, \infty)$, of positive, contractive linear maps

$$\varphi_t \colon C_0(X) \to \mathcal{K}$$

such that $t \mapsto \varphi_t(f)$ is continuous for each f and, for all $f, g \in C_0(X)$,

$$\|\varphi_t(f)\, \varphi_t(g) - \varphi_t(fg)\| \to 0.$$

We will denote such an asymptotic representation by (φ_t). Two asymptotic representations (φ_t) and (ψ_t) are called *equivalent* if, for all f,

$$\|\varphi_t(f) - \psi_t(f)\| \to 0.$$

(The definition of asymptotic representation is stronger than just requiring an asymptotic momorphism from $C(X)$ to $\mathcal{K}$. However, it is known [10] that every asymptotic momorphism from $C(X)$ to $\mathcal{K}$ is equivalent to an asymptotic representation.)

The choice of a real parameter is important here for detecting torsion. By different techniques, we obtained results [12] regarding the existence of enough integer parameterized "quasi-representations," to detect nontorsion elements of $K^0(X)$ for many spaces.

Before defining homotopy, we need to make clear the notion of a family of asymptotic representations parameterized by a compact space S. By the notation

$$\{(\varphi_t^{(s)}) : C_0(X) \to \mathcal{K} \mid s \in S\}$$

we mean that, for each $(s, t) \in S \times [1, \infty)$, there is a positive, linear contraction

$$\varphi_t^{(s)} : C_0(X) \to \mathcal{K},$$

for each $f \in C_0(X)$ the function $(s, t) \mapsto \varphi_t^{(s)}(f)$ is jointly continuous, and, for all $f, g \in C_0(X)$, the function

$$(s, t) \mapsto \|\varphi_t^{(s)}(fg) - \varphi_t^{(s)}(f)\, \varphi_t^{(s)}(g)\|$$

is in $C_0(S \times [1, \infty))$. Equivalence of families is defined by requiring the pointwise difference to lie in $C_0(S \times [1, \infty), \mathcal{K})$.

Two asymptotic representations (φ_t) and (ψ_t) are *homotopic* if there exists a family $\{(\varphi_t^{(s)}) \mid s \in [0, 1]\}$ with $(\varphi_t^{(0)})$ equal to (φ_t) and $(\varphi_t^{(1)})$ equal to (ψ_t). Two families, parameterized by S, are homotopic if there is a family parameterized by $S \times [0, 1]$ with the original families arising from restriction to $S \times \{0\}$ and $S \times \{1\}$.

As a consequence of our main theorems, we have the following descriptions of $K_0(X)$ and $K_1(X)$.

THEOREM 1.1. *Let X be a finite, connected CW complex with one point removed. Then $K_0(X)$ is naturally isomorphic to the group of all asymptotic representations of $C_0(X)$ modulo homotopy. Also, $K_1(X)$ is naturally isomorphic to the group of all families of asymptotic representations of $C_0(X)$, indexed by the circle and vanishing at the basepoint, modulo homotopy. In both cases, the addition comes from the direct sum of asymptotic representations.* (*See also Corollary* 5.5.)

For Mod-n K-homology we have a nice description in degree one. Again with X a finite, connected CW complex with one point removed, we show that $K_1(X; \mathbf{Z}/n)$ is isomorphic to the group of homotopy classes of families of asymptotic representations of $C_0(X)$, indexed by $[0, 1]$, that are zero at one end and, at the other end, are an n-fold multiple of an asymptotic representation.

Let $\mathbf{T}^2$ denote the two dimensional torus. It follows from Theorem 1.1 that $[[C_0(\mathbf{T}^2 \setminus \{pt\}), \mathscr{K}]] \cong \mathbf{Z}$. This explains once more Voiculescu's example of asymptotically commuting finite rank unitaries without commuting approximants [24].

Our main objective is to determine when it is possible to dispense with the suspensions used in the definition of E-theory. This cannot be true in general since $[[A, B \otimes \mathscr{K}]]$ is not always a group (e.g., for $A = \mathbf{C}$). We are also going to need at least an asymptotic momorphism from $A \to S^2 A \otimes \mathscr{K}$ which induces a homotopy equivalence. This, together with the "Bott map," a specific asymptotic momorphism from $C_0(\mathbf{R}^2)$ to $\mathscr{K}$, will at least give us a way of realizing all of $E(A, B)$ without suspensions.

To prove that dispensing with these suspensions does not enlarge the group we will have to prove some partial forms of excision via asymptotic momorphisms. In general, given an exact sequence of separable C^*-algebras $0 \to I \to A \xrightarrow{p} B \to 0$, the mapping cone C_p becomes homotopy equivalent to I via asymptotic momorphisms only after suspending. Such a homotopy equivalence holds without the need for suspensions in enough special cases for us to derive our results. For instance that holds true when p has a right inverse.

