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When are two commutative C*-algebras stably homotopy equivalent?

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1 Introduction

The purpose of this paper is to investigate the homotopy theory of C*-algebras of the form $C_0(X)\otimes\mathcal{K}$, where X is a finite connected CW complex with base point; here $C_0(X)$ is the C*-algebra of continuous complex-valued functions which vanish at the base point, $\mathcal{K}$ is the C*-algebra of compact operators on a separable infinite dimensional complex Hilbert space, and a *homotopy* between two C*-algebra maps is a path in the space of such maps.

It is customary in the theory of C*-algebras to refer to tensoring with $\mathcal{K}$ as "stabilization," because it gives the same result as tensoring with the matrix algebra $M_n(\mathbb{C})$ and then letting n go to ∞. (Note that since $\mathcal{K}\otimes\mathcal{K}\cong\mathcal{K}$, $A\otimes\mathcal{K}\cong A$ whenever A is already stabilized.) We can therefore restate our goal as investigating the "stable" homotopy theory of C*-algebras of the form $C_0(X)$.

Of course, this meaning of the word "stable" is very different from the way topologists use the word. Our first main result, Theorem 3.5, and its supplement Proposition 6.2 show that there is, nevertheless, a close connection with the theory of module spectra (as defined in [17]) over the connective K-theory spectrum bu: if $\mathcal{C}$ is the category of C*-algebras of the form $C_0(X)\otimes\mathcal{K}$ and homotopy classes of $*$-homomorphisms between them and $\mathcal{B}$ is the category of bu-modules of the form $\mathrm{bu}\wedge X$ (see [17, Theorem III.1.1(i)]) and homotopy classes of bu-module maps between them then $\mathcal{C}$ is equivalent to the opposite category of $\mathcal{B}$. Our proof of this relies on work of Segal [30]. The analogous result (and everything else in the first three sections) holds for real C*-algebras and bo-modules.

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One consequence of Theorem 3.5 is that the homotopy classes of maps from $C_0(X)\otimes\mathcal{K}$ to $C_0(Y)\otimes\mathcal{K}$ can be calculated by using a spectral sequence originally introduced by Robinson and perfected by the authors of [17]. In Sects. 4–6 we develop tools for calculating with this spectral sequence, and we apply them to the question of when $C_0(X)\otimes\mathcal{K}$ is homotopy equivalent to $C_0(Y)\otimes\mathcal{K}$. As background for this question, we remark that there is a variant of the notion of homotopy equivalence for C*-algebras, called asymptotic homotopy equivalence, and that $C_0(X)\otimes\mathcal{K}$ is asymptotically homotopy equivalent to $C_0(Y)\otimes\mathcal{K}$ if and only if $K_*(X)\cong K_*(Y)$ (see [13],[7], [21]); thus it is natural to ask whether there is a similar phenomenon for ordinary homotopy equivalence. In the positive direction, we show that if $\mathrm{bu}_*(X)\cong\mathrm{bu}_*(Y)$ and $\mathrm{bu}_*(X)$ has projective dimension 0 or 1 as a module over the coefficient ring bu_* (which has global dimension 2,) then $C_0(X)\otimes\mathcal{K}$ is homotopy equivalent to $C_0(Y)\otimes\mathcal{K}$. On the other hand, we give in Theorem 4.1 an example of two finite based CW complexes X and Y for which $\mathrm{bu}_*(X)\cong\mathrm{bu}_*(Y)$ as bu_*-modules but $C_0(X)\otimes\mathcal{K}$ is not homotopy equivalent to $C_0(Y)\otimes\mathcal{K}$. (We should mention here that Wolbert [33] has defined an invariant which detects bu-modules that have the same homotopy groups but are not equivalent; we will discuss the relation between Wolbert's theory and our example at the end of Sect. 6).

Theorem 4.1 also contains an example of two spaces X and Y which are not stably homotopy equivalent as spaces but for which $C_0(X)$ and $C_0(Y)$ are "stably" homotopy equivalent as C*-algebras.

A number of related results were announced in [10]. That announcement is now superseded by the present paper. For other applications of connective K-theory to C*-algebras we refer the reader to [28], [14], [12],[16], [9], [15] and [11].

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2 The groups $kk(X,Y)$

For a compact space X with a base-point x_0, let $C_0(X)$ denote the C*-algebra of continuous complex-valued functions on X which vanish at x_0. If X and Y are spaces with base-point, let $\mathrm{Map}(X,Y)$ denote the set of continuous functions from X to Y which preserve the base-point. The homotopy classes of these functions are denoted by $[X,Y]$.

If A and B are C*-algebras let $Hom(A,B)$ denote the space of $*$-homomorphisms from A to B with the topology of pointwise-norm convergence. Its base point is the null homomorphism. The homotopy classes $[A,B]$ correspond to the path components of $Hom(A,B)$. The tensor product $C_0(X)\otimes B$ is isomorphic to the C*-algebra of continuous functions from X to B which vanish at x_0. One has $C_0(X) \otimes C_0(Y) \cong C_0(X \wedge Y)$. The suspension of a C*-algebra A is defined by $\Sigma A = C_0(S^1) \otimes A$. If $\varphi \in Hom(A, B)$ then its suspension is defined by $\Sigma\varphi = id_{C_0(S^1)} \otimes \varphi \in Hom(\Sigma A, \Sigma B)$. This induces a map $[A, B] \to [\Sigma A, \Sigma B]$.

For compact spaces X, Y with base points we define the semigroup

$$kk(X, Y) = [C_0(Y), C_0(X) \otimes \mathcal{K}]$$

where $\mathcal{K}$ stands for the compact operators acting on a infinite dimensional separable (complex) Hilbert space. The addition is defined by taking direct sum of $*$-homomorphisms and identifying $M_2(\mathcal{K})$ with $\mathcal{K}$. Using the suspension map we define the groups

$${}^s kk_q(X, Y) = \lim_{n \to \infty} kk(\Sigma^{q+n} X, \Sigma^n Y),\ q \in \mathbb{Z}.$$

Proposition 2.1 *Let $i : A \hookrightarrow X$ be a pair of finite connected CW complexes and let $p : X \to X/A$ denote the quotient map. Then for any finite CW complex Y there are long exact sequences:*

$$\begin{aligned} \dots \to {}^s kk_n(Y, A) &\xrightarrow{i_*} {}^s kk_n(Y, X) \xrightarrow{p_*} {}^s kk_n(Y, X/A) \\ &\to {}^s kk_{n-1}(Y, A) \to \dots \end{aligned}$$

$$\begin{aligned} \dots \to {}^s kk_n(X/A, Y) &\xrightarrow{p^*} {}^s kk_n(X, Y) \xrightarrow{i^*} {}^s kk_n(A, Y) \\ &\to {}^s kk_{n+1}(X/A, Y) \to \dots \end{aligned}$$

Since we are dealing with finite CW complexes, the mapping cone of $C_0(X) \to C_0(A)$ is homotopic to $C_0(X/A)$. The two sequences correspond to the Puppe sequences for stable homotopy classes of $*$-homomorphisms of [28].

There is a natural identification

$$\chi : \mathrm{Map}(X, Hom(A, B)) \to Hom(A, C_0(X) \otimes B) \tag{1}$$

$\chi(f)(a)(x) = f(x)(a)$, $f : X \to Hom(A, B)$, $x \in X$ and $a \in A$. This induces a bijection

$$[X, Hom(A, B)] \to [A, C_0(X) \otimes B].$$

Let $F(X)$ denote the space $Hom(C_0(X), \mathcal{K})$. In particular

$$kk(Y, X) = [C_0(X), C_0(Y) \otimes \mathcal{K}] \cong [Y, Hom(C_0(X), \mathcal{K})] \cong [Y, F(X)].$$

Throughout the paper we identify f with $\chi(f)$. Let M_n denote the C*-algebra of $n \times n$ complex matrices and let $F_0(X) = \cup_{n=1}^{\infty} Hom(C_0(X), M_n)$ with union taken by embedding M_n in M_{n+1} by $a \mapsto a \oplus 0$. The natural inclusion $F_0(X) \to F(X)$ is a homotopy equivalence by [30, Proposition

1.2], hence $kk(Y,X) \cong [Y, F_0(X)]$. G. Segal [30, Proposition 1.5] showed that if $A \subset X$ is a pair of finite connected CW-complexes, then the natural map $F_0(X) \to F_0(X/A)$ is a quasifibration with all the fibers homeomorphic to $F_0(A)$. This is a key result which leads to the following (see [14]).

Theorem 2.2 *Let X, Y be finite connected CW complexes. Then the suspension map induces an isomorphism $kk(X,Y) \to kk(\Sigma X, \Sigma Y)$.*

A complete proof in given in [14, Corollary 3.1.8]. Here we just sketch the argument. Let CX denote the cone over X and ΩZ the loop space of a based space Z. One checks that the inverse of the boundary map δ in the homotopy exact sequence

$$\to [Y, \Omega F_0(CX)] \to [Y, \Omega F_0(\Sigma X)] \xrightarrow{\delta} [Y, F_0(X)] \to [Y, F_0(CX)] \to$$

associated with the quasifibration $F_0(CX) \to F_0(\Sigma X)$ with fiber $F_0(X)$ is given by the suspension map. Since $F_0(CX)$ is contractible the conclusion follows by exactness. QED

The above theorem shows that if X, Y are finite connected CW complexes, then $kk(X,Y)$ is a group isomorphic to ${}^s kk(X,Y)$. Letting

$$kk_q(X,Y) = \begin{cases} kk(\Sigma^q X, Y), & \text{for } q \geq 0 \\ kk(X, \Sigma^{-q} Y), & \text{for } q \leq 0, \end{cases}$$

we have that $kk_q(X,Y)$ is isomorphic to ${}^s kk_q(X,Y)$. An important consequence of this isomorphism is that the exact sequences of Proposition 2.1 become available for the groups $kk_q(X,Y)$. The groups $kk_q(X,Y)$ were introduced in [14] as a connective version of the Kasparov group $KK(C_0(Y), C_0(X))$. They represent a very convenient framework for dealing with the multiplicative structure of connective K-theory. The relation to connective K-theory will be explained in the next section.

