COHOMOLOGICAL OBSTRUCTIONS TO GROUP STABILITY WITH RESPECT TO THE OPERATOR NORM

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In this article, we survey recent results concerning non-stability of discrete groups with respect to the operator norm. We focus on topological obstructions to perturbing almost representations of a discrete group Γ into unitary groups $U(n)$ to true representations. Several natural notions of stability are discussed: local-to-local stability, uniform-to-uniform stability, uniform-to-local stability, and C^*-stability.

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1. INTRODUCTION

Throughout the paper, Γ denotes a countable discrete group. We consider unital maps of Γ into unitary groups $U(n)$ that are almost representations, in the sense that they are approximately multiplicative. For such maps, it is natural to inquire whether they can be perturbed to true representations. One can assess the degree of multiplicativity and the closeness of potential perturbations both locally and globally (uniformly). We formalize these possibilities in the following sections.

There are several ways to construct almost representations of a discrete group Γ which are far from genuine representations. Ultimately, all of these constructions revolve around certain cohomological invariants of G. Voiculescu [43] and Kazhdan [33] constructed almost representations, implicitly or explicitly utilizing central extensions of Γ, which correspond to non-torsion classes in the second cohomology of Γ. Burger, Ozawa, and Thom in [4] constructed uniform almost representations that are not uniformly perturbable to true representations, based on nonzero elements in the kernel of the comparison map $H^2_b(\Gamma,\mathbb{R}) \to H^2(\Gamma,\mathbb{R})$ between bounded group cohomology and regular group

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cohomology. Connes, Gromov, and Moscovici utilized parallel transport in almost flat bundles to construct almost representations capturing K-theory invariants in [7]. These almost representations were then used to prove the Novikov conjecture for large classes of groups. One is naturally led to the question of constructing nontrivial almost flat bundles. Gromov described in [24], [25] geometric methods for constructing nontrivial almost flat K-theory.

We introduced in [8] and [9] a functional analytic method for constructing almost representations and almost flat bundles by exploiting the concept of quasidiagonal K-homology classes of group C^*-algebras and fundamental results of Kasparov [32], Yu [44], and Tu [42] on the Novikov conjecture and the Baum–Connes conjecture. The applicability of this technique, revisited in [13], was significantly broadened by Kubota in [36] through the consideration of quasidiagonal C^*-algebras that are intermediate between the full and the reduced group C^*-algebras. We discuss briefly this topic in the last section of this article.

In the following sections, we survey results on stability and non-stability involving almost representations constructed using the methods mentioned above. An important role is played by the approximate monodromy correspondence between almost flat bundles and almost representations introduced in [7]. This correspondence was further studied in [28], [27], [5], [29] and [35].

2. LOCAL-TO-LOCAL STABILITY

A countable discrete group Γ is *local-to-local* stable if for any sequence of unital maps $\{\rho_n : \Gamma \to U(n)\}$ such that

$$\lim_{n\to\infty} \|\rho_n(st) - \rho_n(s)\rho_n(t)\| = 0, \text{ for all } s, t \in \Gamma,$$

there is a sequence of unitary representations $\{\pi_n : \Gamma \to U(n)\}$ satisfying

$$\lim_{n\to\infty} \|\rho_n(s) - \pi_n(s)\| = 0, \text{ for all } s \in \Gamma.$$

Examples of local-to-local stable groups. Local-to-local stability is referred to as *matricial stability* in the paper by Eilers, Shulman and Sørensen [17], where it was systematically studied. Finite groups and finitely generated free groups $\mathbb{F}_n$, $n \geq 1$ are local-to-local stable. More generally, it was shown in [17] that finitely generated virtually free groups are also local-to-local stable. Among the crystallographic groups, only the line groups, $\mathbb{Z}$ and $\mathbb{Z}/2 * \mathbb{Z}/2$ and 12 wall-paper groups (out of 17), namely those with $H^2(\Gamma, \mathbb{Q}) = 0$, are local-to-local stable. It was shown recently by Gerasimova and Shchepin that the class of local-to-local stable groups is closed under amalgamated free products and HNN extensions over finite subgroups [20].

Examples of groups which are not local-to-local stable.

