Appendix B

Complex numbers

$z = a + bi = a + ib$ where a, b in $\mathbb{R}$

$\boxed{i^2 = -1}$

$z = 4 - 3i$ $a = 4$, $b = -3$

$(a+bi) + (c+di) = (a+c) + (b+d)i$

$(a+bi)(c+di) = (ac - bd) + (ad + bc)i$

$bd(i^2) = -bd$

Conjugate $z = a + bi$ $\bar{z} = a - bi$

absolute value (modulus) $|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$

$|z|^2 = z\bar{z}$

$1 = z \cdot \dfrac{\bar{z}}{|z|^2} \leftarrow \dfrac{1}{z}$ $z \neq 0$

Properties:

1. $\bar{z} = z$ if and only if z is a real number.
2. $\overline{w + z} = \bar{w} + \bar{z}$.
3. $\overline{wz} = \bar{w}\,\bar{z}$; in particular, $\overline{rz} = r\bar{z}$ if r is a real number.
4. $z\bar{z} = |z|^2 \geq 0$.
5. $|wz| = |w||z|$.
6. $|w + z| \leq |w| + |z|$.

If $z \neq 0$, then $|z| > 0$ and z has a multiplicative inverse, denoted by $1/z$ or z^{-1} and given by

$$\frac{1}{z} = z^{-1} = \frac{\bar{z}}{|z|^2}$$

$$\frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-4i)} = \frac{3-4i}{3^2+4^2} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i$$

$$\frac{1+i}{3+4i} = (1+i)\cdot\left(\frac{3}{25} - \frac{4}{25}i\right) = \frac{3}{25} - \frac{4}{25}i^2 + \frac{3}{25}i - \frac{4}{25}i$$

$$= (1+i)\cdot\frac{1}{3+4i} = \frac{7}{25} - \frac{1}{25}i = \frac{7-i}{25}$$

$$(a+bi)(a-bi) = a^2 + (bi)(-bi) + \cancel{abi} - \cancel{abi}$$

$$= a^2 + b^2 \qquad z\cdot\bar{z} = |z|^2$$

Ex 1 Let $p(x) = 1 + 2x + 7x^2 + 100x^3 + x^4$

Suppose that $z = a + bi$ is a root a, b in $\mathbb{R}$

Show that $\bar{z} = a - bi$ is also a root

Given $p(z) = 0 \Rightarrow \overline{p(z)} = \bar{0} = 0$

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

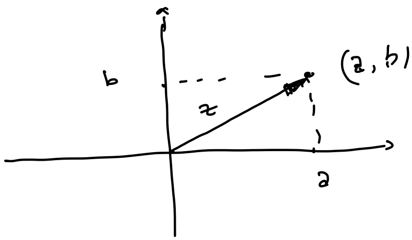
$$0 = \overline{p(z)} = \overline{1 + 2z + 7z^2 + 100z^3 + z^4}$$
$$= 1 + \overline{2z} + \overline{7z^2} + \overline{100z^3} + \overline{z^4}$$
$$= 1 + 2\bar{z} + 7(\bar{z})^2 + 100(\bar{z})^3 + (\bar{z})^4$$

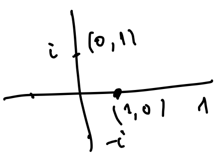
$\Rightarrow \bar{z} = a - bi$ is also a root.

Geometric interpretation of complex numbers

Idea identify $z = a + bi$ a, b in $\mathbb{R}$

with a vector (point) in the plane or $\mathbb{R}^2$



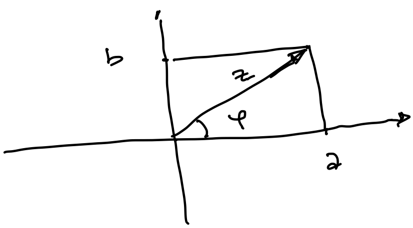


$1 = 1 + 0 \cdot i$

addition = addition of vectors

what about multiplication?

Can we visualize that geometrically?



$z = a + bi$

φ = argument of z

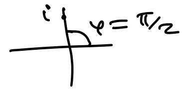
$0 \leq \varphi < 2\pi$

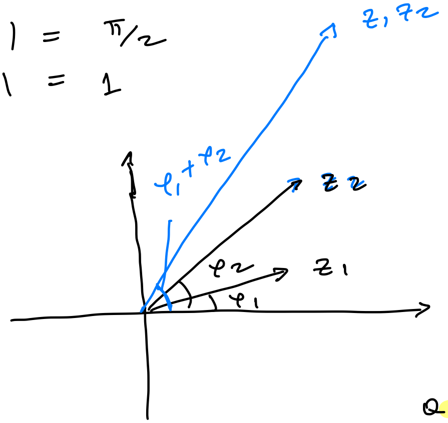
$(-\pi < \varphi \leq \pi)$

$|z| = r$ = the length of vector

$\arg(i) = \pi/2$

$|i| = 1$





$|z_1| = r_1$

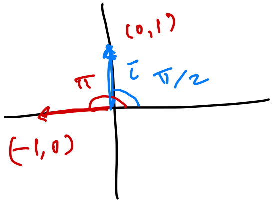
$|z_2| = r_2$

$|z_1 z_2| = r_1 r_2$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$= \varphi_1 + \varphi_2$

$i^2 = -1$?