The groups kk_* have a rich multiplicative structure. We begin by describing the multiplication on kk.
Given $\varphi_0 \in Hom(C_0(X), C_0(Y) \otimes \mathcal{K})$ and $\psi_0 \in Hom(C_0(Y), C_0(Z) \otimes \mathcal{K})$, we define the "composition" $\psi_0 \boxtimes \varphi_0 \in Hom(C_0(X), C_0(Z) \otimes \mathcal{K})$ to be the $*$-homomorphism $id_{C_0(Z)} \otimes \lambda(\psi_0 \otimes id_{\mathcal{K}})\varphi_0$.

$$C_0(X) \xrightarrow{\varphi_0} C_0(Y) \otimes \mathcal{K} \xrightarrow{\psi_0 \otimes id_{\mathcal{K}}} C_0(Z) \otimes \mathcal{K} \otimes \mathcal{K} \xrightarrow{id_{C_0(Z)} \otimes \lambda} C_0(Z) \otimes \mathcal{K}.$$

The map λ is a fixed $*$-isomorphism whose specific choice is irrelevant as the automorphism group of $\mathcal{K}$ is path-connected. This composition induces a bilinear product

$$kk(Y,X) \otimes kk(Z,Y) \to kk(Z,X)$$

$$[\varphi_0] \otimes [\psi_0] \mapsto [\psi_0 \boxtimes \varphi_0].$$

The bilinearity is a general fact [28], [19]. One checks that $\Sigma^n\psi_0 \boxtimes \Sigma^n\varphi_0 = \Sigma^n(\psi_0 \boxtimes \varphi_0)$. Thus the diagram

$$\begin{array}{ccc} kk(Y,X) \otimes kk(Z,Y) & \to & kk(Z,X) \\ \downarrow \Sigma^n \otimes \Sigma^n & & \downarrow \Sigma^n \\ kk(\Sigma^n Y, \Sigma^n X) \otimes kk(\Sigma^n Z, \Sigma^n Y) & \to & kk(\Sigma^n Z, \Sigma^n X) \end{array} \tag{2}$$

is commutative. Using the operation "$\boxtimes$″ and the suspension functor we define a product

$$kk(\Sigma^p Y, \Sigma^m X) \otimes kk(\Sigma^q Z, \Sigma^n Y) \to kk(\Sigma^{p+q} Z, \Sigma^{m+n} X),$$

$[\varphi] \otimes [\psi] \mapsto [\varphi][\psi] = [\Sigma^p\psi \boxtimes \Sigma^n\varphi]$. The various maps involved in this definition are illustrated in the diagram

$$\begin{array}{c} kk(\Sigma^p Y, \Sigma^m X) \otimes kk(\Sigma^q Z, \Sigma^n Y) \\ \downarrow \Sigma^n \otimes \Sigma^p \\ kk(\Sigma^{p+n} Y, \Sigma^{m+n} X) \otimes kk(\Sigma^{p+q} Z, \Sigma^{p+n} Y) \\ \xrightarrow{\boxtimes} kk(\Sigma^{p+q} Z, \Sigma^{m+n} X) \ . \end{array}$$

Since $kk_r(X,Y) = \lim_m kk(\Sigma^{r+m}X, \Sigma^m Y)$, by using the commutativity of the diagram 2, one checks that we obtain a well defined product

$$kk_r(Y,X) \otimes kk_s(Z,Y) \to kk_{r+s}(Z,X).$$

In particular, $T = kk_*(S^1, S^1)$ is a ring which is seen to be isomorphic to $\mathbb{Z}[u]$ [14]. Here u has degree 2 and corresponds to the Bott element $\beta \in Hom(C_0(S^1), C_0(S^3) \otimes \mathcal{K})$ which is a generator of $\pi_3(U(\infty)) \cong [S^3, F(S^1)] \cong [C_0(S^1), C_0(S^3) \otimes \mathcal{K}] \cong \mathbb{Z}$. Moreover $k_*(X) := \oplus_{q \in \mathbb{Z}} k_q$ (X), where $k_q(X) = \lim_r kk(S^{q+r}, \Sigma^q X)$, is a T-module and we have a map

$$\Gamma : kk(Y,X) \to Hom_T(k_*(Y), k_*(X)).$$

In Sect. 3 we need the following description of the product structure of kk_*. If we identify φ and ψ with $*$-homomorphisms $\varphi \in Hom(C_0(S^m \wedge X), C_0(S^p \wedge Y) \otimes \mathcal{K})$ and $\psi \in Hom(C_0(S^n \wedge Y), C_0(S^q \wedge Z) \otimes \mathcal{K})$, (using homeomorphisms of the type $\Sigma^m X \cong S^m \wedge X$, etc.) then $\Sigma^p\psi \boxtimes \Sigma^n\varphi$ will correspond to the "composition" $\psi \boxtimes \varphi \in Hom(C_0(S^m \wedge S^n \wedge X), C_0(S^p \wedge S^q \wedge Z) \otimes \mathcal{K})$ given by

$$\begin{array}{c} C_0(S^m \wedge S^n \wedge X) \xrightarrow{\gamma} C_0(S^n \wedge S^m \wedge X) \\ \xrightarrow{1_{C_0(S^n)} \otimes \varphi} C_0(S^n \wedge S^p \wedge Y) \otimes \mathcal{K} \xrightarrow{\gamma \otimes id_{\mathcal{K}}} \end{array} \tag{3}$$

$$C_0(S^p \wedge S^n \wedge Y) \otimes \mathcal{K}$$
$$\xrightarrow{1_{C_0(S^p)} \otimes \psi \otimes id_{\mathcal{K}}} C_0(S^p \wedge S^q \wedge Z) \otimes \mathcal{K} \otimes \mathcal{K}$$
$$\xrightarrow{id \otimes \lambda} C_0(S^p \wedge S^q \wedge Z) \otimes \mathcal{K} .$$

The maps γ are induced by flip homeomorphisms $S^a \wedge S^b \cong S^b \wedge S^a$.

Let $e \in \mathcal{K}$ be a one dimensional orthogonal projection. If A is a C*-algebra, let $\iota_A : A \to A \otimes \mathcal{K}$ be the map $\iota_A(a) = a \otimes e$. The unit of the ring $kk_*(X, X)$, denoted by 1, is given by the class of $\iota_{C_0(X)}$.

Proposition 2.3 *Suppose that X, Y are finite, connected CW complexes. Then $C_0(X) \otimes \mathcal{K}$ is homotopy equivalent to $C_0(Y) \otimes \mathcal{K}$ if and only if there are $\alpha \in kk(X, Y)$ and $\beta \in kk(Y, X)$ such that $\alpha\beta = 1$ and $\beta\alpha = 1$.*

Proof. For C*-algebras A, B let $\theta : Hom(A, B \otimes \mathcal{K}) \to Hom(A \otimes \mathcal{K}, B \otimes \mathcal{K})$ denote the map $\theta(\varphi_0) = (id_B \otimes \lambda) \circ (\varphi_0 \otimes id_{\mathcal{K}})$. It was shown in [32] that θ induces a bijection $[A, B \otimes \mathcal{K}] \to [A \otimes \mathcal{K}, B \otimes \mathcal{K}]$ and that $\theta(\psi_0 \boxtimes \varphi_0)$ is homotopic to $\theta(\psi_0) \circ \theta(\varphi_0)$ for any $\varphi_0 \in Hom(A, B \otimes \mathcal{K})$ and $\psi_0 \in Hom(B, C \otimes \mathcal{K})$. Moreover $\theta(\iota_A)$ is homotopic to $id_{A \otimes \mathcal{K}}$. The statement is obtained by applying these facts for $A = C_0(X)$ and $B = C_0(Y)$. QED

Theorem 2.4 *Suppose that X, Y are finite, connected CW complexes. Then $C_0(X) \otimes \mathcal{K}$ is homotopy equivalent to $C_0(Y) \otimes \mathcal{K}$ if and only if there is $\alpha \in kk(X, Y)$ such that $\Gamma(\alpha) : k_*(X) \to k_*(Y)$ is an isomorphism of groups.*