Two-cohomology obstructions. In his groundbreaking paper, Voiculescu showed that $\mathbb{Z}^2$ is not local-to-local stable, by employing a Fredholm index argument [43]. He proposed that this instability stems from the non-zero 2-cohomology of $\mathbb{T}^2$. Given that $B\mathbb{Z}^2$ equals $\mathbb{T}^2$, this essentially refers to the nonvanishing of $H^2(\mathbb{Z}^2, \mathbb{Q})$. Voiculescu's examples involve the sequence of pairs of unitaries

$$u_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad v_n = \begin{pmatrix} \lambda_n & 0 & 0 & 0 & 0 \\ 0 & \lambda_n^2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_n^3 & \cdot & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_n^n \end{pmatrix}, \qquad \lambda_n = e^{2\pi i/n}$$

which appear in the representation theory of the integral Heisenberg group

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{Z} \right\}.$$

The canonical central extension $0 \to \mathbb{Z} \to \mathbb{H}_3 \to \mathbb{Z}^2 \to 0$ represents a nonzero element of $H^2(\mathbb{Z}^2, \mathbb{Z})$. Using the notation $[s,t] = sts^{-1}t^{-1}$ for multiplicative commutators, one verifies that

$$[u_n, v_n] = \exp(-2\pi i/n) \cdot 1_n, \quad \|[u_n, v_n] - 1_n\| = |\exp(2\pi i/n) - 1| < 2\pi/n.$$

The winding number in $\mathbb{C} \setminus \{0\}$ of the loop $t \mapsto \det((1-t)1_n + t\,[u_n, v_n])$ is given by

$$\operatorname{wn}\det\big((1-t)1_n + t\,[u_n, v_n]\big)\big) = \frac{1}{2\pi i}\operatorname{Tr}\big(\log(\exp(-2\pi i/n)1_n)\big) = -1.$$

As noted in [33] and rediscovered by Exel and Loring, the nonvanishing of the winding number implies that the pair of unitaries u_n and v_n does not admit commuting approximants. Subsequently, Exel and Loring highlighted the significance of K-theory in understanding Voiculescu's example by proving a formula which asserts that two topological invariants associated to a pair of almost commuting unitary matrices coincide, see [18]. Specifically, there exist a universal constant $\varepsilon > 0$ and certain universal continuous functions $f, g, h : \mathbb{T} \to [0, 1]$ such that given any unitaries $u, v \in U(n)$ with $\|uv - vu\| < \varepsilon$, the selfadjoint matrix

$$e(u, v) = \begin{pmatrix} f(v) & g(v) + h(v)u^* \\ g(v) + uh(v) & 1 - f(v) \end{pmatrix}$$

satisfies $\|e(u,v)^2 - e(u,v)\| < 1/5$ and hence, the spectrum of $e(u,v)$ is contained in $[0,1/3) \cup (2/3,1]$. It follows by functional calculus that

$$e(u,v)_\sharp = \chi_{(\frac{2}{3},1]}\big(e(u,v)\big)$$

is a selfadjoint projection such that $\|e(u,v)_\sharp - e(u,v)\| < 1/3$.

The functions f, g, and h are chosen such that

$$e\big(e^{2\pi ix}, e^{2\pi iy}\big) \in M_2\big(C(\mathbb{T}^2)\big)$$

is a projection representing a rank-one vector bundle over the 2-torus with first Chern class equal to one. The Bott element associated with u, v is the K_0-class

$$\mathrm{Bott}(u,v) = e\,(u,v)_\sharp - n \in K_0(\mathbb{C}) = \mathbb{Z}$$

Exel and Loring proved that

$$\mathrm{Bott}(u,v) = \frac{1}{2\pi i}\mathrm{Tr}\log\big([u,v]\big). \tag{1}$$

Eilers, Shulman and Sørensen [17] showed that certain concrete groups with homogeneous relations are not local-to-local stable by using winding number invariants of Kazhdan/Exel–Loring type. These invariants are connected to the two-homology of the groups as we are going to explain briefly below. Many other examples are implicit in the papers [9], [13] and [15].

In [11], we extended the Exel–Loring formula (1) from $\mathbb{Z}^2$ to arbitrary countable discrete groups Γ as follows. Hopf's formula expresses the second homology of Γ as

$$H_2(\Gamma,\mathbb{Z}) = \frac{R \cap [F,F]}{[R,F]}$$

in terms of a free presentation

$$1 \to R \to F \xrightarrow{q} \Gamma \to 1,$$

where $q(a) = \bar{a}$. Consequently, each element $x \in H_2(\Gamma,\mathbb{Z})$ can be represented by a product of commutators $\prod_{i=1}^g [a_i, b_i]$ with $a_i, b_i \in F$, for some integer $g \geq 1$, such that $\prod_{i=1}^g [\bar{a}_i, \bar{b}_i] = 1$.

