MA265 Linear Algebra — Exam 1

Date: March 3, Spring 2021 *Duration:* 60 min

Name: ____________________

PUID: ____________________

- All answers must be justified and you must show all your work in order to get credit.
- The exam is open book. Each students should work independently, Academic integrity is strictly observed.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. Find all the numbers x for which the vector $\begin{bmatrix} -3 \\ x \\ 13 \end{bmatrix}$ belongs to the subspace of $\mathbb{R}^3$ [10pt]

generated by the vectors $\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -4 \\ 9 \end{bmatrix}$.

$\vec{b}$

$\bar{u}$ $\quad$ $\bar{v}$

$\bar{b}$ belongs to $\operatorname{span}\{\bar{u}, \bar{v}\} \iff \operatorname{rank}[\bar{u}, \bar{v}, \bar{b}] = \operatorname{rank}[\bar{u}, \bar{v}]$

Since $\operatorname{rank}[\bar{u}, \bar{v}] = 2$ as $\bar{u}, \bar{v}$ are linearly independent

we must have $\operatorname{rank}[\bar{u}, \bar{v}, \bar{b}] = 2$

$$\operatorname{rank}\begin{bmatrix} 1 & -2 & -3 \\ 3 & -4 & x \\ -5 & 9 & 13 \end{bmatrix} = 3 \iff \begin{vmatrix} 1 & -2 & -3 \\ 3 & -4 & x \\ -5 & 9 & 13 \end{vmatrix} = 0$$

determinant $= x+5$

$x + 5 = 0 \Rightarrow \boxed{x = -5}$

answer

$$\begin{vmatrix} 1 & -2 & -3 \\ 3 & -4 & x \\ -5 & 9 & 13 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 2 & x+9 \\ 0 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 0 & x+5 \\ 0 & -1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & x+5 \end{vmatrix} = x + 5$$

2. Consider the matrix $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Let S be the subspace of $\mathbb{R}^3$ consisting of those vectors $\mathbf{x}$ such that $A^2\mathbf{x} = \mathbf{0}$. Find a basis of S. [10pt]

$$A^2 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{x_3 = 0}$$

$\text{rank}(A^2) = 1$

$\dim(S) = 2$

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \text{ in } \mathbb{R} \right\}$$

$$\text{basis} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad ; \quad \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3. If two matrices B and C have inverses $B^{-1} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$, and $A = BC$, compute A^{-1}. [10pt]

$$A^{-1} = (BC)^{-1} = C^{-1}B^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 4 \\ -2 & 2 \end{bmatrix}$$

not relevant

$$B^{-1}C^{-1} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & -8 \end{bmatrix}$$

4. Let $L : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation such that [10pt]

$$L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\2\end{bmatrix}, \quad L\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\\2\end{bmatrix}. \quad \text{Compute} \quad L\left(\begin{bmatrix}1\\2\end{bmatrix}\right).$$

$$\begin{bmatrix}1\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} \qquad L\begin{bmatrix}1\\2\end{bmatrix} = 2L\begin{bmatrix}1\\1\end{bmatrix} - L\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$= 2\begin{bmatrix}2\\3\\2\end{bmatrix} - \begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}4\\6\\4\end{bmatrix} - \begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}3\\5\\2\end{bmatrix}$$

Or $\quad \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} \qquad L\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\3\\2\end{bmatrix} - \begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix}$

$L\bar{x} = A\bar{x}$ The columns of A are $L\begin{bmatrix}1\\0\end{bmatrix}$ and $L\begin{bmatrix}0\\1\end{bmatrix}$

$$A = \begin{bmatrix}1 & 1\\1 & 2\\2 & 0\end{bmatrix}$$

$$L\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 2\\2 & 0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\5\\2\end{bmatrix}.$$

5. Let A be a 4×5 matrix. Which of the following statements are true and which are false? Indicate clearly your answers. For this question you do not need to include explanations for your answers. [10pt]

A. It is possible that the rank of A is 3.

B. It is possible that the null space of A has dimension 1.

C. It is possible that $A\mathbf{x} = \mathbf{0}$ has only one solution, the trivial solution.

D. It is possible that there exists two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^5$ such that $A\mathbf{u} = A\mathbf{v} = \mathbf{0}$.

E. It is possible that the columns of A are linearly independent.

A. TRUE $\longleftrightarrow$ Ex. $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

B. TRUE $\longleftrightarrow$ Ex $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ rank$(A) = 4$

C. FALSE $\longleftrightarrow$ rank$(A) \leq 4 \Rightarrow \dim(\mathrm{Nul}(A)) \geq 5 - 4 = 1$

D. TRUE $\longleftrightarrow$ Ex. If rank$(A) = 3$ then Nul(A) is 2-dimensional

$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\bar{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

E. FALSE

The columns of A are vectors in $\mathbb{R}^4$
There are at most 4 linearly independent vectors in $\mathbb{R}^4$ since $\dim(\mathbb{R}^4) = 4$.
Thus any 5 vectors in $\mathbb{R}^4$ are linearly dependent.