2. E-Theory

We will now establish our notation and state those results from [5, 10] that we shall need. Henceforth, all C^*-algebras will be separable and $[A, B]$ will denote the homotopy classes of $*$-homomorphisms from A to B. We will denote by $\mathscr{Sep}$ the category with objects separable C^*-algebras and morphism sets $[A, B]$.

Connes and Highson [5] defined an *asymptotic momorphism* from A to B, denoted $(\varphi_t): A \to B$, to be a family of functions $\varphi_t: A \to B$, parameterized by $t \in [1, \infty)$, such that $t \mapsto \varphi_t(a)$ is continuous for all a and, as $t \to \infty$,

$$\|\varphi_t(ab) - \varphi_t(a)\,\varphi_t(b)\| \to 0,$$

$$\|\varphi_t(a^*) - \varphi_t(a)^*\| \to 0,$$

$$\|\varphi_t(\lambda a + b) - \lambda\varphi_t(a) - \varphi_t(b)\| \to 0$$

for all $a, b \in A$ and $\lambda \in \mathbf{C}$.

Suppose (φ_t) and (ψ_t) are both asymptotic momorphisms. If, for all $a \in A$,

$$\|\varphi_t(a) - \psi_t(a)\| \to 0$$

then (φ_t) and (ψ_t) are called *equivalent*. A *homotopy* from φ to ψ is asymptotic momorphism (Φ_t) from A to $C[0, 1] \otimes B$ such that $\delta_0 \circ \Phi_t$ and $\delta_1 \circ \Phi_t$ are asymptotic momorphisms equal to φ and ψ.

We will let $\mathscr{A}sym$ denote the category of separable C^*-algebras but with arrows homotopy classes of asymptotic momorphisms. We shall let $[[A, B]]$ denote the set of homotopy classes of asymptotic momorphisms from A to B and $[[\varphi]]$ will denote the homotopy class of an asymptotic momorphism $\varphi = (\varphi_t)$.

The connection between $\mathscr{S}ep$ and $\mathscr{A}sym$ is that we will identify a $*$-homomorphism $\varphi: A \to B$ with the asymptotic momorphism $\varphi_t = \varphi$.

Asymptotic homomorphisms can be composed, the result being well defined only up to a homotopy equivalence. This involves restriction to a dense subalgebra and reparameterization. Since at most of the time we will only compose an asymptotic momorphism with a $*$-homomorphism, which is easy, we simply refer the reader to [5] for details.

A useful observation from [5] is that the equivalence classes of asymptotic momorphisms from A to B correspond to the $*$-homomorphisms

$$A \to \frac{C_b([1, \infty), B)}{C_0([1, \infty), B)}.$$

There is a reduction one can make to asymptotic momorphisms and homotopies that have better properties. We used this in the last section to get better descriptions of K-homology. Specifically, it follows immediately from the Choi–Effros lifting theorem [2] that if A is nuclear, any asymptotic momorphism $(\varphi_t): A \to B$ is equivalent to (φ'_t) so that φ'_t is linear, contractive, and positive.

When S is compact, we may identify

$$C_b([1, \infty), C(S, B)) = C_b([1, \infty) \times S, B)$$

which explains the definition of homotopy in terms of families of asymptotic representations.

There is an addition defined on $[[A, B \otimes \mathscr{K}]]$ given by direct summation and composing with the $*$-homomorphism $\mathscr{K} \oplus \mathscr{K} \hookrightarrow M_2(\mathscr{K}) \cong \mathscr{K}$. Without more structure, all one can say is that $[[A, B \otimes \mathscr{K}]]$ is an abelian semigroup with unit. When A is a suspension, for example, this is a group.

Connes and Higson define $E(A, B)$ to be the abelian group $[[SA \otimes \mathscr{K}, SB \otimes \mathscr{K}]]$. There are isomorphisms

$$E(A, B) \cong E(A, S^2 B) \cong E(SA, SB) \cong E(S^2 A, B)$$

reflecting the fact that $C_0(\mathbf{R}^1) \otimes \mathscr{K}$ and $C_0(\mathbf{R}^3) \otimes \mathscr{K}$ are isomorphic as objects of $\mathscr{A}sym$. It is also easy to see that $E(A, B)$ is isomorphic to $[[SA, SB \otimes \mathscr{K}]]$.

For K-nuclear C^*-algebras [23], E-theory coincides with Kasparov's [19] KK-theory, a fact we shall use when convenient.

