Review for Exam 1 (March 3 exam date)

Q1 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find A^2.

$$\boxed{\begin{aligned} A^2 &= A \cdot A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a^2 & ab \\ 0 & 0 \end{bmatrix} \end{aligned}}$$

10 p

$$\boxed{\begin{aligned} A^2 &= A \cdot A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a^2 & ab \\ 0 & 0 \end{bmatrix} \end{aligned}}$$

5 p

- open book exam but
- Do not use computer software to compute inverses, determinants!
- Work independently

Linear systems

$A\vec{x} = \vec{b}$

A	$\vec{x}$	$\vec{b}$
$m \times n$	$n \times 1$	$m \times 1$

m eq's

n unknowns

$$\begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Ex

$$\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 5 \end{bmatrix}}\underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix}}$$

$$\begin{cases} x_1 + 2x_2 = 3 \\ -x_1 + 3x_2 = 10 \\ 4x_1 + 5x_2 = 7 \end{cases}$$

$$\underset{2\times 3}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}}\underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$\begin{cases} 0 = 1 \\ x_1 - x_2 + x_3 = 0 \end{cases}$$

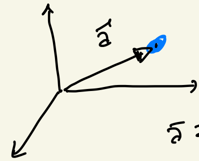
$$\begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$

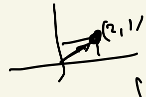
$A = [\,\vec{a}_1 \quad \vec{a}_2 \; \cdots \; \vec{a}_n\,]$

column vectors are in $\mathbb{R}^m$

vectors in $\mathbb{R}^m$



$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$



$m=2$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

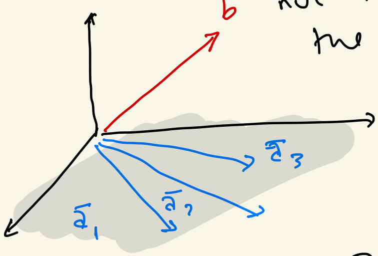
$$\boxed{A\vec{x} = \vec{b}}$$

$$\Updownarrow$$

$$\boxed{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}}$$

Seek to write $\vec{b}$ as a linear combination of $\vec{n}$ vectors, the columns of A

$m = n = 3$



$\vec{b}$ not in the xy-plane

all belong to "xy-plane"

Is it possible to write $\vec{b}$ as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$?

The xy-plane is a linear subspace of $\mathbb{R}^3$

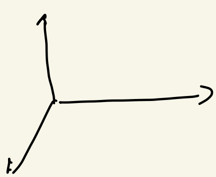
$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$ in xy-plane

system is not consistent

$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & b_3 \end{array}\right] \quad b_3 \neq 0$$

$\vec{a}_1$

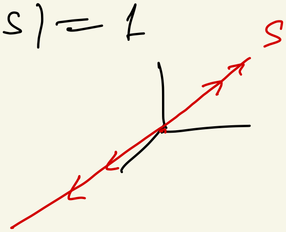
b not in xy-plane



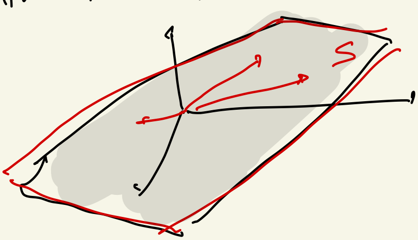
Subspaces S of $\mathbb{R}^3$ can have dimensio $0, 1, 2,$ or 3

$S = \{0\}$ $\dim(S) = 0$

$S =$ line through origin $\dim(S) = 1$



$S =$ plane through the origin



$\dim(S) = 2$

$S = \mathbb{R}^3$ $\dim(S) = 3.$

Linear subspace of $\mathbb{R}^3$

S subset of $\mathbb{R}^3$ s.t.

