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BOUNDED SOLUTIONS FOR SINGULAR BOUNDARY VALUE PROBLEMS

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1. Introduction

M. Ďurikovičová [2] and M. Greguš Jr. [3] have proved the existence of bounded solutions for certain singular boundary value problems associated with the equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, t\, \frac{\mathrm{d}}{\mathrm{d}t}\, x(t) = f(t, x(t), x'(t)) \tag{1.1}$$

respectively

$$\frac{\mathrm{d}}{\mathrm{d}t}\, t\, \frac{\mathrm{d}}{\mathrm{d}t}\, t\, \frac{\mathrm{d}}{\mathrm{d}t}\, x(t) = f(t, x(t), x'(t)) \tag{1.2}$$

where $t \in]0, 1]$ and f is a real valued continuous and bounded function defined on $]0, 1] \times \mathbf{R}^2$. These problems are called singular because the leading coefficients of the involved equations have zeros at the boundary point $t = 0$.

The aim of the present paper is to improve the quoted results by considering more general coefficients and also to give some generalizations for the case of the higher order differential equations and systems with several singularities

Let T be the set obtained by removing the points $t_1 \leqslant t_2 \leqslant \dots \leqslant t_k$ from the closed unit interval. These points will be the singular points of our problem. Suppose that $k \geqslant 1$, $p \geqslant 0$, and let us consider the following differential operators:

$$\boldsymbol{L}\boldsymbol{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\, \boldsymbol{A}_1(t) \dots \frac{\mathrm{d}}{\mathrm{d}t}\, \boldsymbol{A}_k(t)\, \frac{\mathrm{d}^p}{\mathrm{d}t^p}\, \boldsymbol{x}(t)$$

and

$$\boldsymbol{P}\boldsymbol{x}(t) = \boldsymbol{C}_0(t)\, \boldsymbol{x}^{(p)}(t) + \dots + \boldsymbol{C}_p(t)\, \boldsymbol{x}(t)$$

acting between the spaces $C^{k+p}(T, \mathbf{R}^n)$ and $C(T, \mathbf{R}^n)$. As usual $C^r(T, \mathbf{R}^n)$ denotes the space of all functions defined on T with values in the n-dimensional Euclidian space $\mathbf{R}^n$ and which are r-times continuously differentiable. The coefficients $\boldsymbol{A}_1, \dots, \boldsymbol{A}_k$ are real matrix valued functions such that for any $1 \leqslant i \leqslant k$, $\boldsymbol{A}_i$ is an invertible element of the algebra $C^i(T, \mathbf{R}^{n\times n})$ or equivalently $\boldsymbol{A}_i \in C^i(T, \mathbf{R}^{n\times n})$ and $\boldsymbol{A}_i(t)^{-1}$ exists for all t in T. The coefficients $C_0, \ldots, C_p$ are taken from $C(T, \mathbf{R}^{n\times n})$.

Let $\boldsymbol{f}\colon T\times \mathbf{R}^{n(p+l)}\times \mathbf{R}^{n(q-l)} \to \mathbf{R}^n$ be a continuous functions, $1 \leqslant q \leqslant k$. For any points $l_0, \ldots, l_{p-1}$ from the closed unit interval (if $p = 0$, no such points are to be considered) we considered the following boundary value problem which will be the main subject of our study:

$$\boldsymbol{Lx}(t) + \boldsymbol{Px}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \ldots, \boldsymbol{x}^{(p+q-1)}(t)), \quad t\in T \tag{1.3}$$

$$\boldsymbol{x}(l_0) = \ldots = \boldsymbol{x}^{(p-1)}(l_{p-1}) = 0 \tag{1.4}$$

The boundary conditions (1.4) will be imposed only if $p > 0$.

If some l_j is equal to some singular point t_i, then $\boldsymbol{x}^{(j)}(l_j) = 0$ means $\lim \boldsymbol{x}^{(j)}(t) = 0$ when $t \to t_i$.