An asymptotic momorphism, and hence an E-theory element, induces a map on K-theory. Given (φ_t): $A \to B$ and a projection p in A, the class of $\varphi_*([p])$ in $K_0(B)$ is determined by any projection that is close to $\varphi_t(p)$ for some sufficiently large value of t. For unitaries, a similar construction is used.

For the general theory, suspensions are inevitable. A suspension is needed for the group structure. Also, Bott periodicity requires an element in $E(\mathbf{C}, C_0(\mathbf{R}^2))$ inducing an isomorphism on K-theory. However, there is no such map at the level of an asymptotic momorphism from $\mathbf{C}$ to $C_0(\mathbf{R}^2)$.

We will also make use of an asymptotic momorphism

$$(\alpha_t) : C_0(\mathbf{R}^2) \to \mathscr{K}$$

inducing an isomorphism in K-theory. We do not need to know anything about α but that it exists. It is described in various ways in [5, 10, 12].

3. Excision Results

Given a short exact sequence $0 \to I \to A \xrightarrow{p} B \to 0$, there is a long exact sequence, called the Puppe sequence,

$$\cdots \to [[D, SA]] \to [[D, SB]] \to [[D, C_p]] \to [[D, A]] \to [[D, B]]$$

for any D. The objects here are, in general, just pointed sets. The base point corresponds to the zero morphism. See [8, 10] for related sequences. The reason that there are exact sequences in E-theory involving I is that SI and SC_p are equivalent in $\mathscr{A}sym$. In this section, we examine cases where I and C_p are equivalent in $\mathscr{A}sym$ without the need to suspend. If this is the case, we will say the given exact sequence is *excisive*.

One obvious place to look is at the commutative case. An important case is

$$0 \to C_0(\mathbf{R}^n) \to C(S^n) \to \mathbf{C} \to 0.$$

Since the mapping cone construction and tensor products commute, the extension

$$0 \to C_0(\mathbf{R}^n) \otimes B \xrightarrow{\iota} C(S^n) \otimes B \xrightarrow{p} \mathbf{C} \otimes B \to 0 \tag{1}$$

is excisive. Thus we have the following lemma which we used implicitly in the first section when we ignored the difference between free and base-point fixing homotopies.

LEMMA 3.1. *The sequences*

$$0 \to [[A, S^nB]] \xrightarrow{\iota_*} [[A, C(S^n) \otimes B]] \xrightarrow{p_*} [[A, B]] \to 0,$$
$$0 \to [A, S^nB] \xrightarrow{\iota_*} [A, C(S^n) \otimes B] \xrightarrow{p_*} [A, B] \to 0$$

are exact for all A and B.

Proof. Since (1) is a split-exact sequence the Puppe sequence gives us a split-exact sequence

$$0 \to [[A, C_p]] \to [[A, C(S^n) \otimes B]] \to [[A, B]] \to 0.$$

By excision, we may replace C_p by $C_0(\mathbf{R}^n) \otimes B$ to get the result. Since the homotopy equivalence of the mapping cone and ideal is actually given by $*$-homomorphisms, the second, well-known, sequence is also exact. ∎

A key element in the proof of our main result is the analogous result for the unitization $\tilde{\mathcal{K}}$ of the compact operators.

PROPOSITION 3.2. *Any split-exact sequence of separable C^*-algebras is excisive. Therefore, if $\sigma: B \to A$ is a splitting for*

$$0 \to J \to A \xrightarrow{\pi} B \to 0$$

then, for D any C^*-algebra, σ_* is a splitting for the exact sequence

$$0 \to [[D, J]] \to [[D, A]] \to [[D, B]] \to 0.$$

Proof. Let $CB = C_0(0, 1] \otimes B$. Following the usual construction we find

$$C_\pi = \{(x + \sigma(f(1)), f) \in A \oplus CB \mid x \in J\}$$

and the canonical inclusion $\iota: J \to C_\pi$ is given by $\iota(x) = (x, 0)$.

Let $0 \leqslant u_t \leqslant 1$ be a quasicentral approximate unit in J, indexed by $[1, \infty)$, with $t \mapsto u_t$ continuous. Define

$$\rho_t: C[0, 1] \otimes B \to A$$

by setting

$$\rho_t(f \otimes b) = f(u_t)\, \sigma(b)$$

and extending. To do that we use Lemma 5 in [5]. Notice that $\rho_t(CB) \subseteq J$. Now define $\varphi_t: C_\pi \to J$ by

$$\varphi_t(x + \sigma(f(1)), f) = x + \rho_t(f).$$

This is clearly an asymptotically linear, selfadjoint map with $t \mapsto \varphi_t(a)$ continuous for any a.