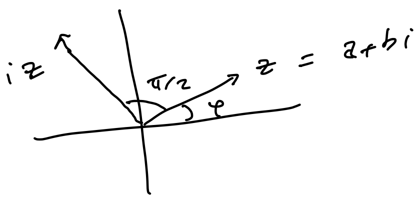
$|i^2| = |i|^2 = 1^2 = 1$

$\arg(i^2) = \arg i + \arg i = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$

Obs Multiplication by i corresponds to rotation (counterclockwise) by $\frac{\pi}{2}$

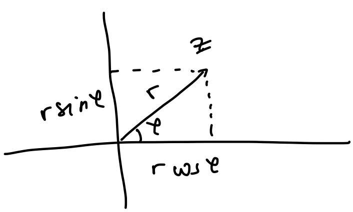
$|iz| = |i||z| = |z|$

$\arg(iz) = \arg i + \arg z = \frac{\pi}{2} + \arg z = \frac{\pi}{2} + \varphi$



$iz = i(a+bi)$

$= -b + ai$



$z = a+bi = r\cos\varphi + i r\sin\varphi$

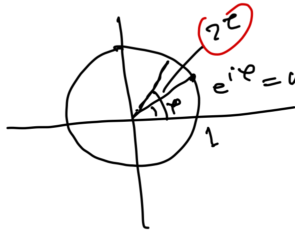
$= r(\cos\varphi + i\sin\varphi)$

Using series one proves that

$$\cos\varphi + i\sin\varphi = e^{i\varphi}$$

$$z = re^{i\varphi} = r(\cos\varphi + i\sin\varphi)$$

$$|\cos\varphi + i\sin\varphi| = \cos^2\varphi + \sin^2\varphi = 1$$



De Moivre's theorem

$$z^n = r^n(\cos\varphi + i\sin\varphi)^n$$

$$z^n = r^n(\cos(n\varphi) + i\sin(n\varphi))$$

$$z^n = (re^{i\varphi})^n = r^n e^{in\varphi}$$

$$(\cos\varphi + i\sin\varphi)^n = \cos(n\varphi) + i\sin(n\varphi)$$

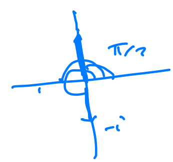
$$(\cos\varphi + i\sin\varphi)\cdot(\cos\varphi + i\sin\varphi) = \cos(2\varphi) + i\sin(2\varphi)$$

Ex 2 Solve $z^2 = 1$

$z^2 = -1$

$1 \quad -1$

$\underline{i} \quad \underline{-i}$



$$\frac{3\pi}{2} + \frac{3\pi}{2}$$

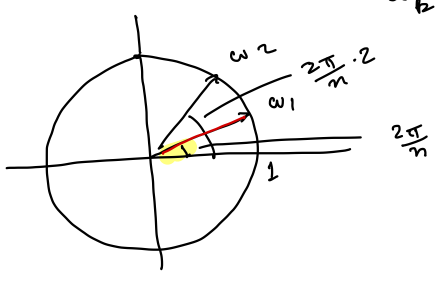
$$3\pi$$

Ex3 Solve $z^n = 1$ $\quad n \geq 2$

has n roots $\omega_1, \omega_2, \ldots, \omega_n$

$$\omega_k = \cos\left(\frac{2\pi}{n}k\right) + i\sin\left(\frac{2\pi}{n}k\right)$$

$k = 1, 2, \ldots, n$



$$\omega_1 = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$$

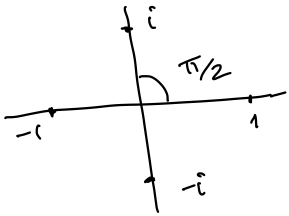
$$= e^{i\frac{2\pi}{n}}$$

$$\omega_1^n = \left(e^{i\frac{2\pi}{n}}\right)^n = e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1$$

Ex 4 $\qquad z^4 = 1 \qquad z^4 - 1 = 0$

$(z^2-1)(z^2+1)$

$\pm 1 \qquad \pm i$



$\dfrac{2\pi}{4} = \dfrac{\pi}{2} \qquad n = 4$

— END —

$z^4 = 16 = 2^4 \qquad \left(\dfrac{z}{2}\right)^4 = 1$

$k = 1, 2, 3, 4$

$\dfrac{z}{2} = \omega_k$

$w^4 = 1$

$\dfrac{z}{2} = w$