Proof. Suppose that $C_0(X) \otimes \mathcal{K}$ is homotopy equivalent to $C_0(Y) \otimes \mathcal{K}$ and let α and β be given by Proposition 2.3. Then $\Gamma(\alpha)$ is an isomorphism with inverse $\Gamma(\beta)$. Conversely, suppose that $\Gamma(\alpha)$ is an isomorphism. From the very definition of $kk(X, Y)$ there is $\varphi \in Hom(C_0(Y), C_0(X) \otimes \mathcal{K})$ with $[\varphi] = \alpha$. Then φ induces a map $f : Hom(C_0(X), \mathcal{K}) \to Hom(C_0(Y), \mathcal{K})$, $f(\psi) = \psi \boxtimes \varphi$. At the level of homotopy groups this gives a map

$$f_* : \pi_* Hom(C_0(X), \mathcal{K}) \to \pi_* Hom(C_0(Y), \mathcal{K}).$$

Since we can make the identification $\Gamma(\alpha) = f_*$, f_* is an isomorphism, hence f is a weak homotopy equivalence. Consequently, for any finite, connected CW complex Z, the map

$$(f_Z)_* : [Z, Hom(C_0(X), \mathcal{K})] \to [Z, Hom(C_0(Y), \mathcal{K})]$$

induced by f is an isomorphism. Since $(f_Z)_*$ can be identified with the map

$$kk(Z, X) \xrightarrow{\times\alpha} kk(Z, X),$$

by abstract nonsense, we obtain that there is $\beta \in kk(Y, X)$ such that $\alpha\beta = 1$ and $\beta\alpha = 1$. We conclude the proof by applying Proposition 2.3. QED

3 The isomorphism $[\Sigma^* X, \mathrm{bu} \wedge Y] \to kk_*(X, Y)$

The reader is referred to [2] for the basic theory of spectra. Let BU denote the spectrum of (reduced) complex K-theory. The spectrum of (reduced) complex connective K-theory will be denoted by bu. There is a map of spectra $\mathrm{bu} \to \mathrm{BU}$ such that $\pi_r(\mathrm{bu}) \to \pi(\mathrm{BU})$ is an isomorphism for $r \geq 0$ and $\pi_r(\mathrm{bu}) = 0$ for $r < 0$. These conditions determine bu uniquely up to a weak equivalence. Let X, Y be finite connected CW complexes. The Kasparov groups $KK_*(C_0(Y), C_0(X))$ are isomorphic to $[\Sigma^* X, \mathrm{BU} \wedge Y]$ [20]. It is then natural to consider the connective version of $KK_*(C_0(Y), C_0(X))$ which is $[\Sigma^* X, \mathrm{bu} \wedge Y]$. Remarkably, one can give these groups a very nice realization in terms of C*-algebras and homotopy classes of $*$-homomorphisms. To be specific, one shows that

$$[X, \mathrm{bu} \wedge Y] \cong kk(X, Y) \cong [C_0(Y), C_0(X) \otimes \mathcal{K}].$$

This isomorphism follows from a remarkable result of Segal [30] who identified bu_n with $F(S^n) = Hom(C_0(S^n), \mathcal{K})$. The structure map $S^1 \wedge F(S^n) \to F(S^{n+1})$ takes $t \wedge \varphi$ to

$$\delta_t \otimes \varphi : C_0(S^{n+1}) \cong C_0(S^1) \otimes C_0(S^n) \to \mathbb{C} \otimes \mathcal{K} \cong \mathcal{K}$$

where $\delta_t : C_0(S^1) \to \mathbb{C}$ is the evaluation map $\delta_t(a) = a(t)$. This result gives a nice realization of bu the (Ω-)spectrum of complex connective K-theory in terms of homotopy classes of homomorphisms of C*-algebras. Let $bu_*(X)$ denote the reduced connective K-homology and $\mathrm{bu}^*(X)$ denote the reduced connective K-theory. If X is a finite connected CW-complex then we have isomorphisms $k_*(X) \cong bu_*(X)$ and $k^*(X) \cong bu^*(X)$. Recall that we denoted by $\beta \in Hom(C_0(S^1), C_0(S^3) \otimes \mathcal{K})$ a morphism representing a generator of $\pi_3(U(\infty)) \cong [S^3, F(S^1)] \cong [C_0(S^1), C_0(S^3) \otimes \mathcal{K}] \cong \mathbb{Z}$. The Bott operation $\mathrm{bu}_{n+2} \to \mathrm{bu}_n$ is given by the map $Hom(C_0(S^{n+2}), \mathcal{K}) \to Hom(C_0(S^n), \mathcal{K})$ which sends φ to the composite

$$C_0(S^n) \xrightarrow{S^{n-1}\beta} C_0(S^{n+2}) \otimes \mathcal{K} \xrightarrow{\varphi \otimes id_{\mathcal{K}}} \mathcal{K} \otimes \mathcal{K} \xrightarrow{\lambda} \mathcal{K}.$$

As before the map λ is a fixed $*$-isomorphism whose specific choice is irrelevant as the automorphism group of $\mathcal{K}$ is arcwise connected. Let $u : k_n(X) \to k_{n+2}(X)$ denote the Bott operation. The ring structure of bu is induced by the multiplication $\mu : F(S^m) \times F(S^n) \to F(S^{m+n})$, $\varphi \wedge \psi \mapsto \lambda \circ (\varphi \otimes \psi)$ as this is easily seen to be compatible with the multiplicative structure of BU.

Recall that for a fixed Y, $kk_q(-, Y)$ is a generalized cohomology theory. The spectrum of this theory denoted by $\mathbf{F}_Y$ is given by the sequence of spaces $F(\Sigma^n Y) = Hom(C_0(\Sigma^n Y), \mathcal{K})$. The structure map $S \wedge F(\Sigma^n Y) \to F(\Sigma^{n+1} Y)$ is given by $t \wedge \varphi \mapsto \delta_t \otimes \varphi$. The dual of this map $F(\Sigma^n Y) \to$

$\Omega F(\Sigma^{n+1}Y)$ takes a $*$-homomorphism φ to its suspension $\Sigma\varphi$. Recall that the suspension map $kk(X,Y) \to kk(\Sigma X, \Sigma Y)$ is an isomorphism. Thus $\mathbf{F}_Y$ is an Ω spectrum. We want to show that $\mathrm{bu} \wedge Y$ is equivalent to $\mathbf{F}_Y$. To this purpose we define maps $t_n : F(S^n) \wedge Y \to F(S^n \wedge Y)$, given by $t_n(\varphi \wedge y) = \varphi \otimes \delta_y$, where $\delta_y : C(Y) \to \mathbb{C}$ is the evaluation map at y.

Proposition 3.1 *The maps* (t_n) *induce a weak equivalence of spectra* $\tau : \mathrm{bu} \wedge Y \to \mathbf{F}_Y$. *Thus there is an isomorphism* $\tau_* : [\Sigma^* X, \mathrm{bu} \wedge Y] \to kk_*(X,Y)$.

Proof. The diagram

$$\begin{array}{ccc} S^1 \wedge F(S^n) \wedge Y & \xrightarrow{1_S \wedge t_n} & S^1 \wedge F(S^n \wedge Y) \\ \downarrow & & \downarrow \\ F(S^{n+1}) \wedge Y & \xrightarrow{t_{n+1}} & F(S^{n+1} \wedge Y) \end{array}$$

is commutative. The vertical arrows are given by the structure maps. Therefore τ is a map of spectra. We need to show that τ induces isomorphisms on the homotopy groups: $\pi_q(\mathrm{bu} \wedge Y) \to \pi_q(\mathbf{F}_Y)$ The above map can be identified with a map $k_q(Y) \to kk_{q-1}(S^1, Y)$. Since these are reduced homology theories it is enough to check the isomorphism for $Y = S^1$. But this is certainly clear since for $Y = S^1$, the map $t_n : F(S^n) \wedge S^1 \to F(S^{n+1})$ coincides up to a flip with the structure map of bu. QED

Next observe that the ring structure of bu gives a multiplication

$$[\Sigma^s Y, \mathrm{bu} \wedge X] \otimes [\Sigma^r Z, \mathrm{bu} \wedge Y] \to [\Sigma^{s+r} Z, \mathrm{bu} \wedge X].$$

On the other hand we have seen in the second section that the composition of $*$-homomorphisms gives rise to a multiplication

$$kk_s(Y,X) \otimes kk_r(Z,Y) \to kk_{s+r}(Z,X)$$

We are going to show that the isomorphism τ_* from Proposition 3.1 preserves the multiplicative structure.

If we identify bu_n with $F(S^n)$, then the product $\mathrm{bu}_m \times \mathrm{bu}_n \to \mathrm{bu}_{m+n}$ is given by $(\alpha, \beta) \mapsto \lambda \circ (\alpha \otimes \beta)$ where $\alpha \otimes \beta \in Hom(C_0(S^{m+n}), \mathcal{K} \otimes \mathcal{K})$ and $\lambda : \mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$.

The product

$$[\Sigma^s Y, \mathrm{bu} \wedge X] \otimes [\Sigma^r Z, \mathrm{bu} \wedge Y] \to [\Sigma^{s+r} Z, \mathrm{bu} \wedge X].$$

$\omega \otimes \eta \mapsto \omega\eta$ is defined as follows. If ω and η are represented by

$$f \in \mathrm{Map}(S^p \wedge Y, F(S^m) \wedge X), \quad g \in \mathrm{Map}(S^q \wedge Z, F(S^n) \wedge Y)$$

with $r = p - m$ and $s = q - n$, then $\omega\eta$ is represented by the composite $g * f$:

$$S^p \wedge S^q \wedge Z \xrightarrow{1_{S^p}\wedge g} S^p \wedge F(S^n) \wedge Y \xrightarrow{\gamma} F(S^n) \wedge S^p \wedge Y \xrightarrow{1_{F(S^n)}\wedge f}$$
$$F(S^n) \wedge F(S^m) \wedge X \xrightarrow{\mu\wedge 1_X} F(S^n \wedge S^m) \wedge X \xrightarrow{\gamma} F(S^m \wedge S^n) \wedge X$$
(4)

The maps γ correspond to various flip homeomorphisms.

