PART II

4.2 Null spaces, Column spaces, Row spaces, and Linear transformations

Let A be a $m \times n$ matrix

m rows n columns

rows are vectors in $\mathbb{R}^n$

The columns of A are vectors in $\mathbb{R}^m$

Ex. $\underset{3 \times 4}{A} = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 2 & 5 & 3 \\ 1 & 3 & 2 & 10 \end{bmatrix}$ columns are vectors in $\mathbb{R}^3$

$\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad \vec{a}_4$

$$\begin{array}{cc} A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 2 & 5 & 3 \\ 1 & 3 & 2 & 10 \end{bmatrix} & \begin{array}{c} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{array} \end{array}$$

rows of A are vectors in $\mathbb{R}^4$

A $\boxed{m \times n}$ matrix

consider the homogeneous system: m equations, n unknowns

$$\boxed{\underset{m \times n}{A} \ \underset{n \times 1}{\vec{x}} = \underset{m \times 1}{\vec{0}}}$$

$\text{Nul}(A) = \{ \vec{x} \text{ in } \mathbb{R}^n : A\vec{x} = \vec{0}_m \}$ subspace of $\mathbb{R}^n$

$\text{Col}(A) = \{ c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n \mid c_1, \dots, c_n \text{ are in } \mathbb{R} \}$

column space of A is a subspace of $\mathbb{R}^m$.

$$\underbrace{x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{0}}_{A\vec{x} = 0}$$

- Thus $\vec{u}$ in $\mathbb{R}^n$ belongs to $\mathrm{Nul}(A) \iff$
  $A\vec{u} = 0$ ($\bar{u}$ is a sol'n of $A\bar{x} = \bar{0}$)

- A vector $\bar{b}$ in $\mathbb{R}^m$ belongs to $\mathrm{Col}(A) \iff$
  there exist $c_1, c_2, \dots, c_n$ in $\mathbb{R}$ such that
  $$\underline{c_1 \bar{a}_1 + c_2 \bar{a}_2 + \dots + c_n \bar{a}_n} = \bar{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\iff \underbrace{\begin{bmatrix} | & | & | & | & | \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\iff \vec{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is a sol'n of $A\bar{x} = \bar{b}$

$\iff A\bar{x} = \bar{b}$ has solution's (is consistent)

$\iff \mathrm{rank}(A) = \mathrm{rank}[A \,\vdots\, \bar{b}]$

Note: $\dim \mathrm{Col}(A) = \mathrm{rank}(A)$

Remark
$$2x_1 + 3x_2 = b_1 \qquad \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
$$-x_1 + 2x_2 = b_2$$
$$x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ lin. indep.

$\mathrm{Col}(A) = \mathbb{R}^2$
subspace of $\mathbb{R}^2$

$$\left[\underbrace{\begin{matrix} 2 & 3 \\ -1 & 2 \end{matrix}}_{\text{rank}=2} \;\vdots\; \begin{matrix} b_1 \\ b_2 \end{matrix}\right]$$

augmented matrix 2×3
its rank ≤ 2

Ex. $A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$ 4×5

2^*

Null(A) subspace of $\mathbb{R}^5$

Nul(A) ?

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 9 & \frac{19}{2} \\ 0 & 1 & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Col(A) subspace of $\mathbb{R}^4$

rank(A) = 3 = dim (Col(A))

dim (Nul(A)) = 5 − rank(A) = 5 − 3 = 2

Col(A) is spanned by the first 3-columns

Nul(A) is spanned by which vectors?

$x_4 = s$ $x_5 = t$

$$x_1 + 9x_4 + \frac{19}{2}x_5 = 0$$

$$x_1 = -9s - \frac{19}{2}t$$

$$x_2 = \frac{17}{4}s + \frac{5}{2}t$$

$$x_3 = -\frac{3}{2}s - 2t$$

$$x_4 = s$$

$$x_5 = t$$

basis of Null(A)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -9 \\ \frac{17}{4} \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{19}{2} \\ \frac{5}{2} \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$

Does the vector $\begin{bmatrix} 10 \\ 11 \\ 12 \\ 13 \end{bmatrix}$ belong to the column space of A

what about $\begin{bmatrix} 10 \\ 11 \\ 12 \\ 14 \end{bmatrix}$?

$$\left[\, A \;\middle|\; \begin{matrix} 10 \\ 11 \\ 12 \\ 13 \end{matrix} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 9 & \frac{19}{2} & 23 \\ 0 & 1 & 0 & -\frac{17}{2} & -\frac{5}{2} & -\frac{13}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 & \frac{11}{2} \\ \underbrace{0 \quad 0 \quad 0 \quad 0}_{A} & & & & 0 & 0 \end{array}\right]$$

$\underbrace{\text{rank}(A) = 3}$

$\text{rank}(A \mid \bar{b}) = 3$

Row(A)

Row space is spanned by $\bar{r}_1, \bar{r}_2, \dots, \bar{r}_m$

dim (Row space) = dim(Col(A))

Linear maps $\quad V, W$ vector space

$T : V \to W$ is linear if

$T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$

$T(c\bar{u}) = c\,T(\bar{u})$

FACT: If $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear then there is a unique $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$

Moreover the columns of A are

$$T\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right),\ T\left(\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}\right),\ \dots,\ T\left(\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right)$$

More examples: (of linear maps)

$V = C[0,1] = \{f : [0,1] \to \mathbb{R} \text{ continuous}\}$

$W = \mathbb{R}$

- $T_1 : V \to W \qquad T_1 : C[0,1] \to \mathbb{R}$

  $T_1(f) = f(0)$

- $T_2 : C[0,1] \to \mathbb{R}, \qquad T_2(f) = \int_0^1 f(t)\,dt$

Ex: $V = W = \mathbb{P}_3$ poly's deg ≤ 3

$T : \mathbb{P}_3 \to \mathbb{P}_3 \qquad T(p) = p''$ linear map

$T(\underbrace{a + bt + ct^2}_{p(t)}) = 2c$

$p' = b + 2ct, \quad p'' = 2c$

$(p+q)'' = p'' + q''$

$(cp)'' = c\,p''$

Kernel of T is a <u>subspace of V</u>

consisting of all vectors $\bar{u}$ such that $T(\vec{u}) = 0$.

Range of T is a <u>subspace of W</u>

consisting of all vectors $\bar{w}$ such that

$\bar{w} = T(\bar{u})$ for some $\bar{u}$ in V.

<u>Remark</u> If $T: \mathbb{R}^n \to \mathbb{R}^m$ linear

$$T(\bar{x}) = A\vec{x}$$

- Kernel $(T) = \mathrm{Nul}(A)$
- Range $(T) = \mathrm{Col}(A)$

<u>Ex:</u> $T: \mathbb{P}_3 \to \mathbb{P}_3$ $\quad T(p) = p''$

Kernel $(T) = ?$

Range $(T) = ?$

$T(p) = 0 \qquad p(t) = a + bt + ct^2$

<u>Kernel (T)</u>

$T(p) = p''(t) = 2c = 0$

$c = 0$

consists of those poly $\deg(P) \le 1$ $\quad p(t) = a + bt$

Range (T) = constant poly's that is of degree zero

is $p(t) = 1$ in the range of T? Yes

can we find $p(t)$ $\quad p'' = 1$

$$p(t) = \frac{1}{2}t^2$$

$$T\left(\frac{1}{2}t^2\right) = 1$$

END

— END —

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\, A \quad \mathbf{0} \,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\, A \quad \mathbf{v} \,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.