4.3 LINEARLY INDEPENDENT SETS; BASES

V vector space

$\{v_1, v_2, \dots, v_p\}$ subset of vectors of V

Def'n $v_1, v_2, \dots, v_p$ are linearly independent

if the vector equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_p \vec{v}_p = \vec{0}$$ where c_i are in $\mathbb{R}$

has only one solution $c_1 = c_2 = \dots = c_p = 0$

Ex 1 $V = \mathbb{R}^2$ $\underset{\vec{v}_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}1\\1\end{bmatrix}}$ are linearly independent.

$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$ $\quad$ $c_1 \begin{bmatrix}1\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}c_1 + c_2\\c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\quad$ $c_2 = 0$, $c_1 + c_2 = 0$, $c_1 = 0$

Ex 2 $\underset{v_1}{\begin{bmatrix}1\\1\end{bmatrix}}$ in $\mathbb{R}^2$ $\quad$ $c_1 \vec{v}_1 = \vec{0}$ $\quad$ $c_1 \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\quad$ $\begin{bmatrix}c_1\\c_1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\boxed{c_1 = 0}$

lin. indep.

Ex 3 $\vec{v}_1$ in V is lin. indep.

$\iff \vec{v}_1 \neq \vec{0}$.

Ex 4 $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ in $\mathbb{R}^3$ lin. indep.

$c_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $\quad$ $\left.\begin{matrix}c_1 + c_2 = 0\\ c_2 = 0\\ c_2 = 0\end{matrix}\right\}$

$\Downarrow$

$c_1 = 0 \quad c_2 = 0.$

Ex5 $\bar{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\bar{v}_2 = \begin{bmatrix}2\\2\\2\end{bmatrix}$ in $\mathbb{R}^3$

are linearly dependent

$$\bar{v}_2 = 2\bar{v}_1 \qquad \underbrace{2}_{c_1}\cdot\bar{v}_1 + \underbrace{(-1)}_{c_2}\bar{v}_2 = 0$$

Ex6 $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}$ ℓ. ind. or ℓ.. ~~dependent~~?

$$c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\1\\1\end{bmatrix} + c_3\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 0\\1 & 1 & 1\\1 & 1 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix}1 & 2 & 0\\0 & -1 & 1\\0 & -1 & -1\end{bmatrix} \to \begin{bmatrix}1 & 2 & 0\\0 & 1 & -1\\0 & 1 & 1\end{bmatrix} \to \begin{bmatrix}1 & 2 & 0\\0 & 1 & -1\\0 & 0 & -2\end{bmatrix}$$

rank $= 3$

$\det[\ \ ] \neq 0$

$\Rightarrow$ unique sol'n $c_1 = c_2 = c_3 = 0$.

Ex7 $\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}2\\1\\1\end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix}3\\2\\2\end{bmatrix}}_{v_3}$ Same question

linearly dependent

$$(1)v_1 + (1)v_2 + (-1)v_3 = 0$$ done

$$\text{rank}\begin{bmatrix}1 & 2 & 3\\1 & 1 & 2\\1 & 1 & 2\end{bmatrix} = 2$$

Ex8 $V = \mathbb{P}$ polynomials

$p_1(t) = 1 \qquad p_2(t) = t \qquad p_3(t) = 2 - 10t$

lin ~~ind~~ or lin. dep. ?

$$p_3 = 2p_1 - 10p_2$$

Ex 10 n vectors $\bar{a}_1, \dots, \bar{a}_n$ in $\mathbb{R}^n$

are linearly independent $\iff$ the matrix $A = [\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n]$ is invertible

Indeed

$c_1 \bar{a}_1 + \dots + c_p \bar{a}_p = \bar{0}$

$$A \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\iff \operatorname{rank}(A) = n$

$\iff \det(A) \neq 0.$

FACT: $v_1, \dots, v_p$ are linearly dependent

(suppose $v_1 \neq 0$) $\iff$ some $\bar{v}_j$ $(j > 1)$ is a linear combination of $\bar{v}_1, \dots, \bar{v}_{j-1}$.

$$\underline{\underline{v_j}} = \underline{\underline{c_1 \bar{v}_1 + \dots + c_{j-1} \bar{v}_{j-1}}}$$

$$c_1 \bar{v}_1 + \dots + c_{j-1} \bar{v}_{j-1} + (-1) \vec{v}_j = 0$$

$\boxed{\text{BASIS}}$ Let H be a subspace of V

A set of vectors $\mathcal{B}$ is a basis of H if

1. $\mathcal{B}$ is linearly independent
2. The subspace of V spanned by $\mathcal{B}$ is H : $\operatorname{span}(\mathcal{B}) = H$.

$\boxed{\text{Ex 11}}$ $H = V = \mathbb{R}^3$

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is this a basis of $\mathbb{R}^3$?

although they are linearly independent, they don't span $\mathbb{R}^3$.

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ not in span $\mathcal{B}$

$c_1 = 1$ $\underline{c_2 = 2}$ $\underline{c_2 = 3}$ $\lightning$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ basis.

independent rank $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} = 3$

Why is it that any other vector $\bar{b}$ in $\mathbb{R}^3$ is a linear combination of $\mathcal{B}$?

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \bar{b}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \bar{b}$$

consistent since A inv.

The spanning theorem

$S = \{\bar{v}_1, \dots, \bar{v}_p\}$ in V

$H = \text{span}\{\bar{v}_1, \dots, \bar{v}_p\}$

Then one can select a basis consisting of elements of S.

Ex: $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ subset of $\mathbb{R}^2$

$\text{Span}(S) = \mathbb{R}^2$

basis : $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

or $\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

Basis for <u>Col A</u>, <u>Row A</u>

$$A \xrightarrow{\text{RREF}} B$$

A basis of $\text{Col}(A)$ is given by the pivot columns of A

Row A subspace generated by the rows of A

Fact $A \underset{\text{row equiv to } B}{\sim} B$

then $\text{Row } A = \text{Row } B$

If B is a matrix in echelon form that is row equivalent to A then the nonzero rows of B form a basis of $\text{Row } A = \text{Row } B$

Ex

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & -4 & -3 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis of $\text{Row } A = \text{Row } B$ is given by

Ex:

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix}} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix} \sim \underset{B}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}} \Big\} \text{basis}$$

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