Proposition 3.2 *With f and g as above, $t_{m+n} \circ (g * f) = (t_n \circ g) \boxtimes (t_m \circ f)$.*

Proof. Recall that $t_m : F(S^m) \wedge X \to F(S^m \wedge X)$ maps an element $\alpha \wedge x$ to $\alpha \otimes \delta_x \in Hom(C_0(S^m \wedge X), \mathcal{K})$ and $t_m \circ f : S^p \wedge Y \to F(S^m \wedge X)$ is identified via χ with a $*$-homomorphism $\varphi \in Hom(C_0(S^m \wedge X), C_0(S^p \wedge Y) \otimes \mathcal{K})$. Similarly we identify $t_n \circ g$ with some $\psi \in Hom(C_0(S^n \wedge Y), C_0(S^q \wedge Z) \otimes \mathcal{K})$ and $t_{m+n} \circ (g * f)$ with some $\pi \in Hom(C_0(S^m \wedge S^n \wedge X), C_0(S^p \wedge S^q \wedge Z) \otimes \mathcal{K})$. Let $a \otimes b \otimes c \in C_0(S^m) \otimes C_0(S^n) \otimes C_0(X) \cong C_0(S^m \wedge S^n \wedge X)$ be an elementary tensor. It suffices to show that

$$\pi(a \otimes b \otimes c) = (\psi \boxtimes \varphi)(a \otimes b \otimes c) \tag{5}$$

when both sides are evaluated at any point $t \wedge z$ where $t \in S^p$ and $z \in S^q \wedge Z$. In order to fix the notation let's say that

$$\begin{aligned} g(z) = \beta \wedge y \in F(S^n) \wedge Y,\ \ &(1_{F(S^n)} \wedge f)(\beta \wedge t \wedge y) \\ &= \beta \wedge \alpha \wedge x \in F(S^n) \wedge F(S^m) \wedge X \end{aligned} \tag{6}$$

Then using (4) we compute $(g * f)(t \wedge z)$:

$$t\wedge z \mapsto t\wedge\beta\wedge y \mapsto \beta\wedge t\wedge y \mapsto \beta\wedge\alpha\wedge x \mapsto (\lambda\circ(\beta\otimes\alpha)\wedge x)\circ\gamma = (g*f)(t\wedge z)$$
(7)

Therefore

$$\begin{aligned} \pi(a \otimes b \otimes c)(t \wedge z) &= (\lambda \circ (\beta \otimes \alpha) \otimes \delta_x)(b \otimes a \otimes c) \\ &= \lambda(\beta(b) \otimes \alpha(a))c(x) \end{aligned} \tag{8}$$

In order to compute the right-hand side of (5) we show first that, with the notation as in (6)

$$(1_{C_0(S^p)} \otimes \psi \otimes id_{\mathcal{K}})\gamma(b \otimes w)(t \wedge z) = \beta(b) \otimes w(t \wedge y) \in \mathcal{K} \otimes \mathcal{K} \tag{9}$$

for all $b \in C_0(S^n)$ and $w \in C_0(S^p \wedge Y) \otimes \mathcal{K}$. The map γ is as in (1.3). Since the both sides of (8) depend linearly and continuously on w, we may assume that w is an elementary tensor $w = u \otimes v \otimes \kappa \in C_0(S^p) \otimes C_0(Y) \otimes \mathcal{K}$. We have $\gamma(b \otimes u \otimes v \otimes \kappa) = u \otimes b \otimes v \otimes \kappa$, and using (6)

$$(1_{C_0(S^p)} \otimes \psi \otimes id_{\mathcal{K}})(u \otimes b \otimes v \otimes \kappa)(t \wedge z) = u(t) \otimes \psi(b \otimes v)(z) \otimes \kappa$$
$$= u(t) \otimes \beta(b) \otimes v(y) \otimes \kappa = \beta(b) \otimes u(t)v(y)\kappa = \beta(b) \otimes w(t \wedge y)$$

We have identified $\mathbb{C} \otimes \mathcal{K} \cong \mathcal{K} \otimes \mathbb{C} \cong \mathcal{K}$. After this preparation we are able to compute $(\psi \boxtimes \varphi)(a \otimes b \otimes c)(t \wedge z)$ using (2) and (6). We have

$$a \otimes b \otimes c \mapsto b \otimes a \otimes c \mapsto b \otimes \varphi(a \otimes c) \xrightarrow{(1_{C_0(S^p)} \otimes \psi \otimes id_{\mathcal{K}})_{t \wedge z}}$$
$$\beta(b) \otimes \varphi(a \otimes c)(t \wedge y) \text{ (by (8) with } w = \varphi(a \otimes c))$$
$$= \beta(b) \otimes \alpha(a) \otimes c(x) \mapsto \lambda(\beta(b) \otimes \alpha(a))c(x)\ .$$

Combining (8) and (10) we see that (5) holds true and this concludes the proof of the Proposition. QED

As a consequence of Proposition 3.2 we obtain the following.

Corollary 3.3 *The map* $\tau_* : [\Sigma^* X, \mathrm{bu} \wedge Y] \to kk_*(X, Y)$ *preserves the products.*

Corollary 3.4 *Let* X, Y *be finite connected CW complexes. Then* $C_0(X) \otimes \mathcal{K}$ *is homotopy equivalent to* $C_0(Y) \otimes \mathcal{K}$ *if and only if the image of the natural map*

$$\Phi : [X, \mathrm{bu} \wedge Y] \to Hom_{bu_*}(bu_* X, bu_* Y)$$

contains an isomorphism.

Proof. By Proposition 3.1 and Corollary 3.3, there is a commutative diagram

$$\begin{array}{ccc} kk(X,Y) & \xrightarrow{\tau_*} & [X, bu \wedge Y] \\ \downarrow \Gamma & & \downarrow \Phi \\ Hom_T(k_*(X), k_*(Y)) & \xrightarrow{\tau_*} & Hom_{bu_*}(bu_* X, bu_* Y) \end{array}$$

with both horizontal arrows being bijections. We conclude the proof by applying Theorem 2.4. QED

Our next result gives a categorical interpretation of the results of this section. Let $\mathcal{C}$ be the category whose objects are C*-algebras of the form $C_0(X) \otimes \mathcal{K}$, with homotopy classes of maps between them. Let $\mathcal{B}$ be the category whose objects are finite connected CW-complexes X, with the maps from X to Y defined to be $[X, \mathrm{bu} \wedge Y]$. The composition in $\mathcal{B}$ is the multiplication

$$[X, \mathrm{bu} \wedge Y] \otimes [Y, \mathrm{bu} \wedge Z] \to [X, \mathrm{bu} \wedge Z]$$

which has already been defined.

Theorem 3.5 *The category $\mathcal{C}$ is equivalent to the opposite of the category $\mathcal{B}$.*

Proof. We have seen in the proof of Proposition 2.3 that there is a natural isomorphism

$$[C_0(Y) \otimes \mathcal{K}, C_0(X) \otimes \mathcal{K}] \cong kk(X, Y)$$

which takes composition on the left to the multiplication on the right. Proposition 3.1 and Corollary 3.3 give a natural isomorphism

$$kk(X, Y) \cong [X, \mathrm{bu} \wedge Y]$$

which preserves multiplication. The result follows immediately. QED

In Sect. 6 we will see that the category $\mathcal{B}$ can also be described as a certain full subcategory of the homotopy category of bu-modules as defined in [17].

4 A spectral sequence and some examples.

Recall that we are using $\simeq$ to denote homotopy equivalence of C*-algebras.

Let bu_* denote $\pi_*\mathrm{bu} \cong \mathbb{Z}[u]$, where u is the Bott operation and has degree 2. Since $\mathbb{Z}[u]$ has global dimension two, the homological projective dimension of the bu_* module $\mathrm{bu}_*(X)$ is at most two. In this section we will begin the proof of the following:

Theorem 4.1 (a) *Let X and Y be finite connected CW complexes. Suppose that the homological projective dimension of the bu_* module $\mathrm{bu}_*(X)$ is at most one. Then $C_0(X) \otimes \mathcal{K} \simeq C_0(Y) \otimes \mathcal{K}$ if and only if $\mathrm{bu}_*(X)$ is isomorphic to $\mathrm{bu}_*(Y)$ as bu_* modules.*

(b) *There exists a pair of finite connected CW-complexes X and Y such that $\mathrm{bu}_*X \cong \mathrm{bu}_*Y$ as bu_*-modules but $C_0(X) \otimes \mathcal{K} \not\simeq C_0(Y) \otimes \mathcal{K}$.*

(c) *There exists a pair of finite connected CW-complexes X and Y (each having two cells) such that $C_0(X) \otimes \mathcal{K} \simeq C_0(Y) \otimes \mathcal{K}$ but X and Y are not stably homotopy equivalent.*

It has been shown in previous sections that $C_0(X) \otimes \mathcal{K} \simeq C_0(Y) \otimes \mathcal{K}$ if and only if the image of the natural map

$$\Phi : [X, \mathrm{bu} \wedge Y] \to \mathrm{Hom}_{\mathrm{bu}_*}(\mathrm{bu}_*X, \mathrm{bu}_*Y)$$

contains an isomorphism. Thus to prove part (a) we will show that Φ is surjective, to prove part (b) we will show that while $\mathrm{Hom}_{\mathrm{bu}_*}(\mathrm{bu}_*X, \mathrm{bu}_*Y)$

contains isomorphisms, no isomorphism is in the image of Φ, and to prove part (c) we will show that there is an isomorphism in the image of Φ.