For $f \in CB$ and $x \in J$,

$$\|\rho_t(f)x - \sigma(f(1))x\| \to 0$$

since u_t is an approximate unit in J. Therefore, with

$$\begin{aligned} a &= (x + \sigma(f(1)), f) \\ b &= (y + \sigma(g(1)), g) \\ ab &= (xy + x\sigma(g(1)) + \sigma(f(1))y + \sigma(f(1)\,g(1)), fg) \end{aligned}$$

we calculate

$$\begin{aligned} &\|\varphi_t(a)\,\varphi_t(b) - \varphi_t(ab)\| \\ &\quad = \|(x + \rho_t(f))(y + \rho_t(g)) - [xy + x\sigma(g(1)) + \sigma(f(1))y + \rho_t(fg)]\| \\ &\quad \leqslant \|x\rho_t(g) - x\sigma(g(1))\| + \|\rho_t(f)y - \sigma(f(1))y\| \\ &\qquad + \|\rho_t(f)\,\rho_t(g) - \rho_t(fg)\| \to 0. \end{aligned}$$

Thus (φ_t) is an asymptotic momorphism which is clearly a left-inverse to ι. Composing in the other direction we find

$$\iota \circ \varphi_t(x + \sigma(f(1)), f) = (x + \rho_t(f), 0)$$

which we will endeavor to show is homotopic to the identity.

Let us define $*$-homomorphisms θ_s and η_s from CB to $C[0,1] \otimes B$ by

$$\begin{aligned} \theta_s(f)(r) &= f(sr) \\ \eta_s(f)(r) &= f(s + (1-s)r). \end{aligned}$$

(Thus $\theta_0(f) = 0$, $\theta_1(f) = f$ and $\eta_0(f) = f$, $\eta_1(f) = f(1)$.) Define

$$\varphi_t^{(s)}(x + \sigma(f(1)), f) = (x + \rho_t \circ \eta_s(f), \theta_s(f)).$$

These are easily seen to define a collection of asymptotically linear, selfadjoint maps from C_π to C_π. With a and b as above, and using a similar calculation, one finds

$$\begin{aligned} &\|\varphi_t^{(s)}(a)\,\varphi_t^{(s)}(b) - \varphi_t^{(s)}(ab)\| \\ &\quad \leqslant \|x\rho_t(\eta_s(g)) - x\sigma(g(1))\| + \|\rho_t(\eta_s(f))y - \sigma(f(1))y\| \\ &\qquad + \|\rho_t(\eta_s(f))\,\rho_t(\eta_t(g)) - \rho_t(\eta_s(fg))\| \end{aligned}$$

which converges to zero uniformly to zero in s. This shows that $(\varphi_t^{(s)})$ is a homotopy. Since $\varphi_t^{(0)} = \iota \circ \varphi_t$ and $\varphi_t^{(1)} = \mathrm{id}_{C_\pi}$ we are done. ∎

4. Unsuspending E-Theory

If, for some A, it happens that

$$[[A, B\otimes\mathcal{K}]]\cong E(A, B)$$

for all B, then in particular $[[\mathrm{id}_A]]$ must have an additive inverse in $[[A, A\oplus\mathcal{K}]]$. (More precisely, the class of the map $a\mapsto a\otimes e_{11}$ has an inverse.) Remarkably, the converse holds.

For simplicity, we will assume now that *A and B are stable.* The assumption on A is now that there exists an asymptotic momorphism $\eta: A\to A$ such that

$$\mathrm{id}_A\oplus\eta : A\to M_2(A)$$

is null-homotopic.

Lemma 4.1. *If $[[\mathrm{id}_A]]$ has an additive inverse then $[[A, B]]$ is always a group.*

We will denote the obvious maps (corresponding to inclusion or evaluation at a point) as

$$i_0 : C_0(\mathbf{R}^1)\otimes C_0(\mathbf{R}^2)\otimes M_2\to C_0(\mathbf{R}^1)\otimes C(S^2)\otimes M_2$$
$$i : A\otimes C_0(\mathbf{R}^2)\otimes M_3\to A\otimes C(S^2)\otimes M_3$$
$$j=Si : A\otimes C_0(\mathbf{R}^1)\otimes C_0(\mathbf{R}^2)\otimes M_3\to A\otimes C_0(\mathbf{R}^1)\otimes C(S^2)\otimes M_3$$
$$\varepsilon : A\otimes C(S^2)\otimes M_3\to A\otimes M_3.$$

