3.3 Cramer's rule, adjugate matrix, volume under linear transformations

Cramer's rule

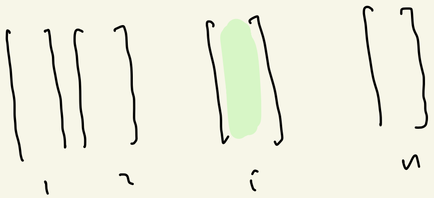
For a linear system $A\vec{x} = \vec{b}$

where A $n \times n$ (square matrix) $\quad \vec{x}$ $n \times 1$

$\vec{b} = n \times 1$

$$A = [\vec{a}_1, \vec{a}_2, \dots, \underset{i}{\vec{a}_i}, \dots, \vec{a}_n]$$

$A =$



$$A_i(\vec{b}) = [\vec{a}_1, \vec{a}_2, \dots \underset{i}{\vec{b}} \dots \vec{a}_n]$$

Cramer's rule: suppose that A is invertible

then $$x_i = \frac{\det(A_i(\vec{b}))}{\det(A)} \qquad i = 1, 2, \dots, n.$$

Ex 1

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & -3 \\ -3 & 4 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -7 \end{bmatrix}$$

$\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \qquad \vec{x} \qquad \vec{b}$

Is A invertible? Yes because

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & -3 \\ -3 & 4 & 7 \end{vmatrix} = 40 \neq 0$$

2

$$A_1(\bar b) = \begin{bmatrix} -5 & 2 & 3 \\ 4 & 1 & -3 \\ -7 & 4 & 7 \end{bmatrix} \qquad A_2(\bar b) = \begin{bmatrix} 1 & -5 & 3 \\ 3 & 4 & -3 \\ -3 & -7 & 7 \end{bmatrix} \qquad A_3(\bar b) = \begin{bmatrix} 1 & 2 & -5 \\ 3 & 1 & 4 \\ -3 & 4 & -7 \end{bmatrix}$$

$\uparrow \bar b$ $\qquad \uparrow \bar b$

$\det|A_1(\bar b)| = -40 \qquad \det|A_2(\bar b)| = 40 \qquad \det(A_3(\bar b)) = -80$

$$x_1 = \frac{|A_1(\bar b)|}{|A|} = -\frac{40}{40} = -1 \qquad x_2 = \frac{|A_2(\bar b)|}{|A|} = \frac{40}{40} = 1$$

$$x_3 = \frac{|A_3(\bar b)|}{|A|} = -\frac{80}{40} = -2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}.$$

Adjugate matrix for a given $n \times n$ square matrix A.

Notation: $\mathrm{adj}(A) \quad n \times n$

Key property:

$$\boxed{A \cdot \mathrm{adj}(A) = \mathrm{adj}(A) \cdot A = \det(A)\, I_n}$$

$$= \det(A) \cdot \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots\; 1 \end{pmatrix} = \begin{pmatrix} \det(A) & & 0 \\ & \ddots & \\ 0 & & \det(A) \end{pmatrix}$$

Thus $$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

whenever A is invertible
$\det(A) \neq 0$.

③

By definition adj(A) is the $n \times n$ matrix whose (i,j) entry is the cofactor C_{ji}.

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

The matrix of cofactors of A is

$$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$\xrightarrow{\text{transpose}}$ to get adj(A)

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex: Use adj(A) to find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

check $\det(A) = -5$
$\Rightarrow A$ is invertible

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

sol'n

The matrix of cofactors is

$$\begin{bmatrix} -1 & -5 & 2 \\ -1 & 5 & -3 \\ 2 & 5 & -4 \end{bmatrix}$$

transpose

$$\operatorname{adj}(A) = \begin{bmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = -\frac{1}{5}\begin{bmatrix} -1 & -1 & 2 \\ -5 & 5 & 5 \\ 2 & -3 & -4 \end{bmatrix}$$

Linear transformations and volume

Review $T: \mathbb{R}^n \to \mathbb{R}^n$ is linear

if
$$\begin{cases} T(\bar{x}+\bar{y}) = T(\bar{x}) + T(\bar{y}) \\ T(c\vec{x}) = cT(\vec{x}) \end{cases}$$

Fact: Any linear map is given by a $n \times n$ matrix A

$$T(\bar{x}) = A\bar{x}$$

How do we construct the matrix A starting from T?