Our main tool for doing this will be the following spectral sequence, which was discovered by Robinson [25] and improved by Elmendorf, Kriz, Mandell and May [17, Theorem IV.3.1].

Theorem 4.2 *(a) There is a spectral sequence $E_r(X,Y)$ which converges to $[\Sigma^{-s-t}X, \mathrm{bu}\wedge Y]$ and is natural in X and Y.*
(b)

$$E_2(X,Y) = \mathrm{Ext}^{s,t}_{\mathrm{bu}_*}(\mathrm{bu}_*X, \mathrm{bu}_*Y),$$

and the edge homomorphism

$$[X, \mathrm{bu}\wedge Y] \to E_\infty^{0,0} \hookrightarrow E_2^{0,0} = \mathrm{Hom}_{\mathrm{bu}_*}(\mathrm{bu}_*X, \mathrm{bu}_*Y)$$

is equal to Φ.
(c) $E_2^{s,t} = 0$ for $s > 2$.
(d) The only possible nonzero differential is

$$d_2 : \mathrm{Hom}^t_{\mathrm{bu}_*}(\mathrm{bu}_*X, \mathrm{bu}_*Y) \to \mathrm{Ext}^{2,t-1}_{\mathrm{bu}_*}(\mathrm{bu}_*X, \mathrm{bu}_*Y)$$

Here $\mathrm{Ext}^{s,t}$ means the part of Ext^s which has internal degree t; for example $\mathrm{Ext}^{0,t} = \mathrm{Hom}^t$ denotes the homomorphisms of graded groups which *lower* degree by t. The filtration of $[\Sigma^* X, \mathrm{bu}\wedge Y]$ is described in Sect. 6. We have exact sequences:

$$\begin{aligned} 0 &\to F^{s+1}[\Sigma^{-s-t}X, \mathrm{bu}\wedge Y] \\ &\to F^s[\Sigma^{-s-t}X, \mathrm{bu}\wedge Y] \to E_\infty^{s,t} \to 0 \end{aligned} \tag{10}$$

and $F^s[\Sigma^{-s-t}X, \mathrm{bu}\wedge Y] = 0$ for $s > 2$.

Corollary 4.3 *Let X and Y be finite connected CW complexes. Suppose that the homological projective dimension of the bu_* module $\mathrm{bu}_*(X)$ is at most one. Then there is an exact sequence*

$$\begin{aligned} 0 \;\to\; &\mathrm{Ext}^{1,-1}_{\mathrm{bu}_*}(\mathrm{bu}_*X, bu_*Y) \to [X, \mathrm{bu}\wedge Y] \\ \xrightarrow{\Phi}\; &\mathrm{Hom}_{\mathrm{bu}_*}(\mathrm{bu}_*X, \mathrm{bu}_*Y) \to 0 \end{aligned} \tag{11}$$

Proof. The assumption on the homological projective dimension of bu_*X implies that $\mathrm{Ext}^{s,t}(\mathrm{bu}_*X, \mathrm{bu}_*Y) = 0$ for $s > 1$. Thus all the d_2 differentials are zero and the edge homomorphism Φ is surjective. The statement follows now from (10). QED

Remark Notice that if we let X and Y be the spaces described in part (b) of 4.1 then $d_2 \neq 0$ in the spectral sequence $E_r(X,Y)$. It will become apparent from our arguments below that if X, Y are finite connected CW complexes, then $C_0(X) \otimes \mathcal{K} \simeq C_0(Y) \otimes \mathcal{K}$ if and only if there is an isomorphism $\alpha \in E_2^{0,0} = \mathrm{Hom}_{\mathrm{bu}_*}(\mathrm{bu}_*X, \mathrm{bu}_*Y)$ with $d_2(\alpha) = 0$.

The statement of Theorem 4.2 does not agree precisely with the corresponding statements in [25] and [17]; we shall explain how to deduce it from the work of those authors in Sect. 6, except that we will show part (c) in Sect. 5.

It is now easy to complete the proofs of part (a) and (c) of Theorem 4.1. Part (a) is a straightforward consequence of Corollary 4.3 and Corollary 3.4.

Next we deal with part (c). Let $\alpha : S^m \to S^n$ be any map which is not stably trivial. Let X be the cofiber of α and let $Y = S^{m+1} \vee S^n$ (i.e., Y is the cofiber of the trivial map from S^m to S^n). The stable maps from S^m to S^n form a finite group, but bu_*S^n is either zero or torsion free, so α must induce the zero map of bu-homology. Now the long exact sequence for bu_*X shows at once that

$$\mathrm{bu}_*X \cong \mathrm{bu}_*S^{m+1} \oplus \mathrm{bu}_*S^n \cong \mathrm{bu}_*Y.$$

In particular, bu_*X is a free bu_*-module, and so we have that $\mathrm{Ext}^{s,t}(\mathrm{bu}_*X, \mathrm{bu}_*Y) = 0$ for $s > 0$. Thus the spectral sequence collapses and Φ is an isomorphism, implying $C_0(X) \otimes \mathcal{K} \simeq C_0(Y) \otimes \mathcal{K}$. On the other hand, if X were stably homotopic to Y then the composite of stable maps

$$S^m \xrightarrow{\alpha} S^n \hookrightarrow X \xrightarrow{\simeq} Y \to S^n$$

(where the last map is the projection of Y on its wedge-summand S^n) would be nullhomotopic (since the first two of these maps are consecutive maps in a cofiber sequence), while the composite of the last three would be a stable equivalence of S^n (since it is clearly an isomorphism in homology). This would imply that α is stably nullhomotopic, contrary to our initial assumption.

We remark that if α is chosen to be the nontrivial map $\eta : S^{n+1} \to S^n$ then an easy calculation shows that KO_*X is not isomorphic to KO_*Y. Thus in this case we have an example where $C_0(X) \otimes \mathcal{K}$ is homotopy equivalent to $C_0(Y) \otimes \mathcal{K}$ over the complex numbers but not over the reals.

We now turn to part (b) of Theorem 4.1. Fix an odd prime p and an integer $k \geq 2$, and let M be defined by the cofiber sequence

$$S^k \xrightarrow{p} S^k \to M.$$

By [1, Theorem 1.7] and [8], there is a map

$$A : \Sigma^{2p-2}M \to M$$

which induces an isomorphism in K-homology. Let N be defined by the cofiber sequence

$$\Sigma^{2p-2}M \xrightarrow{A} M \to N.$$

Let $f : N \to \Sigma^{2p-1}M$ be the next map in this cofiber sequence, and let $g : \Sigma^{2p-1}M \to S^{2p+k}$ be the next map in the cofiber sequence defining M. Finally, let $h = g \circ f$, let X be the cofiber of h, and let $Y = \Sigma N \vee S^{2p+k}$ (i.e., Y is the cofiber of the trivial map from N to S^{2p+k}).

(It is not difficult to show that X is homotopic to a 3-cell complex, but we shall not need to know this).

In order to verify the first assertion of part (b) we begin by noting that $\mathrm{bu}_{k+n}M$ is $\mathbb{Z}/p$ if n is a nonnegative even integer and 0 otherwise. The long exact sequence for bu_*N now shows that $\mathrm{bu}_{k+n}(N)$ is $\mathbb{Z}/p$ if n is a nonnegative even integer less than $2p-2$, and zero otherwise. Hence h induces the zero map of bu-homology, and we conclude that

$$\mathrm{bu}_*X \cong \mathrm{bu}_*\Sigma N \oplus \mathrm{bu}_*S^{2p+k} \cong \mathrm{bu}_*Y$$

as bu_*-modules.

Before proceeding, we need to introduce some notation. Given any spectra W and Z and a map $\alpha : W \to \mathrm{bu} \wedge Z$, let $\tilde{\alpha}$ denote the composite

$$\mathrm{bu} \wedge W \xrightarrow{1\wedge\alpha} \mathrm{bu} \wedge \mathrm{bu} \wedge Z \xrightarrow{\mu} \mathrm{bu} \wedge Z,$$

where μ is the multiplication of the ring spectrum bu. Note that $\Phi(\alpha)$ is the map of homotopy groups induced by $\tilde{\alpha}$. Also, given a map $\beta : W \to Z$, let β' denote the composite

$$W \xrightarrow{\beta} Z = S^0 \wedge Z \xrightarrow{\eta\wedge 1} \mathrm{bu} \wedge Z,$$

where η is the unit map of the ring spectrum bu.