Let $p_0\in C(S^2)\otimes M_2$ denote a projection such that

$$\varepsilon(p_0)=\begin{bmatrix}0&0\\0&1\end{bmatrix}\qquad\text{and}\qquad [p_0]=(1, 1)\in K_0(C(S^2)),$$

given the usual identification of $K_0(C(S^2))$ with $\mathbf{Z}^2$. Let β_0 denote a $*$-homomorphism

$$\beta_0 : C_0(\mathbf{R}^1)\to C_0(\mathbf{R}^3)\otimes M_2$$

that induces an isomorphism in K_1. (There are two choices here for the isomorphism. We will specify which we want later.) Finally, we define

$$\gamma_1 : A\to A\otimes C(S^2)\otimes M_2\qquad\text{and}\qquad\gamma_2 : A\to A\otimes C(S^2)$$

by $\gamma_1(a)=a\otimes p_0$ and $\gamma_2(a)=\eta_t(a)\otimes 1$.

PROPOSITION 4.2. *If* $[[\mathrm{id}_A]]$ *has an additive inverse in* $[[A, A]]$ *then there exists an asymptotic morphism*

$$(\beta_t^A) : A \to S^2 A \otimes M_3$$

such that (a) *the suspension of* β^A *is homotopic to* $(\mathrm{id}_A \otimes \beta_0) \oplus 0$ *and* (b) *the diagram*

$$\begin{array}{ccc} & & A \otimes C(S^2) \otimes M_3 \\ & \overset{\gamma_1 \oplus \gamma_2}{\nearrow} & \uparrow i \\ A & \xrightarrow[\beta^A]{} & A \otimes C(\mathbf{R}^2) \otimes M_3 \end{array}$$

commutes up to homotopy.

Proof. The first step is to compute

$$\varepsilon(\gamma_1 \oplus \gamma_2)(a) = \begin{bmatrix} 0 & & \\ & a & \\ & & \eta(a) \end{bmatrix}$$

for $a \in A$. Our hypothesis implies $\varepsilon(\gamma_1 \oplus \gamma_2)$ is null-homotopic. The existence of β^A satisfying (b) now follows from Lemma 3.1.

Now suspend this picture. By Lemma 3.1, the map induced by j,

$$[[A \otimes C_0(\mathbf{R}^1), A \otimes C_0(\mathbf{R}^1) \otimes C_0(\mathbf{R}^2) \otimes M_3]]$$
$$\to [[A \otimes C_0(\mathbf{R}^1), A \otimes C_0(\mathbf{R}^1) \otimes C(S^2) \otimes M_3]],$$

is one-to-one (notice we are dealing with groups). Therefore, we are done if we can show

$$S(\gamma_1 \oplus \gamma_2) \sim Si \circ (\mathrm{id}_A \otimes \beta_0) \oplus 0.$$

Consider

$$(i_0 \circ \beta_0) \oplus (\mathrm{id}_{C_0(\mathbf{R})} \otimes 1) : C_0(\mathbf{R}^1) \to C_0(\mathbf{R}^1) \otimes C(S^2) \otimes M_3.$$

This has the same K-theory as $S\gamma_0$ where we define $\gamma_0 : \mathbf{C} \to C(S^2) \otimes M_2$ by $\lambda \mapsto \lambda p_0$. (Here is where we fix our choice of β_0.) This implies that these two $*$-homomorphisms are stably homotopic because, as is well-known,

$$[C_0(\mathbf{R}^1), D \otimes \mathcal{K}] \cong K_1(D) \cong \mathrm{Hom}(K_1(C_0(\mathbf{R}^1)), K_1(D)).$$

Since A is stable, we conclude

$$\begin{aligned} S\gamma_1 &= S(\mathrm{id}_A \otimes \gamma_0) \\ &\sim \mathrm{id}_A \otimes S\gamma_0 \\ &\sim \mathrm{id}_A \otimes ((i_0 \circ \beta_0) \oplus (\mathrm{id}_{C_0(\mathbf{R})} \otimes 1)) \\ &= Si \circ (\mathrm{id}_A \otimes \beta_0) \oplus (\mathrm{id}_A \otimes \mathrm{id}_{C_0(\mathbf{R})} \otimes 1) \end{aligned}$$

and therefore

$$\begin{aligned} S(\gamma_1 \oplus \gamma_2) &\sim Si \circ (\mathrm{id}_A \otimes \beta_0) \oplus ((\mathrm{id}_A \oplus \eta) \otimes \mathrm{id}_{C_0(\mathbf{R})} \otimes 1) \\ &\sim Si \circ (\mathrm{id}_A \otimes \beta_0) \oplus 0. \quad \blacksquare \end{aligned}$$