Let $\beta^\Gamma : H_2(\Gamma,\mathbb{Z}) \cong H_2(B\Gamma,\mathbb{Z}) \to RK_0(B\Gamma)$ be the (rationally injective) homomorphism studied in [2], [39] and let $\alpha^\Gamma : H_2(\Gamma,\mathbb{Z}) \to K_0(\ell^1(\Gamma))$ be the composition $\alpha^\Gamma = \mu_1^\Gamma \circ \beta^\Gamma$ where μ_1^Γ is the ℓ^1-version of the assembly map of Lafforgue [37].

We showed in [11] that the linear extension $\rho : \ell^1(\Gamma) \to M_n(\mathbb{C})$ of a sufficiently multiplicative unital map $\rho : \Gamma \to U(n)$ satisfies the following.

THEOREM 2.1. *Let $x \in H_2(\Gamma,\mathbb{Z})$ be represented by a product of commutators $\prod_{i=1}^g [a_i, b_i]$ with $a_i, b_i \in F$ and $\prod_{i=1}^g [\bar{a}_i, \bar{b}_i] = 1$. There exist a finite set $S \subset \Gamma$ and $\varepsilon > 0$ such that if $\rho : \Gamma \to U(n)$ is unital map with*

$\|\rho(st) - \rho(s)\rho(t)\| < \varepsilon$ *for all* $s, t \in S$, *then*

$$\rho_\sharp(\alpha^\Gamma(x)) = \mathrm{wn}\det\left((1-t)1_n + t\prod_{i=1}^{g}[\rho(\bar{a}_i), \rho(\bar{b}_i)]\right) = \frac{1}{2\pi i}\mathrm{Tr}\log\left(\prod_{i=1}^{g}[\rho(\bar{a}_i), \rho(\bar{b}_i)]\right).$$

Here, if we write $\alpha^\Gamma(x) = [p_0] - [p_1]$, where p_i are projections in matrices over $\ell^1(\Gamma)$, then $\rho_\sharp(\alpha^\Gamma(x)) = \rho_\sharp(p_0) - \rho_\sharp(p_1)$, where $\rho_\sharp(p_i) \in \mathbb{Z}$ is the K-theory class (i.e., the rank) of the perturbation of $(\mathrm{id} \otimes \rho)(p_i)$ to a projection via analytic functional calculus. Moreover, we show in [11] that if Γ is a quasidiagonal group which admits a γ-element and $x \in H_2(\Gamma, \mathbb{Z})$ is of infinite order, then there exist unitary almost representations $\rho_n : \Gamma \to U(n)$ for which the winding number in the formula above is nonzero and hence, ρ_n are not perturbable to genuine representations. The proof of Theorem 2.1 relies crucially on the corresponding result for surface groups of genus ≥ 1 from [8], where we noted that the Exel–Loring formula is related to an index theorem of Connes, Gromov and Moscovici [7] and to its extensions studied thereby. The joint work with Carrión [5] led to a generalization of the result above for almost representations into unitary groups of tracial C^*-algebras, see [11].

Ioana, Spaas and Wiersma [30] used nonvanishing of 2-cohomology groups in conjunction with the relative Property (T) to exhibit obstructions to completely positive lifting for full group C^*-algebras. In particular, it follows from their arguments that if $\Gamma = \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ then there is a unital $*$-homomorphism $C^*(\Gamma) \to \prod_n M_n / \bigoplus_n M_n$ which does not even have a unital completely positive lifting and hence, Γ is far from being matricially stable. However, it should be noted that the absence of a unital completely positive lifting is only one aspect of nonstability. Indeed, by [10, Theorem 1.2] there are unital $*$-homomorphisms $C^*(\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})) \to \prod_n M_n / \bigoplus_n M_n$ which admit unital completely positive liftings but do not lift to $*$-homomorphisms.

Higher-dimensional cohomology obstructions. After observing obstructions to stability originating in two-cosmology, it is natural to inquire about the role of higher-dimensional cohomology. We showed in [10] that rational cohomology in higher even dimensions also obstructs group stability for large classes of groups.

THEOREM 2.2. *Let* Γ *be a countable discrete MF-group that admits a* γ*-element. If* $H^{2k}(\Gamma, \mathbb{Q}) \neq 0$ *for some* $k \geq 1$*, then* Γ *is not local-to-local stable.*

The theorem is proved by using the dual assembly map of Kasparov to map quasidiagonal K-homology classes of $C^*(\Gamma)$ to almost flat K-theory classes of the classifying space $B\Gamma$. For local-to-local stable groups, these classes must be flat and hence, their Chern classes vanish over $\mathbb{Q}$ by a classic result of Milnor, see [40].