The proof of the remaining assertion of part (b) will be by contradiction, so suppose that $C_0(X) \otimes \mathcal{K} \simeq C_0(Y) \otimes \mathcal{K}$. Then there is a map $\alpha : X \to \mathrm{bu} \wedge Y$ for which $\Phi(\alpha)$ is an isomorphism; this implies that $\tilde{\alpha}$ induces an isomorphism of homotopy groups. Now consider the following composite, which we shall denote by Γ:

$$\mathrm{bu} \wedge S^{2p+k} \xrightarrow{1\wedge i} \mathrm{bu} \wedge X \xrightarrow{\tilde{\alpha}} \mathrm{bu} \wedge Y \xrightarrow{1\wedge j} \mathrm{bu} \wedge S^{2p+k}$$

(here i is the inclusion of S^{2p+k} in X, and j is the projection of Y to its wedge-summand S^{2p+k}). The map Γ induces an isomorphism of homotopy groups, since $(1\wedge i)_*$ and $(1\wedge j)_*$ are isomorphisms in all dimensions where $\pi_*(\mathrm{bu} \wedge S^{2p+k})$ is nonzero. But $\Gamma \circ h'$ is homotopically trivial, since h and i are consecutive maps in a cofiber sequence. It follows that h' is trivial. To establish a contradiction, we shall show that h' is nontrivial.

For this purpose we need to use a product pairing in the spectral sequence of Theorem 4.2. Given $\alpha : V \to \mathrm{bu} \wedge W$ and $\beta : W \to \mathrm{bu} \wedge Z$ let us define the “composition” $\beta * \alpha \in [V, \mathrm{bu} \wedge Z]$ to be the composite

$$V \xrightarrow{\alpha} \mathrm{bu} \wedge W \xrightarrow{1 \wedge \beta} \mathrm{bu} \wedge \mathrm{bu} \wedge Z \xrightarrow{\mu \wedge 1} \mathrm{bu} \wedge Z.$$

(Note that this composition operation corresponds to ordinary composition of homomorphisms of C^* algebras). The following result is due to Robinson [27] and Elmendorf, Kriz, Mandell and May [17, Theorem IV.3.4]; we state it in a somewhat different form which we shall reconcile with their version in Sect. 6.

Theorem 4.4 *There is a pairing of spectral sequences*

$$E_r(V, W) \otimes E_r(W, Z) \to E_r(V, Z)$$

which is the Yoneda product ([22, Sect. 3.5]) when $r = 2$ and is induced by the $*$ operation when $r = \infty$.

Notice that $h' = g' * f'$. To complete the proof that h' is nontrivial, we shall need the following facts.

Proposition 4.5 (a) $E_2^{1,-1}(N, \Sigma^{2p-1}M)$, $E_2^{1,-1}(\Sigma^{2p-1}M, S^{2p+k})$ *and $E_2^{2,-2}(N, S^{2p+k})$ are each isomorphic to $\mathbb{Z}/p$.*

(b) *f' and g' are represented by nontrivial elements $x \in E_2^{1,-1}(N, \Sigma^{2p-1} M)$, and $y \in E_2^{1,-1}(\Sigma^{2p-1}M, S^{2p+k})$.*

(c) *The Yoneda product of x and y is a nontrivial element of $E_2^{2,-2}(N, S^{2p+k})$.*

We shall prove parts (a) and (c) in Sect. 5, and part (b) in Sect. 6. Assuming these for the moment, we see that $h' = g' * f'$ is represented by a nontrivial element of $E_2^{2,-2}$. If h' were trivial then this element of E_2 would have to be hit by a differential. But such a differential would have to originate in $\mathrm{Hom}_{\mathrm{bu}_*}(\mathrm{bu}_* N, \mathrm{bu}_* S^{2p+k})$, and this Hom group is zero because there are no nontrivial homomorphisms from $\mathbb{Z}/p$ to $\mathbb{Z}$.

5 Algebraic calculations in the E_2-term.

In this section we prove parts (a) and (c) of Proposition 4.5. The following result will allow us to calculate the relevant Ext groups.

Theorem 5.1 *Let A and B be any two bu_*-modules. Let u denote the generator of $\pi_2\mathrm{bu}$, and let $u_1 : A \to A$ and $u_2 : B \to B$ denote multiplication* *by u. Then there is a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}^t_{\mathrm{bu}_*}(A,B) \to \mathrm{Hom}^t_{\mathbb{Z}}(A,B) &\xrightarrow{u_1^*-u_{2*}} \mathrm{Hom}^{t-2}_{\mathbb{Z}}(A,B) \to \\ \mathrm{Ext}^{1,t}_{\mathrm{bu}_*}(A,B) \to \mathrm{Ext}^{1,t}_{\mathbb{Z}}(A,B) &\xrightarrow{u_1^*-u_{2*}} \mathrm{Ext}^{1,t-2}_{\mathbb{Z}}(A,B) \to \\ \mathrm{Ext}^{2,t}_{\mathrm{bu}_*}(A,B) \to 0 & \end{aligned}$$

and $\mathrm{Ext}^s_{\mathrm{bu}_*}(A,B)=0$ *for* $s>2$.

From now on we will abbreviate bu_* by T.

Proof of Theorem 5.1. Let $u_3 : T \to T$ be multiplication by u, and let $\mu : T\otimes A \to A$ be the action of T on A. It is easy to check that the sequence

$$0 \to T\otimes A \xrightarrow{1\otimes u_1 - u_3\otimes 1} T\otimes A \xrightarrow{\mu} A \to 0$$

is an exact sequence of T-modules in which the first map raises degrees by 2. It therefore induces a long exact sequence of Ext-groups:

$$\begin{aligned} &0 \to \mathrm{Hom}^t_T(A,B) \to \mathrm{Hom}^t_T(A,B) \xrightarrow{(1\otimes u_1)^*-(u_3\otimes 1)^*} \\ &\mathrm{Hom}^{t-2}_T(T\otimes A,B) \xrightarrow{\partial} \\ &\mathrm{Ext}^{1,t}_T(A,B) \to \mathrm{Ext}^{1,t}_T(T\otimes A,B) \xrightarrow{(1\otimes u_1)^*-(u_3\otimes 1)^*} \\ &\mathrm{Ext}^{1,t-2}_T(T\otimes A,B) \xrightarrow{\partial} \mathrm{Ext}^{2,t}_T(A,B) \to \cdots \end{aligned} \tag{12}$$

But a standard change of rings theorem [6, Proposition VI.4.1.3] says that the composite

$$\mathrm{Ext}^*_T(T\otimes A,B) \to \mathrm{Ext}^*_{\mathbb{Z}}(T\otimes A,B) \to \mathrm{Ext}^*_{\mathbb{Z}}(A,B)$$

is an isomorphism (here the first map is the evident forgetful map and the second is induced by the $\mathbb{Z}$-module map which takes m to $1\otimes m$). Using this isomorphism, and the fact that $\mathrm{Ext}^s_Z = 0$ for $s > 2$, it is easy to see that in the long exact sequence we have just given all terms are zero after $\mathrm{Ext}^{2,t}_T(A,B)$ and that the initial part of the sequence is isomorphic to that given in the theorem. QED

We can now prove part (a) of Proposition 4.5. First we consider

$$\mathrm{Ext}^{1,-1}_T(\mathrm{bu}_*N, \mathrm{bu}_*\Sigma^{2p-1}M).$$

Let A denote the graded group bu_*N and let B denote $\mathrm{bu}_*\Sigma^{2p-1}M$. We have seen in the previous section that A_{k+n} is $\mathbb{Z}/p$ if n is even with $0\le n < 2p-2$ and 0 otherwise, while $B_{2p-1+k+n}$ is $\mathbb{Z}/p$ when n is a nonnegative even integer and 0 otherwise. This is described in the following table, the last row of which will be used in the second part of the proof.

n	0	1	2	3	4	$\cdots$	$2p-4$	$2p-3$	$2p-2$	$2p-1$	$2p$	$2p+1$	$2p+2$	$\cdots$
A_{k+n}	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$	$\cdots$	$\mathbb{Z}/p$	0	0	0	0	0	0	$\cdots$
B_{k+n}	0	0	0	0	0	$\cdots$	0	0	0	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$	0	$\cdots$
C_{k+n}	0	0	0	0	0	$\cdots$	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$

By definition, $\mathrm{Hom}^t_{\mathbb{Z}}(A, B)$ is

$$\prod_{i=-\infty}^{\infty} \mathrm{Hom}_{\mathbb{Z}}(A_i, B_{i-t}),$$

so in our case $\mathrm{Hom}^{-1}_{\mathbb{Z}}(A, B)$ is zero, while $\mathrm{Hom}^{-3}_{\mathbb{Z}}(A, B)$ is $\mathbb{Z}/p$, so the cokernel of $u_1^* - u_{2*}$ is $\mathbb{Z}/p$. Since $\mathrm{Ext}^{1,-1}_{\mathbb{Z}}(A, B)$ is zero we conclude that $\mathrm{Ext}^{1,-1}_{T}(A, B)$ is $\mathbb{Z}/p$ as required.

Next we consider $\mathrm{Ext}^{1,-1}_{T}(\mathrm{bu}_*\Sigma^{2p-1}M, \mathrm{bu}_*S^{2p+k})$. Let B be as above and let C denote bu_*S^{2p+k}. Then C_{2p+k+n} is $\mathbb{Z}$ if n is an even integer ≥ 0 and 0 otherwise. It follows that $\mathrm{Hom}^{-3}_{\mathbb{Z}}(B, C) = 0$, so $\mathrm{Ext}^{1,-1}_{T}(B, C)$ is the kernel of

$$\mathrm{Ext}^{1,-1}_{\mathbb{Z}}(B, C) \xrightarrow{u_1^* - u_{2*}} \mathrm{Ext}^{1,-3}_{\mathbb{Z}}(B, C).$$

Now an element of either $\mathrm{Ext}^{1,-1}_{\mathbb{Z}}(B, C)$ or $\mathrm{Ext}^{1,-3}_{\mathbb{Z}}(B, C)$ is a sequence

$$x_{2p+k-1}, x_{2p+k+1}, x_{2p+k+3}, \ldots$$

of elements of $\mathbb{Z}/p$, and

$$\begin{aligned}(u_1^* - u_{2*})&(x_{2p+k-1}, x_{2p+k+1}, x_{2p+k+3}, \ldots) \\ &= (x_{2p+k+1} - x_{2p+k-1}, x_{2p+k+3} - x_{2p+k+1}, \ldots).\end{aligned}$$

Therefore the kernel of $u_1^* - u_{2*}$ is a copy of $\mathbb{Z}/p$ generated by $(1, 1, 1, \ldots)$.