This problem is called singular because of the gaps $\{t_1\}, \ldots, \{t_k\}$ where the coefficients $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ are not defined. In fact even if there are smooth extensions of $\boldsymbol{A}_i$'s to $[0, 1]$, these extensions may take noninvertible values in the points $t_1, \ldots, t_k$. Moreover, if, for example, $\boldsymbol{A}_1$ has its limit at the point t_1 and if this limit is a noninvertible matrix, then it is easy to prove that $\|\boldsymbol{A}_1(t)^{-1}\| \to \infty$ when $t \to t_1$.

This suggests that the solutions of the singular problems may be unbounded (see the examples at the end of the paper) or have other "bad" properties related to the singular points. However, under some suitable assumptions involving the behaviour of the coefficients $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k$ around the singular points and the growing of the function $\boldsymbol{f}$, the boundary problem (1.3)—(1.4) has a solution $\boldsymbol{x}\in C^{k+p}(T, \mathbf{R}^n)$ such that $\boldsymbol{x}, \ldots, \boldsymbol{x}^{(p-1)}$ may be extended to continuous functions on the closed unit interval and $\boldsymbol{x}^{(p)}$ is bounded on T. This is in fact the main result of the paper and it is contained in *Theorem* 3.1. Of a certain interest, in connection with the references [1], [2] and [3] are the *Corollaries* 3.3, 3.4 and the *Theorem* 3.5. Before concluding this introductory part we state some notation and recall two classical theorems needed in the second section of the paper.

We endow the vector space $C^r(T, \mathbf{R}^n)$ with the topology of the uniform convergence on compacta for derivatives, which is a metrizable locally convex topology and is defined by the family of seminorms:

$$\{\|\cdot\|_{r,m}\colon m\in \mathbf{N},\, m > 2b^{-1}\} \quad \text{where} \quad b = \min\{|t_i - t_j|\colon t_i \neq t_j\}$$

For $\boldsymbol{x}\in C^r(T, \mathbf{R}^n)$, $\|\boldsymbol{x}\|_{r,m} = \sup\{\max\{\|\boldsymbol{x}(t)\|, \ldots, \|\boldsymbol{x}^{(r)}(t)\|\}\colon t\in T_m\}$ where

$$T_m = [0, 1]\setminus \bigcup_{i=1}^{k}]t_i - m^{-1}, t_i + m^{-1}[.$$

If $r = 0$ we write $\|\cdot\|_{r,m} = \|\cdot\|_m$. The space $C^r(T, \mathbf{R}^n)$ is complete relative to the uniform topology defined above.

The following theorem is a well-known generalization for the space $C(T, \mathbf{R}^n)$ of the classical Ascoli-Arzela theorem:

Theorem 1.1. *A subset of $C(T, \mathbf{R}^n)$ is relatively compact with respect to the locally convex topology of $C(T, \mathbf{R}^n)$ if and only if it is equibounded and equicontinuous with respect to this topology.*

We shall need also:

Theorem 1.2. (*Tyhonov's fixed point theorem.*) *Let Z be a convex closed subset of a Hausdorff, complete, locally convex space and let Q be a continuous map from Z to Z. If the image $Q(Z)$ is relatively compact, then Q has a fixed point in Z.* (See ref. [4].)

2. A class of continuous and compact integral operators

For each $1 \leqslant i \leqslant k$ let $\boldsymbol{B}_i \in C^i(T, \mathbf{R}^{n \times n})$. Let $\boldsymbol{g}$ be a continuous function from $T \times \mathbf{R}^{n(p+1)} \times \mathbf{R}^{n(q-1)}$ to $\mathbf{R}^n$, $(t, \boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{g}(t, \boldsymbol{u}, \boldsymbol{v})$, where $t \in T$, $\boldsymbol{u} = (u^0, \ldots, u^p)$ and $\boldsymbol{v} = (v^1, \ldots, v^{q-1}) \in \mathbf{R}^{n(q-1)}$. We recall here that every norm on $\mathbf{R}^n$ induces a norm on the space of all $n \times n$ real matrices: if $\boldsymbol{B}$ is such a matrux, then $\|\boldsymbol{B}\| = \sup\{\|\boldsymbol{B}\boldsymbol{u}\| : \boldsymbol{u} \in \mathbf{R}^n, \|\boldsymbol{u}\| \leqslant 1\}$.

