Functional Analysis, Spring 2020
Homework #11
This assignment is due on Wednesday, April 15

1. Define $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$T(x_0, x_1, ...) = (\alpha_0 x_0, \alpha_1 x_1, \alpha_2 x_2 ...).$$

where $(\alpha_n)_{n\geq 0}$ is a bounded sequence of nonzero complex numbers.

Show that T is compact if and only if $\lim_{n\to\infty} |\alpha_n| = 0$.

2) (a) Let $K : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous function. Show that for any $\varepsilon > 0$ there are functions $f_1, g_1, .., f_n, g_n \in C[0,1]$ such that $|K(x,y) - \sum_{i=1}^n f_i(x)g_i(y)| \leq \varepsilon$ for all $x, y \in [0,1]$.

(b) Denote $L(x,y) = \sum_{i=1}^n f_i(x)g_i(y)$. Show that the operator $T : L^2[0,1] \to L^2[0,1]$ defined by

$$(Th)(x) = \int_0^1 L(x,y)h(y)dy$$

is a finite rank rank operator.

(3) Let $K : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous function.
Define $S : L^2[0,1] \to L^2[0,1]$ by

$$(Sh)(x) = h(x) + \int_0^1 K(x,y)h(y)dy$$

Show that the spectrum of S is a countable set.

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Hints
(1) Consider the spectrum of T.

(2) (a) Use the Stone-Weierstrass Theorem.

(3) If $T \in L(X)$ and I is the identity map on X how is $\sigma(I+T)$ related to $\sigma(T)$?