Finally, consider $\mathrm{Ext}^{2,-2}_{T}(A, C)$. We have $\mathrm{Ext}^{1,-2}_{\mathbb{Z}}(A, C) = 0$ and $\mathrm{Ext}^{1,-4}_{\mathbb{Z}}(A, C) = \mathbb{Z}/p$, so we conclude that $\mathrm{Ext}^{2,-2}_{T}(A, C) = \mathbb{Z}/p$. QED

We conclude this section by proving part (c) of Proposition 4.5. We continue with the notation of the previous proof. From what has been shown we know that the boundary map

$$\mathrm{Hom}^{-3}_{T}(T \otimes A, B) \xrightarrow{\partial} \mathrm{Ext}^{1,-1}_{T}(A, B)$$

is an isomorphism, so there is an element x' of $\mathrm{Hom}^{-3}_{T}(T \otimes A, B)$ with $\partial(x') = x$. By [22, Theorem III.9.1] we see that the boundary operator ∂ is (up to sign) the Yoneda product with a certain element, and in particular we have $xy = (\partial x')y = \partial(x'y)$. Since

$$\mathrm{Ext}^{1,-4}_{T}(T \otimes A, C) \xrightarrow{\partial} \mathrm{Ext}^{2,-2}_{T}(A, C)$$

is also an isomorphism, it suffices to show that $x'y \neq 0$.

Next consider the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_T^{-3}(T\otimes A,B)\otimes \mathrm{Ext}_T^{1,-1}(B,C) & \rightarrow & \mathrm{Ext}_T^{1,-4}(T\otimes A,C)\\
\downarrow {\scriptstyle I\otimes\rho} & & \downarrow {\scriptstyle I}\\
\mathrm{Hom}_{\mathbb{Z}}^{-3}(A,B)\otimes \mathrm{Ext}_{\mathbb{Z}}^{1,-1}(B,C) & \rightarrow & \mathrm{Ext}_{\mathbb{Z}}^{1,-4}(A,C)\\
\downarrow r_1\otimes r_2 & & \downarrow r_3\\
\mathrm{Hom}_{\mathbb{Z}}(A_{k+2p-4},B_{k+2p-1})\otimes \mathrm{Ext}_{\mathbb{Z}}(B_{k+2p-1},C_{k+2p}) & \rightarrow & \mathrm{Ext}_{\mathbb{Z}}(A_{k+2p-4},C_{k+2p})
\end{array}$$

Here the horizontal arrows are Yoneda products (which in this case are simply composition operations), ρ is the evident restriction map, and I denotes the isomorphism used in the proof of Theorem 5.1; from the description of I given there, it is easy to see that the upper square commutes. The lower square clearly commutes, and the bottom horizontal map is an isomorphism. Also, the restriction maps r_1 and r_3, and the composite $r_2\circ\rho$, have been shown to be isomorphisms in the proof just given. The result follows.

QED

6 Detection of elements in filtration 1.

In this section we shall prove part (b) of Theorem 4.5.

Let W and Z be any spaces, and let $\beta : W \to Z$ be any map. As in Sect. 1, we write

$$\beta' \in [W, \mathrm{bu}\wedge Z]$$

for the composite

$$W \xrightarrow{\beta} Z = S^0\wedge Z \xrightarrow{\eta\wedge 1} \mathrm{bu}\wedge Z,$$

and we recall that $\Phi(\beta')$ is the homomorphism induced by β in bu-homology.

Now we assume that β induces the zero homomorphism in bu_* homology. (for example, this is true for the maps f and g in Proposition 4.5). Then $\Phi(\beta')=0$, so by part (b) of Theorem 4.2, β' determines an element (which may be zero) in

$$E_\infty^{1,-1}(W,Z)$$

and by parts (b) and (d) of Theorem 4.2 this group is equal to

$$\mathrm{Ext}_T^{1,-1}(\mathrm{bu}_*W, \mathrm{bu}_*Z),$$

where T denotes $\pi_*\mathrm{bu}$. Now by [22, Theorem III.6.4] an element of this Ext-group is identified with an equivalence class of extensions of T-modules

$$0\to \mathrm{bu}_*Z\to A\to \mathrm{bu}_*W\to 0,$$

(where the map $A \to \mathrm{bu}_* W$ raises degrees by 1) so in particular the map β is represented by a class of such extensions. Our aim is to describe explicitly an extension which represents β.

To do this, we let

$$W \xrightarrow{\beta} Z \to C(\beta) \to \Sigma W$$

be the cofiber sequence determined by β. Applying bu_*, and using the fact that $\beta_* = 0$, we get a short exact sequence of T-modules

$$0 \to \mathrm{bu}_* Z \to \mathrm{bu}_* C(\beta) \to \mathrm{bu}_* W \to 0 \tag{13}$$

where the map $\mathrm{bu}_* C(\beta) \to \mathrm{bu}_* W$ raises degrees by 1.

Theorem 6.1 *The extension (13) represents the element of* $\mathrm{Ext}_T^{1,-1}(\mathrm{bu}_* W, \mathrm{bu}_* Z)$ *determined by* β*. In particular, this element of* Ext *is zero if and only if (13) is split as a short exact sequence of* T*-modules.*

Before proving this, we use it to complete the proof of Proposition 4.5(b). From the definition of f, we see that the cofiber sequence determined by f has the form

$$N \xrightarrow{f} \Sigma^{2p-1} M \to \Sigma M \to \Sigma N\ .$$

The short exact sequence induced by this is

$$0 \to \mathrm{bu}_* \Sigma^{2p-1} M \to \mathrm{bu}_* \Sigma M \to \mathrm{bu}_* N \to 0,$$

and this is certainly not split since there is no nontrivial T-module homomorphism from $\mathrm{bu}_* N$ to $\mathrm{bu}_* \Sigma M$ (note that the generator u of T acts nilpotently on the former but not on the latter). The argument for g is similar and is left to the reader.

In the remainder of the section we give the proof of Theorem 6.1. It is necessary first to put this result in a broader context by relating it to the work of Elmendorf, Kriz, Mandell and May.

Let R be an A_∞ ring spectrum (for our purposes it is not necessary to know what this means, we only need to know that bu is one; in fact it has the stronger structure of an E_∞ ring spectrum, by [23, VIII.2.1]). Then Elmendorf, Kriz, Mandell and May, following earlier work of Robinson [24–27], show how to define a category of R-module spectra in which one can do homotopy theory (technically, these are A_∞ R-modules, which is a stricter notion than the R-module spectra used in [2], for instance). The homotopy category is denoted $\mathcal{D}_R$. For example, when R is the sphere spectrum S, $\mathcal{D}_R$ is the usual stable category. An R-module is a spectrum with extra structure, so there is a forgetful functor which takes an R-module to its underlying spectrum (technically, the underlying spectrum functor is $F_{\mathcal{L}}(S, -)$; see [17, Section III.1]), there is also a "free R-module" functor $\mathbb{F}$ which is left adjoint to the forgetful functor. Thus if we write $[A, B]_R$ for the set of homotopy classes in $\mathcal{D}_R$ when A and B are two R-modules, then we have

$$[\mathbb{F}(X), B]_R = [X, B]$$

whenever B is an R-module. In particular, we can let $B = \mathbb{F}(Y)$; the underlying spectrum of B is then $R \wedge Y$, by [17, Proposition III.1.4], and so we have

$$[\mathbb{F}(X), \mathbb{F}(Y)]_R = [X, R \wedge Y].$$

As a first application of these ideas we give a different description of the category $\mathcal{B}$ mentioned in Theorem 3.5. Recall that the objects of $\mathcal{B}$ are finite connected CW-complexes, and the set of morphisms from X to Y is $[X, \mathrm{bu} \wedge Y]$, with the composition operation defined in Sect. 3. Now let $\mathcal{B}'$ be the category whose objects are bu-modules of the form $\mathbb{F}(X)$, with X a finite connected CW-complex, and whose morphisms are homotopy classes of bu-module maps.

Proposition 6.2 *$\mathcal{B}$ is equivalent to $\mathcal{B}'$.*

Proof. The result follows immediately from [17, Theorem III.1.4] and its proof. QED

Next we observe that there is a spectral sequence

$$\mathrm{Ext}^{s,t}_{\pi_* R}(\pi_* A, \pi_* B) \Longrightarrow [\Sigma^{-s-t} A, B]_R,$$

(see [25], [17, Theorem IV.3.1]), and letting $R = \mathrm{bu}$, $A = \mathbb{F}(X)$ and $B = \mathbb{F}(Y)$ gives the spectral sequence of Theorem 4.2. Properties (a) and (b) of Theorem 4.2 are immediate from the corresponding properties in [25] and [17, Theorem IV.3.1].

