Functional Analysis, Spring 2020
Homework # 2
due on Wednesday January 29 in class

1. Consider the space $C[0,1]$ of complex valued continuous functions on $[0,1]$ endowed with the with the norm $\|f\|_1 = \int_0^1 |f(t)|dt$. Show that the linear functional $\lambda : C[0,1] \to \mathbb{C}$, $\lambda(f) = f(1/2)$ is discontinuous.

2. Let T be a nonempty compact metric space and let $C(T)$ denote the linear space of all continuous functions from T to $\mathbb{C}$, endowed with the norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in T\}.$$

Let μ be a Borel probability measure on T *whose support* is equal to T. This means that $\mu(U) > 0$ for every nonempty open subset U of T. For each $f \in C(T)$ define $S : L^2(T,\mu) \to L^2(T,\mu)$ by $S\xi = f\xi$, $\forall$ $\xi \in L^2(T,\mu)$. Show that $\|S\| = \|f\|_\infty$.

(Recall that the norm on $L^2(T,\mu)$ is defined by $\|\xi\|_2 = \big(\int_T |\xi|^2 d\mu\big)^{1/2}$).

3. Consider the Banach spaces

$$\ell^1(\mathbb{N}) = \{\xi : \mathbb{N} \to \mathbb{C} \,:\, \|\xi\|_1 = \sum_{n=1}^{\infty} |\xi(n)| < \infty\},$$

$$\ell^2(\mathbb{N}) = \{\xi : \mathbb{N} \to \mathbb{C} \,:\, \|\xi\|_2 = \big(\sum_{n=1}^{\infty} |\xi(n)|^2\big)^{1/2} < \infty\}.$$

(a) Show that $\ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$.

(b) Show that the linear map $J : \ell^1(\mathbb{N}) \to \ell^2(\mathbb{N})$, defined by $J(\xi) = \xi$ is continuous.