$\bar{u}, \bar{v}$ in $S \Rightarrow \bar{u} + \bar{v}$ in S

$\bar{u}$ in S, c in $\mathbb{R} \Rightarrow c\bar{u}$ in S

The column space of a matrix $A = [\bar{a}_1, a_2, .., \bar{a}_n]$

$\bar{a}_1, \bar{a}_2, .., \bar{a}_n$ in $\mathbb{R}^m$ $m \times n$ m rows, n columns

$\to \mathrm{Col}(A) =$ the linear subspace of $\mathbb{R}^m$ generated by the vectors $\bar{a}_1, .., \bar{a}_n$

Thus $A\bar{x} = \bar{b}$ is consistent $\iff$

$\vec{b}$ belongs to the column space of A

$\left(\dim(\mathrm{Col}(A)) = \mathrm{rank}(A) = \right.$ the number of pivots in the RREF(A) $\left.\right)$

- $A\bar{x} = \bar{b}$ is consistent $\iff$ $\boxed{\mathrm{rank}(A) = \mathrm{rank}[A \mid \bar{b}]}$

- If $A\bar{x} = \bar{b}$ is consistent, do we have a unique solution or infinitely many solutions?

First look at homogeneous systems

$$\boxed{A\bar{x} = \bar{0}}$$

- always consistent $\bar{x} = \bar{0}$ is a sol'n.

- how many solutions?

Look at $\mathrm{Nul}(A) = \{ \vec{x} \text{ in } \mathbb{R}^n : A\bar{x} = \bar{0} \} \subseteq \mathbb{R}^n$ is a linear subspace.

$\dim(\mathrm{Nul}(A)) = ?$

$\underbrace{\dim(\mathrm{Col}(A))}_{\mathrm{rank}(A)} + \dim(\mathrm{Nul}(A)) = n =$ nr of columns

$$\boxed{\dim(\mathrm{Nul}(A)) = n - \mathrm{rank}(A)}$$

$A\bar{x} = \bar{0}$ has only the trivial sol'n

$\iff \dim(\mathrm{Nul}(A)) = \{0\}$

$\iff \mathrm{rank}(A) = n$

$A\bar{x} = \bar{0}$ has infinitely many sol's

$\iff \mathrm{rank}(A) < n.$

<u>Remark</u> If A $m \times n$

then $\mathrm{rank}(A) \leq n$

$\mathrm{rank}(A) \leq m$

Back to $A\bar{x} = \bar{b}$

- consistent $\iff \mathrm{rank}(A) = \mathrm{rank}(A|\bar{b})$

Suppose this is the case:

$$\mathrm{rank}(A) = \mathrm{rank}(A|\bar{b})$$

When do we have

. a unique solutions ?

- infinitely many ?

<u>Remark:</u> Say $\bar{x}_0$ satisfies $A\bar{x}_0 = \bar{b}$

take any $\bar{v}$ in $\mathrm{Nul}(A)$ $\quad A\bar{v} = \bar{0}$

$A(\bar{x}_0 + \bar{v}) = A\bar{x}_0 + A\bar{v} = \bar{b} + \bar{0}$

$= \bar{b}$

Thus all the solutions of $A\bar{x} = \bar{b}$

are of the form:

$$\{ \bar{x}_0 + \bar{v} \ : \ \bar{v} \text{ in } Null(A) \}$$

- unique sol'n $\iff Null(A) = 0$
  $\iff rank(A) = n$

- infinitely many sol's $\iff Null(A) \neq 0$
  $\iff rank(A) < n$

Linear independence

Given $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$ in $\mathbb{R}^m$

they are linearly independent

$\iff rank\{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n\} = n$

If $m = n$ $\iff det(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n) \neq 0$

Say A $n \times n$ matrix c in $\mathbb{R}$

$$det(cA) = c^n \, det(A)$$

$$\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} \qquad \begin{vmatrix} 4c & 2c \\ c & 3c \end{vmatrix} = c \begin{vmatrix} 4 & 2 \\ c & 3c \end{vmatrix}$$

$$= c^2 \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix}$$

$n \times n$ $c = -1$

$$det(-A) = (-1)^n det(A)$$

Say A is $n \times n$ invertible.

What is $\det(\mathrm{adj}(A)) = ?$

Answer:

$$A \cdot \mathrm{adj}(A) = \begin{pmatrix} \det(A) & & 0 \\ & \ddots & \\ 0 & & \det(A) \end{pmatrix} = \Big(\underset{c}{\det(A)} \underbrace{\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \\ & & \vdots \end{pmatrix}}_{I_n}\Big)$$

$$\det(AB) = \det(A)\det(B)$$

$$\det(A) \cdot \det(\mathrm{adj}(A)) = \det(c I_n)$$

$$= \det(A)^n \cdot 1$$

$$\det(\mathrm{adj}(A)) = \frac{\det(A)^n}{\det(A)} = \det(A)^{n-1}$$

— END —