Let us recall that a group Γ is MF if it embeds in $\mathbf{U}/\mathbf{N}$ where we have that $\mathbf{U} = \prod_{n=1}^{\infty} U(n)$ and $\mathbf{N} = \{(u_n)_n \in \mathbf{U} : \|u_n - 1_n\| \to 0\}$. Put simply, a group is MF if it has enough approximate unitary representations that allow to effectively separate its elements. The groups that are locally embeddable in amenable groups are MF as a consequence of [41]. Linear groups are MF as noted in [10]. There are no known examples of discrete, countable groups that are not MF.

The class of groups possessing a γ-element encompasses a wide range, as shown in [31], including those that can be uniformly embedded into a Hilbert space as proved in [42]. Amenable groups, or more generally, the groups with Haagerup's property are uniformly embeddable in a Hilbert space [6] and so are the linear groups [26]. Hilbert space uniform embeddability passes to subgroups and products, direct limits, free products with amalgam, and extensions by exact groups [14].

Using Theorem 2.2, we showed in [10] that the following groups are not local-to-local stable.

(a) Γ a finitely generated torsion free, nilpotent group $\neq 0, \mathbb{Z}$.

(b) Γ an amenable group with $H^{2k}(\Gamma, \mathbb{Q}) \neq 0$ for some $k > 0$.

(c) Γ a linear group with $H^{2k}(\Gamma, \mathbb{Q}) \neq 0$ for some $k > 0$.

(d) Γ a hyperbolic, residually finite group with $H^{2k}(\Gamma, \mathbb{Q}) \neq 0$ for some $k > 0$.

(e) $\mathrm{Aut}(\mathbb{F}_n)$ and $\mathrm{Out}(\mathbb{F}_n)$, $n = 4, 6, 8$.

(f) Mapping class groups $Mod(S_g)$ for $g \geq 3$.

Bader, Lubotzky, Sauer and Weinberger used Theorem 2.2 to demonstrate that lattices in semisimple real Lie groups are typically not local-to-local stable.

THEOREM 2.3 ([1]). *If Γ is a cocompact lattice in a real semisimple Lie group G which is not locally isomorphic to either $SO(n,1)$ for n odd or $SL_3(\mathbb{R})$, then $H^{2k}(\Gamma, \mathbb{Q}) \neq 0$ for some $k \geq 1$ and hence, Γ is not local-to-local stable.*

The important question of constructing concrete approximate representations corresponding to specific cohomology classes was addressed by Glebe in [21]. For a class of groups Γ that includes all finitely generated nilpotent groups, Glebe developed a method and an algorithm that, for any given element of $H^2(\Gamma, \mathbb{Z})$, generates an approximate representation of Γ that cannot be perturbed to a true representation. This conceptualizes the role of Voiculescu's unitaries and extends their construction from $\mathbb{Z}^2$ to nilpotent groups. These methods were refined in [22] to control the growth of the dimension of the almost representations leading to results on non-stability with respect to the Frobenius norm and other unnormalized p-norms.

3. UNIFORM-TO-UNIFORM STABILITY

A group Γ is *uniform-to-uniform* stable if for any sequence of unital maps $\{\rho_n : \Gamma \to U(n)\}$ such that

$$\lim_{n\to\infty}\left(\sup_{s,t\in\Gamma}\|\rho_n(st)-\rho_n(s)\rho_n(t)\|\right)=0$$

there is a sequence of representations $\{\pi_n : \Gamma \to U(n)\}$ satisfying

$$\lim_{n\to\infty}\left(\sup_{s\in\Gamma}\|\rho_n(s)-\pi_n(s)\|\right)=0.$$

Uniform stability is called *Ulam stability* in [4].

Examples of uniform-to-uniform stable groups. Kazhdan proved in [33] that all amenable groups are uniform-to-uniform stable. Burger, Ozawa and Thom proved in [4] that if $\mathcal{O}$ is the ring of integers of a number field, $S \subset \mathcal{O}$ is a multiplicatively closed subset, and $\mathcal{O}_S$ is the localized ring, then the group $\mathrm{SL}_n(\mathcal{O}_S)$ is uniform-to-uniform stable, for $n \geq 3$. In particular, the groups $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, have this property. In a recent paper, Glebsky, Lubotzky, Monod and Rangarajan [23] developed a new cohomology theory, termed asymptotic cohomology, that captures the obstruction to uniform-to-uniform stability. In particular, they extended the stability results of [4] to numerous lattices in higher-rank semisimple Lie groups.

