4.2. Null spaces, Column spaces, Row spaces, and Linear transformations

Let A be a $m \times n$ matrix

m rows n columns

The columns of A are vectors in $\mathbb{R}^m$

Ex $\underset{2\times 3}{A} = \begin{bmatrix} 1 & -1 & 10 \\ 2 & 4 & 11 \end{bmatrix}$ columns are vectors in $\mathbb{R}^2$

A $\boxed{m \times n}$ matrix

consider the homogeneous system: m equations, n unknowns

$$\boxed{\underset{m\times n}{A}\ \underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{0}}}$$

If $A = \begin{bmatrix} 1 & -1 & 10 \\ 2 & 4 & 11 \end{bmatrix}$ $\quad A\bar{x} = \bar{0}$ $\quad \underset{2\times 3}{\begin{bmatrix} 1 & -1 & 10 \\ 2 & 4 & 11 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{cases} x_1 - x_2 + 10x_3 = 0 \\ 2x_1 + 4x_2 + 11x_3 = 0 \end{cases}$$

The solution set of $\boxed{\underset{m\times n}{A}\bar{x} = \bar{0}}$ are vectors in $\mathbb{R}^n$

<u>Def'n</u>

$$\mathrm{Nul}(A) = \{ \vec{x} \text{ in } \mathbb{R}^n : A\vec{x} = \vec{0}_m \}$$

Note $\vec{0}_n$ = the zero vector in $\mathbb{R}^n$ belongs to $\mathrm{Nul}(A)$

Moreover:

Nul(A) is a subspace of $\mathbb{R}^n$

How to determine Nul(A) in concrete examples?

Def'n The column space of a matrix A $m \times n$
$A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n]$ where $\vec{a}_i$ are in $\mathbb{R}^m$
is $\mathrm{Col}(A)$ = the vector subspace of $\mathbb{R}^m$ generated (spanned) by the columns of A

$$\mathrm{Col}(A) = \{ c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_n \vec{a}_m : c_1, c_2, \dots, c_n \text{ in } \mathbb{R} \}$$

Fact $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$ = nr. of pivots in RREF of A

$$\dim(\mathrm{Col}(A)) + \dim(\mathrm{Nul}(A)) = n = \text{nr. of columns}$$

Ex 1 $\underset{1 \times 3}{A} = [0 \ 0 \ 1]$

$\mathrm{Nul}(A) = ?$

$\mathrm{Col}(A) = ?$

$\mathrm{Nul}(A)$ subspace of $\mathbb{R}^3$

$\mathrm{Col}(A)$ subspace of $\mathbb{R}^1$

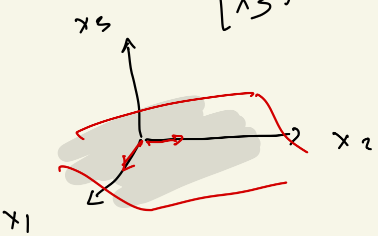
$$\mathrm{Nul}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : [0, 0, 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \right\}$$

$x_1 = s$
$x_2 = t$
$\begin{bmatrix} s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x_3 = 0$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_3 = 0 \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \text{ in } \mathbb{R} \right\}$$

the "$x_1 x_2$-plane"



$\dim(\mathrm{Nul}(A)) = 2$

$\mathrm{rank}(A) = 1$ $A = [0 \ 0 \ 1]$

$\mathrm{Col}(A) = \mathbb{R}$

$Col(A) = \{ c_1 \cdot 0 + \underbrace{c_2 \cdot 0 + c_3 \cdot 1}_{c_3} : c_1, c_2, c_3 \text{ are in } \mathbb{R} \}$ (3)

$= \mathbb{R}$

Ex 2 $\quad A = [1, 2, 3, -1]$ $\quad 1 \times 4$

$Nul(A)$ subspace of $\mathbb{R}^4$

$Col(A)$ subspace of $\mathbb{R}^1$ $= \mathbb{R}$

$\dim(Nul(A)) = 3$

$\text{rank}(A) + \dim(Nul(A)) = 4$

$A\bar{x} = \bar{0}$

$[1 \quad 2 \quad 3 \quad -1 \; \vdots \; 0] \leftarrow$ RREF

$x_1 + 2x_2 + 3x_3 - x_4 = 0$

$x_2 = s$
$x_3 = t$
$x_4 = r$

$$\begin{cases} x_1 = -2s - 3t + r \\ x_2 = s \\ x_3 = t \\ x_4 = r \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$Nul(A)$ is spanned by those 3 vectors
(form a basis)

Ex3

$$A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix} \quad 4\times 5$$

Null(A) subspace of $\mathbb{R}^5$

Nul(A) ?

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 9 & \frac{19}{2} \\ 0 & 1 & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Col(A) subspace of $\mathbb{R}^4$

$\text{rank}(A) = 3 = \dim(\text{Col}(A))$

$\dim(\text{Nul}(A)) = 5 - \text{rank}(A) = 5 - 3 = 2$

Col(A) is spanned by the first 3-columns

Nul(A) is spanned by which vectors ?

$x_4 = s \qquad x_5 = t$

$x_1 + 9x_4 + \frac{19}{2}x_5 = 0$

$x_1 = -9s - \frac{19}{2}t$

$x_2 = \frac{17}{4}s + \frac{5}{2}t$

$x_3 = -\frac{3}{2}s - 2t$

$x_4 = s$

$x_5 = t$

basis of Null(A)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -9 \\ \frac{17}{4} \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{19}{2} \\ \frac{5}{2} \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Questions

If I give us a vector $\bar{u}$ in $\mathbb{R}^5$

how do know if $\bar{u}$ is in $Nul(A)$?

Answer: compute $A\bar{u}$

if $A\bar{u} = 0$ then $\bar{u}$ in $Nul(A)$

What if we give us a vector $\bar{b}$ in $\mathbb{R}^4$?

How do we check if $\bar{b}$ is in $Col(A)$?

This happens $\iff$ $A\bar{x} = \bar{b}$ has solutions

$\iff$ $rank(A) = rank[A \,\vdots\, \bar{b}]$

Ex: $\begin{bmatrix} 10 \\ 11 \\ 12 \\ 13 \end{bmatrix}$ is in $col(A)$ with A as above

$$RREF \left[A \;\; \begin{matrix} 10 \\ 11 \\ 12 \\ 13 \end{matrix} \right] = \begin{bmatrix} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{has rank} = 3$$

— END —

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Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.