Throughout this section we suppose that the following conditions are satisfied:

(2.1) *For every $1 \leqslant i \leqslant k$ there is a strictly positive number λ_i such that $\|(t - t_i) B_i(t)\| \leqslant \lambda_i$ for all t in T. Let*

$$\lambda = \lambda_1 \lambda_2 \ldots \lambda_k .$$

(2.2) *There is $F > 0$ such that $\lambda \|\boldsymbol{g}(t, \boldsymbol{u}, \boldsymbol{v})\| \leqslant F$ whenever $t \in T$, $\|u^0\| \leqslant F, \ldots, \|u^p\| \leqslant F$ and $\boldsymbol{v} \in \mathbf{R}^{n(q-1)}$.*

(2.3) *$\boldsymbol{g}$ is uniformly continuous on $K \times \mathbf{R}^{n(q-1)}$ for every compact subset K of $T \times \mathbf{R}^{n(p+1)}$.*

Remark. If $q = 1$ the condition (2.3) becomes trivial and in this case the condition (2.2) is satisfied by every bounded and continuous map $(t, \boldsymbol{u}) \mapsto g(t, \boldsymbol{u})$. Indeed if $\|\boldsymbol{g}(t, \boldsymbol{u})\| \leqslant M$, then we can choose $F = \lambda M$.

Let us consider the sets:

$$Z_i = \{\boldsymbol{x} \in C^{i-1}(T, \mathbf{R}^n) : \|\boldsymbol{x}(t)\| \leqslant \lambda_1 \ldots \lambda_i \lambda_F^{-1} \text{ for all } t \text{ in } T\}, \quad 1 \leqslant i \leqslant k$$

$$Z_{k+j} = \{\boldsymbol{x} \in C^{k+j-1}(T, \mathbf{R}^n) : \|\boldsymbol{x}(t)\| \leqslant F, \ldots, \|\boldsymbol{x}^{(j)}(t)\| \leqslant F, t \in T\}, \quad 1 \leqslant j \leqslant k.$$

The sets Z_{k+j} will be defined only if $p \geqslant 1$.

We shall study the integral operator $Q : Z_{k+p} \to Z_{k+p}$ given by

$$Q\boldsymbol{x}(t) = \int_{t_0}^{t} \mathrm{d}r_{p-1} \ldots \int_{t_{p-2}}^{r_2} \mathrm{d}r_1 \int_{t_{p-1}}^{r_1} \mathrm{d}s_k \boldsymbol{B}_k(s_k) \int_{t_k}^{s_k} \ldots$$

$$\ldots \boldsymbol{B}_1(s_1)\int_{t_1}^{s_1} \boldsymbol{g}(s, \boldsymbol{x}(s), \ldots, \boldsymbol{x}^{(p+q-1)}(s))\, \mathrm{d}s$$

This expression may be suggestive but not very precise. We define Q rigorously as a product of certain operators. This enables us to prove with little effort some useful facts about Q. For this purpose we considered the sequences of operators $(U_i)_{1 \leqslant i \leqslant k}$ and $(S_j)_{1 \leqslant j \leqslant p}$ acting as described in the following diagram:

$$\begin{tikzcd}
Z_1 \arrow[r, "U_2"] & Z_2 \arrow[r, "U_3"] & \cdots \arrow[r, "U_k"] & Z_k \arrow[r, "S_1"] & \cdots \arrow[r, "S_{p-1}"] & Z_{k+p-1} \\
Z_{k+p} \arrow[u, "U_1"] \arrow[rrrrr, "Q = S_p \ldots S_1 U_k \ldots U_1"'] & & & & & Z_{k+p} \arrow[u, "S_p"']
\end{tikzcd}$$

and whose expressions are:

$$U_1 \boldsymbol{x}(t) = \boldsymbol{B}_1(t)\int_{t_1}^{t} \boldsymbol{g}(s, \boldsymbol{x}(s), \ldots, \boldsymbol{x}^{(p+q-1)}(s))\, \mathrm{d}s$$

$$U_i \boldsymbol{x}(t) = \boldsymbol{B}_i(t)\int_{t_i}^{t} \boldsymbol{x}(s)\, \mathrm{d}s, \quad 2 \leqslant i \leqslant k$$

$$S_j \boldsymbol{x}(t) = \int_{l_{p-j}}^{t} \boldsymbol{x}(s)\, \mathrm{d}s, \quad 1 \leqslant j \leqslant p$$

Finally we close the diagram by choosing $Q = S_p \ldots S_1 U_k \ldots U_1$. (If $p = 0$ we take $Q = U_k \ldots U_1$.)