Similarly, Theorem 6.1 is immediate from the following more general result:

Theorem 6.3 *Let*

$$A \xrightarrow{\gamma} B \to C \to \Sigma A \to$$

be a cofiber sequence of R-modules, with $\pi_* \gamma = 0$. Then γ is in $F^1[A, B]_R$, and the image of γ under the composite

$$F^1[A, B]_R / F^2[A, B]_R = E^{1,-1}_\infty(A, B) \hookrightarrow \mathrm{Ext}^{1,-1}_{\pi_* R}(\pi_* A, \pi_* B)$$

is the element represented by the short exact sequence

$$0 \to \pi_* B \to \pi_* C \to \pi_* \Sigma A \to 0.$$

If $f : M \to N$ is a map of graded modules, then $\Sigma f : \Sigma M \to \Sigma N$ is defined by $(\Sigma f)_i = f_{i-1}$ where ΣM is the module $(\Sigma M)_i = M_{i-1}$. In order to prove Theorem 6.3 we have to recall how the spectral sequence is defined; we use the construction in [17, Sect. IV.4] and the reader is referred to that source for further information. First pick a resolution of $\pi_* A$ by free $\pi_* R$-modules:

$$\cdots \leftarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} = \pi_* A \xrightarrow{d_{-1}} 0 \,. \tag{14}$$

For each $i \geq 0$ let X_i be a wedge of spheres indexed by a basis for M_i. Define R-module spectra A_i for $i \geq 0$, with $A_0 = A$, inductively by the cofiber sequences

$$\Sigma^i \mathbb{F} X_i \xrightarrow{k_i} A_i \to A_{i+1} \xrightarrow{j_{i+1}} \Sigma^{i+1} \mathbb{F} X_i,$$

with the following properties:

(i) k_i induces an epimorphism $\pi_* \Sigma^i \mathbb{F} X_i \to \pi_* A_i$ for $i \geq 0$.
(ii) $\pi_* A_i \cong \Sigma^i(\ker\, d_{i-1})$ for $i \geq 0$.
(iii) k_i realizes $\Sigma^i d_i : \Sigma^i M_i \to \Sigma^i(\ker\, d_{i-1})$ on π_* for $i \geq 0$.
(iv) j_{i+1} realizes the inclusion $\Sigma^{i+1}(\ker\, d_i) \to \Sigma^{i+1} M_i$ on π_* for $i \geq 0$.

Observe that (ii) together with (iii) implies (i). Then the groups

$$D_1^{p,q} = [\Sigma^{-p-q} A_p, B]_R$$

and

$$E_1^{p,q} = [\Sigma^{-q} \mathbb{F} X_i, B]_R$$

form an exact couple, which in the usual way induces the required spectral sequence. To prove that the E_2-term has the desired form, one uses the fact that

$$[\mathbb{F} X_i, B]_R \cong [X_i, B] \cong \mathrm{Hom}_{\pi_* R}(M_i, \pi_* B).$$

It is shown in [17, Sect. IV.4] that the spectral sequence is independent, from E_2 on, of the choices made in this construction.

Now let $\gamma : A \to B$ be a map of R-modules which induces the zero homomorphism of homotopy groups. Since X_0 is a wedge of spheres, the composite

$$X_0 \to A \xrightarrow{\gamma} B$$

is nullhomotopic, and since $\mathbb{F}$ is left adjoint to the forgetful functor the composite

$$\mathbb{F} X_0 \to A \xrightarrow{\gamma} B$$

is also nullhomotopic; thus it is possible to extend γ to a map

$$\Gamma : A_1 \to B.$$

The composite

$$\Sigma \mathbb{F} X_1 \xrightarrow{k_1} A_1 \xrightarrow{\Gamma} B$$

induces a map of homotopy groups

$$\bar{\gamma} : M_1 = \pi_* \mathbb{F} X_1 \to \pi_* B$$

which raises degrees by 1 and is a $\pi_* R$-homomorphism. Using (ii)-(iv) we see that
$j_{1*} k_{1*} d_2 = d_1 d_2 = 0$. Since j_{1*} is injective, this implies that

$$\bar{\gamma} d_2 = \Gamma_* k_{1*} d_2 = 0.$$

Since (14) is a projective resolution, and $\bar{\gamma} d_2 = 0$, $\bar{\gamma}$ represents an element of $\mathrm{Ext}_R^{1,-1}(\pi_* A, \pi_* B)$, and by the definition of the spectral sequence this is the element corresponding to γ.

It remains to show that $\bar{\gamma}$ corresponds to the extension

$$0 \to \pi_* B \to \pi_* C \to \pi_* \Sigma A \to 0.$$

Since the sequence

$$\pi_* \mathbb{F} X_1 \to \pi_* \mathbb{F} X_0 \to \pi_* A \to 0$$

is a partial projective resolution, Theorem III.6.4 of [22] tells us that it suffices to show that there is a commutative diagram

$$\begin{array}{ccc} \pi_* \mathbb{F} X_1 & \xrightarrow{\bar{\gamma}} & \pi_* B \\ \downarrow & & \downarrow \\ \pi_* \mathbb{F} X_0 & \to & \pi_* C \\ \downarrow & & \downarrow \\ \pi_* A & \xrightarrow{=} & \pi_* A. \end{array} \tag{15}$$

To construct such a diagram, we first observe that the composite

$$\mathbb{F} X_0 \xrightarrow{k_0} A \xrightarrow{\gamma} B$$

is nullhomotopic, and hence k_0 lifts to $\Sigma^{-1} C$, where C is the cofiber of γ. Thus we have a homotopy-commutative diagram

$$\begin{array}{ccc} \mathbb{F} X_0 & \to & \Sigma^{-1} C \\ \downarrow k_0 & & \downarrow \\ A & \xrightarrow{=} & A \end{array}$$

Now it is a well-known fact of homotopy theory (see for instance [31, Lemma 8.31]) that any homotopy-commutative diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ \downarrow f & & \downarrow f' \\ Y & \xrightarrow{\beta} & Y' \end{array}$$

extends to a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ \downarrow f & & \downarrow f' \\ Y & \xrightarrow{\beta} & Y' \\ \downarrow & & \downarrow \\ Cf & \rightarrow & Cf' \\ \downarrow & & \downarrow \\ \Sigma X & \xrightarrow{\Sigma\alpha} & \Sigma X' \\ \downarrow \Sigma f & & \downarrow \Sigma f' \\ \Sigma Y & \xrightarrow{\Sigma\beta} & Y' \end{array}$$

where the columns are the cofiber sequences of f and f'. In our case, we obtain a homotopy-commutative diagram

$$\begin{array}{ccc} \mathbb{F}X_0 & \rightarrow & \Sigma^{-1}C \\ \downarrow k_0 & & \downarrow \\ A & \xrightarrow{=} & A \\ \downarrow & & \downarrow \\ A_1 & \rightarrow & B \\ \downarrow & & \downarrow \\ \Sigma\mathbb{F}X_0 & \rightarrow & C \\ \downarrow & & \downarrow \\ \Sigma A & \xrightarrow{=} & \Sigma A\ . \end{array} \tag{16}$$

The map $A_1 \to B$ in this diagram is a candidate for the map Γ mentioned above, and so the composite

$$\pi_*\mathbb{F}X_1 \xrightarrow{(k_1)_*} \pi_*A_1 \to \pi_*B$$

represents $\bar{\gamma}$. Now applying π_* to the bottom half of diagram (17) and pre-composing with $(k_1)_*$ gives the diagram (16), and this concludes the proof.

Remark The filtration of $[\Sigma^*A, B]_R$ in the above spectral sequence is defined by letting $F^i[\Sigma^*A, B]_R$ to be the image of the map

$$[\Sigma^*A_i, B]_R \to [\Sigma^*A, B]_R$$

induced by the evident iterate $A \to A_i$. If $R = \mathrm{bu}$, since $\mathrm{bu}_* = \mathbb{Z}[u]$ has homological projective dimension equal to 2, one can find a free resolution (14) such that $M_i = 0$ for $i \geq 3$. It follows from (ii) that $\pi_*A_i = 0$ for $i \geq 4$ so that the filtration vanishes in all the degrees greater than three. For the spectral sequence from Theorem 4.2 we see that $F^s[\Sigma^*X, \mathrm{bu} \wedge Y] = 0$ for $s > 2$ since $E_\infty^{s,t} = 0$ for $s > 2$.

Remark If A is a fixed bu-module, Wolbert [33] has adapted the method introduced by Bousfield in [3] to classify the bu-modules B for which $\pi_* B \cong \pi_* A$; he shows that the set of such B is in one to one correspondence with the set

$$S_A = \mathrm{Ext}^{2,-1}_{\mathrm{bu}_*}(\pi_* A, \pi_* A)/\mathrm{Aut}(\pi_* A).$$

Our Theorem 4.1(b) can be restated as follows: there exist finite CW-complexes X and Y so that if $A = \mathrm{bu} \wedge X$ and $B = \mathrm{bu} \wedge Y$ then B represents a nontrivial element in the set S_A. For this example it is not difficult to show directly that the set S_A is nontrivial, but our result does not follow from this because Wolbert's theory gives no way of recognizing which (if any) bu modules in the set S_A have the special form $\mathrm{bu} \wedge Z$ for a finite CW complex Z.

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