Functional Analysis, Spring 2020

Homework #6

This assignment is due on Wednesday, February 26

1. Let H be a Hilbert space, let K be a closed subspace of H and let $T \in L(H)$. Let $P = P_K$ be the orthogonal projection of H onto K.

(a) Show that K is an invariant subspace for T i.e. $T(K) \subset K$ if and only if $TP = PTP$.

(b) Show that K is a reducing subspace for T i.e. $T(K) \subset K$ and $T(K^{\perp}) \subset K^{\perp}$ if and only if $TP = PT$.

2. Let H be a Hilbert space and let $x_1, y_1, ..., x_n, y_n \in H$. Let $T \in L(H)$ be defined by $Tx = \sum_{i=1}^{n} \langle x, x_i \rangle y_i$ for $x \in H$. Compute T^*.

3. Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by

$$S(x_0, x_1, ..., x_n, ...) = (0, x_0, x_1, ..., x_n, ...).$$

Find an explicit formula for S^*.

4. Define $T : L^2[0,1] \to L^2[0,1]$ by

$$(Tf)(x) = f(x) + \int_0^x y f(y) dy, \quad f \in L^2[0,1],\ x \in [0,1].$$

a) Show that T is a bounded operator.

b) Show that T is invertible in $L(H)$. (Hint: estimate $\|T - I\|$ where I is the identity operator on $L^2[0,1]$.)