Answer: if $\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\bar{e}_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ .... $\bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

The columns of A are the vectors

$$T(e_1), T(\bar{e}_2), \dots, T(\bar{e}_n)$$

Ex: $T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad T(\bar{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

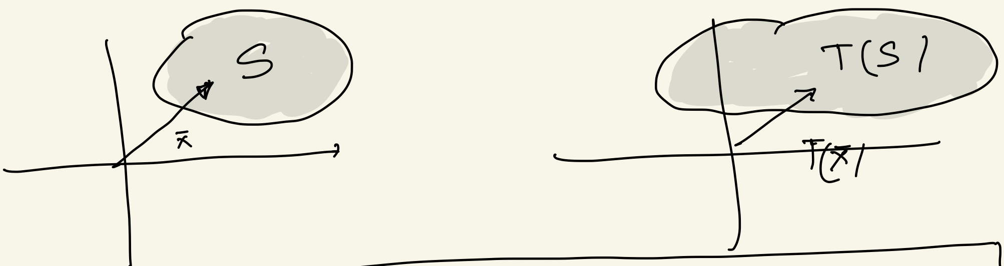
$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$T(\bar{e}_1) \qquad T(\bar{e}_2)$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

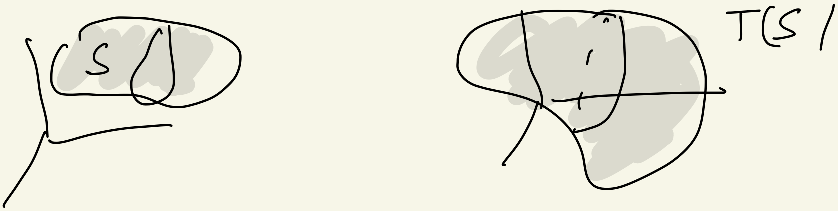
<u>Question:</u> If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear

$$T(\bar{x}) = A\bar{x}$$



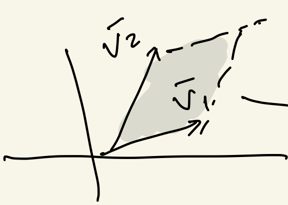
2 $$\boxed{\text{Area}(T(S)) = |\det(A)|\ \text{Area}(S)}$$

3



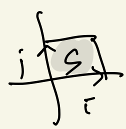
$$\text{Vol}(T(S)) = |\det(A)|\ \text{vol}(S)$$

From this the area of the paralelogram spanned by $v_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $v_2 = \begin{bmatrix} b \\ d \end{bmatrix}$ is $\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$



area

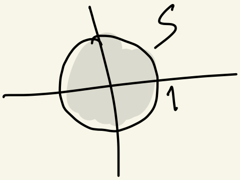
image of

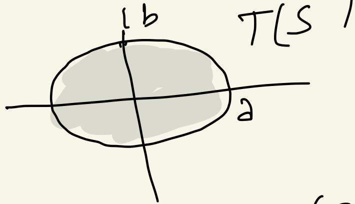


under A

$T(\bar{x}) = A\bar{x}$

$$T\begin{bmatrix}x_1\\x_2\end{bmatrix} = \underbrace{\begin{bmatrix}a & 0\\0 & b\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}a x_1\\b x_2\end{bmatrix} \qquad a, b > 0$$





$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\0\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\b\end{bmatrix}$$

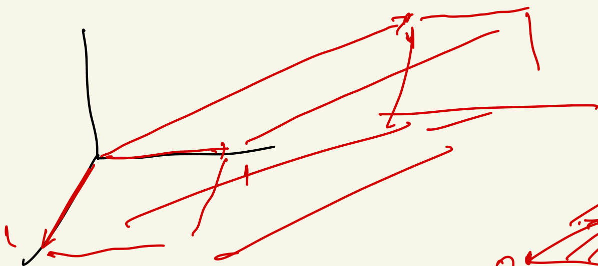
$$\text{area}(S) = \pi$$

$$\det(A) = \begin{vmatrix}a & 0\\0 & b\end{vmatrix} = ab$$

$$\text{area}(T(S)) = ab\,\text{area}(S) = \pi a b$$

Ex.

$$A = \begin{bmatrix}1 & 0 & 2\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix} \qquad \det(A) = 2$$



$$\text{vol} = |\det(A)| = 2$$