THEOREM 4.3. *If A is stable and $[[\mathrm{id}_A]]$ has an inverse in $[[A, A]]$ then*

$$[[A, B]] \cong E(A, B)$$

for all stable B. If we identify $E(A, B)$ with $[[S^2A, S^2B]]$, the isomorphisms are

$$S^2 : [[A, B]] \to [[S^2A, S^2B]]$$

and

$$\Theta : [[S^2A, S^2B]] \to [[A, B]]$$

defined by

$$\Theta([[\varphi]]) = [[\mathrm{id}_B \otimes \alpha \otimes \mathrm{id}_{M_3}]] \circ [[\varphi \otimes \mathrm{id}_{M_3}]] \circ [[\beta^A]].$$

Proof. We first remark that α here denotes any asymptotic momorphism from $C_0(\mathbf{R}^2)$ to $\mathscr{K}$ inducing an isomorphism on K_0 while β^A is defined earlier in this section.

Since $C_0(\mathbf{R}^2)$ is nuclear,

$$\begin{aligned} S\alpha \in [[SC_0(\mathbf{R}^2), S\mathscr{K}]] &= E(C_0(\mathbf{R}^2), K) \\ &\cong KK(C_0(\mathbf{R}^2), \mathscr{K}) \end{aligned}$$

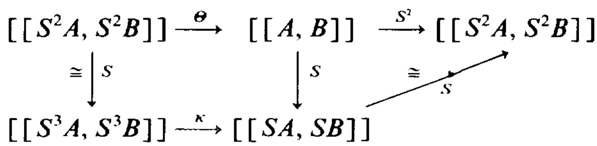
induces a KK-equivalence. This implies that $S\alpha$ is an isomorphism in $\mathscr{A}sym$. Thus, even for nonnuclear, but stable, C and D, the map

$$[[C, S^3D]] \to [[C, SD]]$$

induced by composition with $\mathrm{id}_D \otimes S_\alpha$ is an isomorphism. A similar remark applies to β_0. Therefore, the map κ, defined by

$$\kappa([[\varphi]]) = [[\mathrm{id} \otimes \alpha]] \circ [[\varphi]] \circ [[\mathrm{id} \otimes \beta_0]],$$

is an isomorphism. The diagram

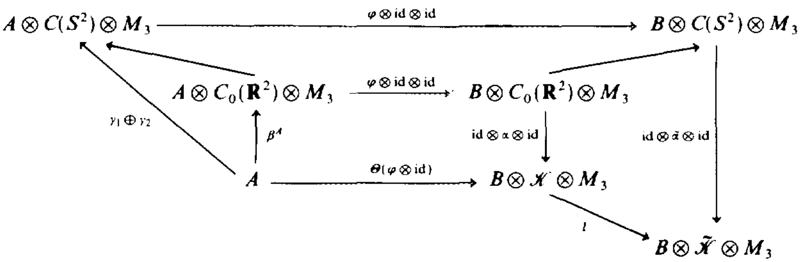


commutes, by Proposition 4.2, which implies that Θ is surjective.

To finish, we must show that $\Theta \circ S^2$ is the identity. That is, given φ in $[[A, B]]$, we must show

$$\Theta([[\varphi \otimes \mathrm{id}_{C_0(\mathbf{R})}]]) = [[\varphi]].$$

Consider the following diagram, which commutes up to homotopy.



Here $\tilde{\mathscr{K}} = \mathscr{K} + \mathbf{C}\mathscr{I}$ denotes the unitization of $\mathscr{K}$ and l is induced by the inclusion $\mathscr{K} \to \tilde{\mathscr{K}}$. We first compute the composition of the outer loop. For $a \in A$,

$$\begin{aligned}(\mathrm{id} \otimes \tilde{\alpha}_t \otimes \mathrm{id}) \circ (\varphi_t \otimes \mathrm{id} \otimes \mathrm{id}) \circ \gamma_1(a) &= (\mathrm{id} \otimes \tilde{\alpha}_t \otimes \mathrm{id}) \circ (\varphi_t \otimes \mathrm{id} \otimes \mathrm{id})(a \otimes p_0) \\ &= (\mathrm{id} \otimes \tilde{\alpha}_t \otimes \mathrm{id})(\varphi_t(a) \otimes p_0) \\ &= \varphi_t(a) \otimes (\tilde{\alpha}_t \otimes \mathrm{id})(p_0) \\ &= \varphi_t(a) \otimes P_t\end{aligned}$$