Examples of groups which are not uniform-to-uniform stable. Kazhdan proved in [33] that the surface groups Γ_g of genus $g > 1$ are not uniform-to-uniform stable. More generally, it was shown in [4] that if the comparison map $J : H_b^2(\Gamma,\mathbb{R}) \to H^2(\Gamma,\mathbb{R})$ is not injective, then Γ is not uniformly stable. Consequently, the non-elementary hyperbolic groups are not uniformly stable, since J is not injective for such groups [19]. We write here a sketch of the argument from [4]. Let $QH_h(\Gamma,\mathbb{R})$ denote the space of homogeneous quasimorphisms $\phi : \Gamma \to \mathbb{R}$. These are maps such that the map $(s,t) \mapsto \phi(st)-\phi(s)-\phi(t)$ is bounded on $\Gamma\times\Gamma$ and $\phi(s^k) = k\phi(s)$. One verifies that $\ker J \cong QH_h(\Gamma,\mathbb{R})/\mathrm{Hom}(\Gamma,\mathbb{R})$. If $\phi \in QH_h(\Gamma,\mathbb{R})$ is not a homomorphism, then the sequence of maps $\{\rho_n : \Gamma \to U(1)\}$, defined by

$$\rho_n(s) = e^{\frac{2\pi i}{n}\phi(s)}$$

is not uniformly perturbable to a sequence of homomorphisms since for any $n \in \mathbb{N}$ and any homomorphism $\pi : \Gamma \to U(1)$ one has $\sup_{s\in\Gamma} \|\rho_n(s) - \pi(s)\| \geq \sqrt{3}$.

4. UNIFORM-TO-LOCAL STABILITY

A countable discrete group Γ is *uniform-to-local* stable if for any sequence of unital maps $\{\rho_n : \Gamma \to U(n)\}$ such that

$$\lim_{n\to\infty} \left(\sup_{s,t\in\Gamma} \|\rho_n(st) - \rho_n(s)\rho_n(t)\| \right) = 0$$

there is a sequence of representations $\{\pi_n : \Gamma \to U(n)\}$ satisfying

$$\lim_{n\to\infty} \|\rho_n(s) - \pi_n(s)\| = 0, \text{ for all } s \in \Gamma.$$

It is clear that the class of uniform-to-local stable groups includes all the local-to-local stable groups and all the uniform-to-uniform stable groups

The notion of uniform-to-local stability was introduced in [12]. This was inspired by the observation that Kazhdan proves more that what he states in Theorem 2 of [33], namely he shows that the surface groups Γ_g of genus $g > 1$ are not only not uniform-to-uniform stable; in fact, they are not even uniform-to-local stable. Uniform-to-uniform stability is a strictly stronger condition than uniform-to-local stability. Indeed, if Γ has a finite index subgroup isomorphic to a free group $\mathbb{F}_k$, then Γ is local-to-local stable [17], [1] and hence uniform-to-local stable, but Γ is not uniform-to-uniform stable if $k \geq 2$, due to the existence of nontrivial homogeneous quasimorphisms $\phi : \mathbb{F}_k \to \mathbb{R}$, [4].

By developing an idea of Gromov [25, p. 166], we prove that many of the cocompact lattices in the Lorentz group $SO_0(n, 1)$, $n > 1$ are not uniform-to-local stable. More generally, we have the following theorem. Let $b_i(M) = \dim_{\mathbb{R}} H^{2i}(M, \mathbb{R})$ denote the Betti numbers of a manifold M.

THEOREM 4.1. *Let M be a closed connected Riemannian manifold with strictly negative sectional curvature and residually finite fundamental group. If $b_{2i}(M) > 0$ for some $i > 0$, then $\pi_1(M)$ is not uniform-to-local stable.*

Concerning the assumption on Betti numbers, observe that if M is orientable and $\dim M = 2m$ then $b_{2m}(M) = 1$, while if M is orientable and $\dim M = 2m+1$, then it suffices to require that $b_i(M) > 0$ for some $1 \leq i \leq 2m$, since $b_i(M) = b_{2m+1-i}(M)$ by Poincaré duality and either i or $2m+1-i$ must be even.

