3.2 $\boxed{\text{Properties of determinants}}$

<u>Review:</u> Input A $n \times n$ square matrix Output $\det(A)$ or $|A|$ number

$$A \longmapsto \det(A) = |A|$$

defined using cofactor expansion

1×1 $\quad A = [a] \quad \det(A) = a$

2×2 $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det(A) = ad - bc$

3×3
$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \qquad \begin{vmatrix} \boxed{a} & b & c \\ d & \boxed{e} & \boxed{f} \\ g & \boxed{h} & \boxed{i} \end{vmatrix} \qquad \begin{vmatrix} a & \boxed{b} & c \\ \boxed{d} & e & \boxed{f} \\ \boxed{g} & h & \boxed{i} \end{vmatrix} \qquad \begin{vmatrix} a & b & \boxed{c} \\ \boxed{d} & \boxed{e} & f \\ \boxed{g} & \boxed{h} & i \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 4 & -1 & 7 \end{vmatrix} = -2 \begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix} + \underbrace{0 \cdot \begin{vmatrix} * & * \\ * & * \end{vmatrix}}_{0} - (-1) \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

cofactor expansion with respect 2nd column.

<u>Triangular matrices</u>

$$A = \begin{bmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & & * \\ 0 & \cdots 0 & \ddots & * \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \qquad \det(A) = \underline{a_{11} a_{22} \cdots a_{nn}}$$

why?

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & & * \\ 0 & \ddots & a_{nn} \end{vmatrix} = a_{11} a_{22} \begin{vmatrix} a_{33} & & * \\ 0 & \ddots & a_{nn} \end{vmatrix} = \cdots$$

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ * & a_{22} & 0 & 0 \\ & & \ddots & 0 \\ * & * & * & a_{nn} \end{bmatrix} \qquad \det(A) = \underline{a_{11} a_{22} \cdots a_{nn}}$$

Properties of determinants concerning row operations

How does the det change if we perform a row operation?

1. Row replacement
2. Interchange two rows
3. rescale a row (replace a row by multiple of itself)

① replace one row by the sum of itself and a multiple of another row

$$A \xrightarrow[①]{} B \qquad \det(B) = \det(A)$$

② Interchange two rows

$$A \xrightarrow[②]{} B \qquad \det(B) = -\det(A)$$

③ replace a row by $k \cdot (\text{itself})$

$$A \xrightarrow[③]{} B \qquad \det(B) = k \det(A)$$

Ex

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \xrightarrow{①} B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 - 10a_1 & b_2 - 10b_1 & c_2 - 10c_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

② ↙

$$\det(B) = \det(B)$$

$$C = \begin{bmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{bmatrix} \qquad \det(C) = -\det(A)$$

$$D = \begin{pmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{pmatrix} \qquad \underline{\det(D)} = \underline{\det(A)}$$

interchange row 2 times

Remark The same rules are valid if we use column operations.

Ex. $\begin{vmatrix} 2 & 5 & -7 \\ 2 & 6 & 5 \\ -2 & -4 & 7 \end{vmatrix} = 2 \begin{vmatrix} 1 & 5 & -7 \\ 1 & 6 & 5 \\ -1 & -4 & 7 \end{vmatrix}$

compute

(2) $= 2 \begin{vmatrix} 1 & 5 & -7 \\ 0 & 1 & 12 \\ 0 & 1 & 0 \end{vmatrix} = 2 \begin{vmatrix} 1 & 5 & -7 \\ 0 & 1 & 12 \\ 0 & 0 & -12 \end{vmatrix}$ ← triangular

$= 2 \cdot (1)(1)(-12) = -24$

Ex $\begin{vmatrix} 0 & 0 & a \\ 0 & b & x \\ c & y & z \end{vmatrix} = \; ? \; = - \begin{vmatrix} a & 0 & 0 \\ x & b & 0 \\ z & y & c \end{vmatrix} = -abc$

1- column interchange

triangular

Ex. $\begin{vmatrix} 0 & 3 & 2 & 0 & 0 \\ 3 & 2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 2 \\ 0 & 3 & 2 & 0 & 5 \end{vmatrix} = \; ? \; = +(1) \begin{vmatrix} 0 & 3 & 0 & 0 \\ 3 & 2 & 2 & 3 \\ 2 & 1 & 1 & 2 \\ 0 & 3 & 0 & 5 \end{vmatrix}$

cofact exp 3rd row

$+ \; - \; +$

$+$

(i, j) $\quad (-1)^{i+j}$

$= (-3) \begin{vmatrix} 3 & 2 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$

$= (-3)(5) \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -15(3-4) = 15$

$\boxed{\text{FACT}}$ Suppose given A $n \times n$ matrix we bring A in an echelon form U using only row operations (1) and (2)

$$\det(A) = \begin{cases} (-1)^r \det(U) & \text{where } r \text{ is the number of row interchanges and } \underline{A \text{ invertible}} \\ 0 & \text{if } \underline{A \text{ not invertible}} \end{cases}$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $\det(A) = 2$

RREF$(A) = ?$ answer $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U$

$\det(U) = 1$

Obs $\det(A^T) = \det(A)$

$\boxed{\text{Key property:}}$ If A, B are $n \times n$ matrices

then $\det(AB) = \det(A)\det(B)$

Suppose A is invertible then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Indeed $A^{-1} \cdot A = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$$\det(A^{-1} \cdot A) = \det(I_n) = 1$$

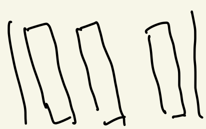
$$\underline{\det(A^{-1})} \det(A) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Geometric meaning:

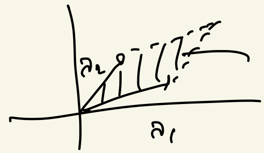
View $\det(A)$ as a functions of the columns of A

$\parallel$ $\det(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$



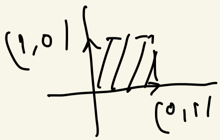
represents $\pm$ volume of region determined by $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$

2×2 $\qquad A = \begin{array}{c} \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \\ \begin{array}{cc} \bar{a}_1 & \bar{a}_2 \end{array} \end{array}$



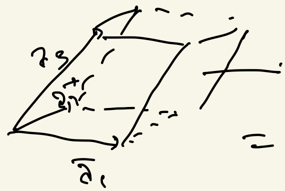
area

$\pm \begin{vmatrix} a & b \\ c & d \end{vmatrix}$



$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

3×3 $\qquad A = \begin{array}{c} \left[\begin{array}{ccc} \; & \; & \; \end{array}\right] \\ \begin{array}{ccc} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \end{array} \end{array}$



volume

$= \pm \det(A)$

Hint: The columns of a matrix are linearly independent $\Longleftrightarrow \det(A) \neq 0$.