6. Let [10pt]

$$A = \begin{bmatrix} x^2 & 1 & 13 & 3 & 4 \\ 3 & 1 & x & 4 & 2 \\ 2 & 2 & 2 & 2 & x^3 \\ 1 & 3x^4 & 4 & 3 & 11 \\ 2 & 2 & 4 & 8 & 1 \end{bmatrix}.$$

The determinant of A is a polynomial in x. What is the coefficient of x^{10} in this polynomial? No use of computer software is allowed for computing this determinant. *Hint:* you don't need to compute all the terms of $det(A)$ in order to answer the question.

After interchanging columns (3-pairs)

$$\det(A) = (-1)^3 \begin{vmatrix} x^2 & 13 & 4 & 1 & 3 \\ 3 & x & 2 & 1 & 4 \\ 2 & 2 & x^3 & 2 & 2 \\ 1 & 4 & 11 & 3x^4 & 3 \\ 2 & 4 & 1 & 4 & 8 \end{vmatrix}$$

$$= (-1)\, x^2 \begin{vmatrix} x & 2 & 1 & 4 \\ 2 & x^3 & 2 & 2 \\ 4 & 11 & 3x^4 & 3 \\ 4 & 1 & 4 & 8 \end{vmatrix} + \text{terms of lower degree}$$

$$= (-1)(x^2)(x) \begin{vmatrix} x^3 & 2 & 2 \\ 11 & 3x^4 & 3 \\ 1 & 4 & 8 \end{vmatrix} + \text{terms of lower degree}$$

$$= (-1)(x^2)(x)(x^3) \begin{vmatrix} 3x^4 & 3 \\ 4 & 8 \end{vmatrix} + \text{terms of lower degree}$$

$$= \underbrace{(-1)(x^2)(x)(x^3)(24x^4)}_{-24x^{10}} + \text{terms of lower degree}$$

Answer: $\boxed{-24}$

7. Let A be an invertible 5×5-matrix such that $2A^2 = -A^T$. Compute $\det(A)$. [10pt]

$$\det(2A^2) = \det((-1)A^T)$$

$$2^5 \det(A)^2 = (-1)^5 \det(A)$$

$$32 \det(A)^2 = -\det(A)$$

A invertible $\Rightarrow$ $\det(A) \neq 0$

$$(32 \det(A) + 1) \det(A) = 0$$

$$\Rightarrow \quad 32 \det(A) + 1 = 0$$

$$\boxed{\det(A) = -\frac{1}{32}}$$

8. Consider the vectors [10pt]

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let A be a 3×4 matrix such that $A\mathbf{u} = \mathbf{0}$, $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{k}$. What are the possible values of the rank of A?

$\text{rank}(A) + \underbrace{\dim(\text{Nul}(A))} = 4$

$\dim(\text{Nul}(A)) \geq 2$ since $\bar{u}, \bar{v}$ in $\text{Nul}(A)$ are linearly independent.

Thus $\underline{\text{rank}(A)} = 4 - \dim(\text{Nul}(A)) \leq 4 - 2 = \underline{2}$

1 $\text{rank}(A) \leq 2$

Since $A\bar{w} = \bar{k} \neq 0 \Rightarrow \text{rank}(A) \geq 1$ 2

Thus 1 & 2 $\Rightarrow$ $\text{rank}(A) = 1$ or $\text{rank}(A) = 2$

<u>Both cases occur</u>

Ex $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ has rank $= 1$

$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ has rank $= 2$

These 2 matrices satisfy the given conditions.

9. Consider a linear system whose augmented matrix can be reduced via row operations to the echelon form [10pt]

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 1 & a & a-3 \\ 0 & 0 & a^2-9 & a-3 \end{array}\right]$$

(i) For which values of a will the system have no solution? $a=-3$

(ii) For which values of a will the system have a unique solution? $a \neq \pm 3$

(iii) For which values of a will the system have infinitely many solutions? $a=3$

- Unique solution if $\begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & a \\ 0 & 0 & a^2-9 \end{vmatrix} = a^2-9 \neq 0$

  thus $a \neq \pm 3$

- If $a=3$ $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ infinitely many solutions

  $x_1 - 2x_3 = 0$ $x_3 = s$

  $x_2 + 3x_3 = 0$

  $x_1 = 2s$

  $x_2 = -3s$ s in $\mathbb{R}$

  $x_3 = s$

- If $a=-3$ $\left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 0 & -6 \end{array}\right]$

  $0 \cdot x_3 = -6$

  Inconsistent

10. Find a matrix A with $\det(A) = 1$ whose adjugate is $\operatorname{adj}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. [10pt]

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{1} \operatorname{adj}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $A = (A^{-1})^{-1}$ is the inverse of $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$B \quad I_3 \quad\quad \underbrace{\quad}_{I_3} \quad \underbrace{\quad}_{B^{-1}}$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$