COROLLARY 4.2. *Let Γ be a torsion free cocompact lattice in $SO_0(n, 1)$.*
(i) *If n is even then Γ is not uniform-to-local stable.*

(ii) *If n is odd and $b_i(\Gamma) > 0$ for some $i > 0$ then Γ is not uniform-to-local stable.*

The corollary reproves Kazhdan's result since $\Gamma_g \subset SO_0(2,1)$, $g > 1$. For other concrete examples, consider

$$G = SO_0\big(x_1^2 + \cdots + x_n^2 - \sqrt{p}\, x_{n+1}^2, \mathbb{R}\big) \cong SO_0(n,1),$$

where p is a square free integer. Let $\mathcal{O}$ be ring of integers of $\mathbb{Q}\sqrt{p}$. Then $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\sqrt{p}$ if $p \not\equiv 1 \pmod 4$ and $\mathcal{O} = \{\frac{a+b\sqrt{p}}{2} : a, b \in \mathbb{Z},\ a - b \equiv 0 \pmod 2\}$ if $p \equiv 1 \pmod 4$ and $G_{\mathcal{O}}$ is a cocompact arithmetic lattice in G, [3]. By a result of Li and Millson [38], any arithmetic lattice in $SO_0(n,1)$, $n \neq 3, 7$ contains a congruence subgroup Γ such that $b_1(\Gamma) > 0$.

5. C^*-STABILITY

A countable group Γ is *C^*-stable* if for any sequence $\{\rho_n : \Gamma \to U(B_n)\}_n$ of unital maps, where B_n are unital C^*-algebras, such that

$$\lim_{n\to\infty} \|\rho_n(st) - \rho_n(s)\rho_n(t)\| = 0, \quad \forall\, s, t \in \Gamma$$

there exist homomorphisms $\{\pi_n : \Gamma \to U(B_n)\}_n$ satisfying

$$\lim_{n\to\infty} \|\rho_n(s) - \pi_n(s)\| = 0, \quad \forall\, s \in \Gamma.$$

If in the definition above we consider only C^*-algebras B_n in a class $\mathcal{B}$, we say that the group Γ is C^*-stable with respect to the class $\mathcal{B}$. We note that local-to-local stability corresponds to C^*-stability relative to finite dimensional C^*-algebras. Eilers, Shulman and Sørensen showed the following.

Theorem 5.1 ([17]). *All finitely generated virtually free groups are C^*-stable.*

Prompted by a question of Shlyakhtenko concerning the possible role of the odd-dimensional cohomology groups in group stability, we showed in [12] the following.

Theorem 5.2. *Let Γ be a countable discrete MF-group that admits a γ-element. If $H^k(\Gamma, \mathbb{Q}) \neq 0$ for some $k \geq 1$, then Γ is not C^*-stable with respect to the class $\{M_n(C(\mathbb{T})) : n \geq 1\}$.*

Theorem 5.2 may be viewed as a weak topological converse of Theorem 5.1: a C^*-stable group must have the rational cohomology of a (virtually) free group.

6. QUASIDIAGONAL GROUPS AND ALMOST REPRESENTATIONS

As mentioned in the Introduction, for a discrete group Γ there are several known methods for producing interesting almost representations with respect to the operator norm. We introduced in [8], [9] a method for constructing almost representations from quasidiagonal K-homology classes in $K^0(C^*(\Gamma)) = KK(C^*(\Gamma), \mathbb{C})$. This relies on the concept of a quasidiagonal group.

A representation $\pi : \Gamma \to U(H)$ is quasidiagonal if there is an increasing sequence $(p_n)_n$ of finite dimensional projections which converges strongly to 1_H and such that $\lim_{n\to\infty} \|[\pi(s), p_n]\| = 0$ for all $s \in \Gamma$. A group Γ is *quasidiagonal* if it admits a *faithful* quasidiagonal unitary representation on a separable Hilbert space. Equivalently, Γ embeds in the unitary group of a unital and quasidiagonal C^*-algebra. Quasidiagonality of Γ is weaker than quasidiagonality of $C^*(\Gamma)$. Indeed, the following assertions are equivalent, see [10]:

(i) Γ is quasidiagonal.

(ii) The left regular representation λ_Γ is weakly contained in some quasidiagonal representation π of Γ.

(iii) The canonical map $C^*(\Gamma) \to C^*_r(\Gamma)$ factors through a unital quasidiagonal C^*-algebra.

Let A and B separable C^*-algebras and assume that B is unital. An element $x \in KK(A, B)$ is quasidiagonal if it is represented by a Cuntz pair $\varphi, \psi : A \to M(K(H) \otimes B)$, $\varphi(a) - \psi(a) \in K(H) \otimes B$, for all $a \in A$, with the property that there exists an approximate unit of projections $(p_n)_n$ of $K(H)$ such that $\lim_{n\to\infty} \|[\psi(a), p_n \otimes 1_B]\| = 0$, for all $a \in A$. The quasidiagonal elements form a subgroup of $KK(A, B)$, denoted by $KK(A, B)_{qd}$. Let $\mathcal{Q} = \bigotimes_{n\in\mathbb{N}} M_n(\mathbb{C})$ denote the universal UHF-algebra. By combining techniques from [9] with Kubota's idea of considering intermediate group C^*-algebras [34], we showed in [10] that the dual assembly map $\nu : KK(C^*(\Gamma), B)) \to RK^0(B\Gamma; B)$ of Kasparov has certain properties for quasidiagonal groups that are crucial in our results on group instability.