for a continuous path P_t in $M_2(\tilde{\mathscr{K}})$ such that

$$\|P_t^2 - P_t\| \to 0 \qquad \text{and} \qquad \|P_t^* - P_t\| \to 0.$$

Moreover, we know that this path represents the class (1, 1) in $K_0(\tilde{\mathscr{K}})$. Therefore, up to homotopy we can replace P_t by $I \oplus e_{11}$ with e_{11} a minimal projection in $\mathscr{K}$. For $a \in A$,

$$\begin{aligned}(\mathrm{id} \otimes \tilde{\alpha}_t) \circ (\varphi_t \otimes \mathrm{id}) \circ \gamma_2(a) &= (\mathrm{id} \otimes \tilde{\alpha}_t) \circ (\varphi_t \otimes \mathrm{id})(\eta_t(a) \otimes 1) \\ &= (\mathrm{id} \otimes \tilde{\alpha}_t)(\varphi_t(\eta_t(a)) \otimes 1) \\ &= \varphi_t(\eta_t(a)) \otimes I.\end{aligned}$$

Therefore,

$$l \circ (\Theta(\varphi \otimes \mathrm{id})) \sim (\varphi \otimes e_{11}) \oplus (\varphi \otimes I) \oplus (\varphi \circ \eta \otimes I) \sim (\varphi \otimes e_{11}) \oplus 0.$$

This implies that $\Theta[[\varphi \otimes \mathrm{id}]] = [[\varphi]]$ by Proposition 3.1. ∎

We thank the referee for pointing out a different way of deriving Theorem 4.3 from Proposition 3.1. This argument uses the abstract characterizations of *KK*-theory and *E*-theory as categories and the Yoneda Lemma.

5. Examples

We shall call a C^*-algebra *homotopy symmetric* if $[[\mathrm{id}_A]]$ has an additive inverse in $[[A, A \otimes \mathscr{K}]]$. An easy example is $C_0(\mathbf{R}^1)$. Another example is the algebra $q\mathbf{C}$ introduced by Cuntz [9]. Notice that $q\mathbf{C}$ is *KK*-equivalent to $C_0(\mathbf{R}^2)$.

LEMMA 5.1. *If A is homotopy symmetric then $A \otimes B$ is homotopy symmetric for all B.*

Using Theorem 4.3 we get the following corollaries.

COROLLARY 5.2. *For all A and B,*

$$E(A, B) \cong [[q\mathbf{C} \otimes A, B \otimes \mathscr{K}]]$$
$$E(SA, B) \cong [[SA, B \otimes \mathscr{K}]].$$

PROPOSITION 5.3. *If X is a locally compact space whose one point compactification X^+ has the homotopy type of a finite, connected CW complex then $C_0(X)$ is homotopy symmetric.*

Proof. By [15], $[C_0(X), C_0(X) \otimes \mathscr{K}]$ is a group $(\cong kk(X, X))$. In particular, $\mathrm{id}_{C_0(X)}$ has an additive inverse. ∎

COROLLARY 5.4. *If X is a locally compact space with X^+ having the homotopy type of a finite, connected CW complex then*

$$E(C_0(X)\otimes A, B)\cong [[C_0(X)\otimes A, B\otimes \mathcal{K}]]$$

for all C^*-algebras A and B.

COROLLARY 5.5. *If X is a locally compact space with X^+ having the homotopy type of a finite, connected CW complex then*

$$\begin{aligned} K_0(X) &\cong [[C_0(X), \mathcal{K}]] \\ K_1(X) &\cong [[C_0(X), C_0(0,1)\otimes \mathcal{K}]] \\ &\cong \operatorname{Ker}([[C_0(X), C(S^1)\otimes \mathcal{K}]] \xrightarrow{\varepsilon_*} [[C_0(X), \mathcal{K}]]). \end{aligned}$$

In a forthcoming paper [11], the first named author shows that the above corollaries are valid for any locally compact, connected, metrizable space X.

6. DIMENSION-DROP ALGEBRAS

For various purposes, such as finding E-theory with torsion coefficients, we are interested in the mapping cone of the (unique) unital $*$-homomorphism $\mathbf{C}\to M_n$. We call this the dimension-drop algebra of order n, denote it by A_n, and notice

$$A_n = \{f\in C_0((0,1], M_n \mid f(1) \text{ is scalar}\}.$$

Adding a unit, we obtain what is usually called the dimension-drop interval:

$$\tilde{A}_n = \{f\in C([0,1], M_n) \mid f(0), f(1) \text{ are scalar}\}.$$

This is the C^*-algebra Elliott used to introduce K_1-torsion to his inductive limits [16].