THEOREM 6.1. *Let Γ be a countable discrete quasidiagonal group and let B be a separable nuclear unital C^*-algebra. If Γ admits a γ-element, then $\gamma KK(C^*(\Gamma), B)) \subset KK(C^*(\Gamma), B)_{qd}$. It follows that $\nu(KK(C^*(\Gamma), \mathcal{Q})_{qd}) = RK^0(B\Gamma; \mathcal{Q})$. If we also assume that Γ is torsion free, then $\nu(K^0(C^*(\Gamma))_{qd}) = RK^0(B\Gamma)$.*

Remark 6.2. (i) All finitely generated linear groups are quasidiagonal.

(ii) All maximally periodic groups are quasidiagonal. All residually amenable groups are quasidiagonal as a consequence of [41].

(iii) If Γ_1 is quasidiagonal and Γ_2 is amenable, then the wreath product $\Gamma_1 \wr \Gamma_2$ is quasidiagonal, [16].

(iv) The class of quasidiagonal groups is strictly larger than the class of residually amenable groups. Indeed, we showed in [10] that the orthogonal group $\mathrm{O}_n(\mathbb{Q})$, $n \geq 5$ is quasidiagonal but not residually amenable. We do not have examples of quasidiagonal groups which are not locally embeddable in amenable groups.

(v) Infinite simple property (T) groups are not quasidiagonal. More generally, Ozawa and Thom showed that an infinite property (T) quasidiagonal group must have an infinite residually finite quotient.

Theorem 6.1 gives a powerful functional analytic method for constructing almost flat K-theory classes which is complementary to the geometric methods of Gromov [24], [25]. If we assume in addition that Γ is local-to-local stable, then the image of $KK(C^*(\Gamma), \mathcal{Q})_{qd}$ under the dual assembly map restricted to compact subspaces Y of $B\Gamma$ consists entirely of flat K-theory classes.

$$\nu_Y(KK(C^*(\Gamma), \mathcal{Q})_{\mathrm{qd}}) \subset RK^0(Y; \mathcal{Q})_{\mathrm{flat}} = RK^0(Y; \mathbb{Q})_{\mathrm{flat}}.$$

Since the Chern classes of flat complex bundles vanish rationally, one deduces that the cohomology groups $H^{2k}(B\Gamma, \mathbb{Q}) \cong H^{2k}(\Gamma, \mathbb{Q})$ must vanish for $k > 0$.

We conclude this section by sketching the construction of almost representations of Γ which are not near true representations starting from quasidiagonal K-homology classes in $K^0(C^*(\Gamma)) = KK^*(C^*(\Gamma), \mathbb{C})$. We assume that Γ has a γ-element (this is the case if, for example, Γ is boundary amenable (exact) or if Γ is linear or if Γ is hyperbolic.) For simplicity, also assume that Γ is torsion free. Under these assumptions, Kasparov has shown that the dual assembly map $\nu : KK(C^*(\Gamma), \mathbb{C}) \to K^0(B\Gamma)$ is surjective with kernel $(1-\gamma)KK(C^*(\Gamma), \mathbb{C})$. In particular, the restriction of ν to $\gamma KK(C^*(\Gamma), \mathbb{C})$ is surjective. If we assume in addition that Γ is quasidiagonal (for example, residually finite), then $\gamma KK(C^*(\Gamma), \mathbb{C}) \subset KK(C^*(\Gamma), \mathbb{C})_{qd}$ by Theorem 6.1 and hence, the restriction map $\nu : KK(C^*(\Gamma), \mathbb{C})_{qd} \to RK^0(B\Gamma)$ is surjective.

Every class $x \in KK(C^*(\Gamma), \mathbb{C})_{qd}$ is represented by a pair of $*$-representations $\varphi, \varphi' : C^*(\Gamma) \to M(K(H)) = B(H)$, such that $\varphi(a) - \varphi'(a) \in K(H)$, for all $a \in C^*(\Gamma)$ and with property that there is an increasing approximate unit $(p_n)_n$ of $K(H)$ consisting of projections such that $(p_n)_n$ commutes asymptotically with both $\varphi(a)$ and $\varphi'(a)$, for all $a \in C^*(\Gamma)$.