We already demonstrated excision for the exact sequence

$$0\to A_n\to \tilde{A}_n\to \mathbf{C}\to 0. \tag{2}$$

A less obvious fact is the equality

$$[[A_n, B]] = [A_n, B] \tag{3}$$

that holds for any B. In [17], we showed that A_n is exactly semiprojective. As a consequence, any $*$-homomorphism

$$A_n \to \frac{C_b([1, \infty), B)}{C_0([1, \infty), B)}$$

lifts, for some m, to a $*$-homomorphism

$$A_n \to \frac{C_b([1, \infty), B)}{C_0([1, m), B)}.$$

This shows that any asymptotic momorphism from A_n to any C^*-algebra is equivalent to a path of $*$-homomorphisms. The equality above follows.

PROPOSITION 6.1. *The (nonunital) dimension-drop C^*-algebras A_n are homotopy symmetric.*

Proof. We leave it to the reader to check that

$$\mathrm{id}_{A_n} \oplus \cdots \oplus \mathrm{id}_{A_n} : A_n \to M_n(A_n)$$

is null-homotopic, implying the result. ∎

COROLLARY 6.2. *For all C^*-algebras A and B,*

$$E(A_n \otimes A, B) \cong [[A_n \otimes A, B \otimes \mathcal{K}]].$$

7. RESULTS ON Mod-p K-THEORY

The usual definition, due to Cuntz and Schochet [21, 22], of K-theory with Mod-n coefficients is

$$K_*(A; \mathbf{Z}/n) = K_*(A \otimes D),$$

where D is any C^*-algebra that is KK-equivalent to a commutative one and such that $K_0(D) = \mathbf{Z}/n$ and $K_1(D) = 0$. More generally, one can define [1, Sect. 23.15] KK with Mod-n coefficients as

$$KK(A, B; \mathbf{Z}/n) = KK(A, B \otimes D) \cong KK^1(A \otimes D, B).$$

(In this section, we will stick with nuclear C^*-algebras so that E-theory and KK-theory coincide.) One may also use such a D but with $K_0(D) = 0$ and $K_1(D) = \mathbf{Z}/n$ so long as one watches for the degree shift.

One common choice is $C\mathbf{n}$, the mapping cone for the degree-n map $C_0(\mathbf{R}^1) \to C_0(\mathbf{R}^1)$. One of the consequences of Corollaries 5.4 and 6.2

is that $SC\mathbf{n} \otimes \mathscr{K}$ and the stable dimension-drop algebra $A_n \otimes \mathscr{K}$ are isomorphic as objects of $\mathscr{A}sym$. The asymptotic momorphism from $SC\mathbf{n}$ to $A_n \otimes \mathscr{K}$ implementing this is interesting in its own right. This is described in detail in [13].

Using the dimension-drop algebra for introducing torsion coefficients may have some advantages because it is both homotopy symmetric (which the Cuntz algebra [8] $\mathcal{O}_{n+1}$ is not) and semiprojective (which $C\mathbf{n}$ is not). In particular, we may describe $K_0(A; \mathbf{Z}/n)$ entirely in terms of $*$-homomorphisms, with A_n playing the role of a noncommutative spectrum.

COROLLARY 7.1. *For B separable,*

$$K_0(B; \mathbf{Z}/n) \cong [A_n, B \otimes \mathscr{K}].$$

Proof. By Theorem 6.2 and (3) we have

$$\begin{aligned} K_0(B; \mathbf{Z}/n) &\cong KK(\mathbf{C}, B; \mathbf{Z}/n) \\ &\cong KK(A_n, B) \\ &\cong [[A_n, B \otimes \mathscr{K}]] \\ &\cong [A_n, B \otimes \mathscr{K}]. \quad \blacksquare \end{aligned}$$

Looking at this from another point of view, this corollary partially explains why the dimension-drop algebras are such important building blocks in inductive limits. There is torsion in $K_1(A_n)$ and the $*$-homomorphisms out of A_n can be classified, at least up to homotopy. We will address related perturbation questions in [14].

Question 7.2. Does there exist a separable, nuclear, homotopy symmetric semiprojective C^*-algebra D_n with $K_0(D) = \mathbf{Z}/n$ and $K_1(D) = 0$?

We also have a nice statement for K-homology in degree one, although involving asymptotic momorphisms instead of $*$-homomorphisms.

COROLLARY 7.3. *If X is a locally compact space with X^+ having the homotopy type of a finite, connected CW complex then*

$$K_1(X; \mathbf{Z}/n) = K^1(C_0(X); \mathbf{Z}/n) \cong [[C_0(X), A_n \otimes \mathscr{K}]].$$

COROLLARY 7.4. *For A and B separable and A nuclear,*

$$KK(A, B; \mathbf{Z}/n) \cong [[A_n \otimes A, B \otimes \mathscr{K}]].$$

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