It follows that the compressions $\varphi_n(a) = p_n\varphi(a)p_n$ and $\varphi'_n(a) = p_n\varphi'(a)p_n$ are completely positive asymptotic homomorphisms $\varphi_n, \varphi'_n : C^*(\Gamma) \to K(H)$.

$$\lim_{n\to\infty}\big(\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| + \|\varphi'_n(ab) - \varphi'_n(a)\varphi'_n(b)\|\big) = 0,\ \forall a, b \in C^*(\Gamma).$$

It is routine to perturb these maps to unital completely positive asymptotic homomorphisms $\varphi_n : C^*(\Gamma) \to M_{k(n)}(\mathbb{C})$ and $\varphi'_n : C^*(\Gamma) \to M_{k'(n)}(\mathbb{C})$. Their restrictions to Γ are sequences of almost representations $\varphi_n : \Gamma \to GL_{k(n)}(\mathbb{C})$ and $\varphi'_n : \Gamma \to GL_{k'(n)}(\mathbb{C})$, which we can further perturb to sequences of almost representations (denoted by the same symbols) $\varphi_n : \Gamma \to U(k(n))$ and $\varphi'_n : \Gamma \to U(k'(n))$, since $\varphi_n(s)$ and $\varphi'_n(s)$ are almost unitaries. We can now use the sequences $(\varphi_n)_n$ and $(\varphi'_n)_n$ that we associated to the class $x \in KK(C^*(\Gamma), \mathbb{C})_{qd}$ to calculate $\nu(x) \in RK^0(B\Gamma)$ as we describe in the sequel.

Mishchenko's line-bundle ℓ_Γ is the canonical flat $C^*(\Gamma)$-bundle over the classifying space of Γ, $E\Gamma \times_\Gamma C^*(\Gamma) \to B\Gamma$, defined by the diagonal action $\Gamma \subset C^*(\Gamma)$. If $B\Gamma$ is compact, then the bundle ℓ_Γ can be equivalently represented by a projection $e_\Gamma \in M_m(C(B\Gamma) \otimes C^*(\Gamma))$. Let id denote the identity map of $M_m(C(B\Gamma))$. As argued in [9, Proposition 2.5],

$$\nu(x) = \nu\big[(\varphi, \varphi')\big] = (\mathrm{id} \otimes \varphi_n)_\sharp(e_\Gamma) - (\mathrm{id} \otimes \varphi'_n)_\sharp(e_\Gamma) \in K^0(B\Gamma)$$

for all sufficiently large n. Here, $(\mathrm{id} \otimes \varphi_n)_\sharp(e_\Gamma)$ and $(\mathrm{id} \otimes \psi_n)_\sharp(e_\Gamma)$ are the K-theory classes of the projections in matrices over $C(B\Gamma)$ obtained by continuous functional calculus from the approximate projections $(\mathrm{id} \otimes \varphi_n)(e_\Gamma)$ and $(\mathrm{id} \otimes \varphi'_n)(e_\Gamma)$. If one assumes that both sequences φ_n, φ'_n can be perturbed to sequences of true representations $\pi_n : \Gamma \to U(k(n))$ and $\pi'_n : \Gamma \to U(k'(n))$ such that

$$\lim_{n\to\infty}\big(\|\varphi_n(s) - \pi_n(s)\| + \|\varphi'_n(s) - \pi'_n(s)\|\big) = 0,\ \text{for all}\ s \in \Gamma,$$

then

$$\nu(x) = (\mathrm{id} \otimes \pi_n)_\sharp(e_\Gamma) - (\mathrm{id} \otimes \pi'_n)_\sharp(e_\Gamma).$$

But $(\mathrm{id} \otimes \pi_n)_\sharp(e_\Gamma)$ and $(\mathrm{id} \otimes \pi'_n)_\sharp(e_\Gamma)$ coincide with the K-theory classes of the complex flat bundles

$$E\Gamma \times_\Gamma \mathbb{C}^{k(n)} \to B\Gamma \quad \text{and} \quad E\Gamma \times_\Gamma \mathbb{C}^{k'(n)} \to B\Gamma$$

associated to the representations π_n and π'_n. Thus, $\nu(x) \in K^0(B\Gamma)_{\mathrm{flat}}$ and hence, all the Chern classes of $\nu(x)$ vanish rationally. In view of the surjectivity of ν, it follows that the cohomology groups $H^{2k}(B\Gamma, \mathbb{Q})$, $k > 0$ must vanish. If $B\Gamma$ is not compact, one proceeds similarly using restrictions of ℓ_Γ to compact subspaces of $B\Gamma$.

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