Lemma 2.1. *The operator U_1 is well defined and continuous; the image $U_1(Z_{k+p})$ is relatively compact.*

Proof. Consider the set

$$W = \{\boldsymbol{x} \in C(T, \mathbf{R}^n) : \|\boldsymbol{x}(t)\| \leqslant \lambda^{-1} F \text{ for all } t \text{ in } T\}$$

and define the map $R: Z_{k+p} \to W$, $\boldsymbol{x} \mapsto R\boldsymbol{x}$, by setting

$$R\boldsymbol{x}(t) = \boldsymbol{g}(t, \boldsymbol{x}(t), \ldots, \boldsymbol{x}^{(p+q-1)}(t)).$$

The condition (2.2) together with the continuity of $\boldsymbol{g}$ imply that $R\boldsymbol{x} \in W$ for all $\boldsymbol{x}$ in Z_{k+p} and therefore the operator R is well defined. Moreover the uniform continuity of $\boldsymbol{g}$ on the sets $I_m = T_m \times \{(u^0, \ldots, u^p) \in \mathbf{R}^{n(p+1)} : \|u^i\| \leqslant F\} \times \mathbf{R}^{n(q-l)}$ (assured by (2.3)) implies the continuity of R. Indeed, for given $m \in \mathbf{N}$ and $\varepsilon > 0$

we find $\delta > 0$ such that for every $\boldsymbol{x}, \boldsymbol{y} \in Z_{k+p}$ with $\|\boldsymbol{x} - \boldsymbol{y}\|_{k+p-1,m} \leqslant \delta$ we have $\|R\boldsymbol{x} - R\boldsymbol{y}\|_m \leqslant \varepsilon$. For we have only to choose $\delta > 0$ such that $\|\boldsymbol{g}(t, \boldsymbol{u}, \boldsymbol{v}) - \boldsymbol{g}(t, \boldsymbol{u}_0, \boldsymbol{v}_0)\| \leqslant \varepsilon$ whenever $(t, \boldsymbol{u}, \boldsymbol{v})$, $(t, \boldsymbol{u}_0, \boldsymbol{v}_0) \in I_m$ and $\|u^j - u_0^j\| \leqslant \delta$, $\|v^i - v_0^i\| \leqslant \delta$. Now we consider the operator $S\colon W \to Z_1$, defined by

$$S\boldsymbol{x}(t) = \boldsymbol{B}_1(t) \int_{t_1}^{t} \boldsymbol{x}(s)\, \mathrm{d}s$$

If $\boldsymbol{x} \in W$ and $s \in T$, we have $\|\boldsymbol{x}(s)\| \leqslant \lambda^{-1}F$ whence, using (2.1), it follows that $\|S\boldsymbol{x}(t)\| \leqslant \|(t - t_1)\boldsymbol{B}_1(t)\|\, \lambda^{-1}F \leqslant \lambda_1\lambda^{-1}F$. Thus $S\boldsymbol{x} \in Z_1$ and the operator S is well defined. In addition we shall prove that S is continuous. Indeed for $\varepsilon > 0$ and $q \geqslant 1$ we find $\delta > 0$ and $m \geqslant 1$ such that for all $\boldsymbol{x}, \boldsymbol{y} \in W$ the inequality $\|\boldsymbol{x} - \boldsymbol{y}\|_m \leqslant \delta$ implies $\|S\boldsymbol{x} - S\boldsymbol{y}\|_q \leqslant \varepsilon$. To do it, it is enough to choose $m > q$ and $\delta > 0$ such that

$$\sup\{\|\boldsymbol{B}_1(t)|: t \in T_q\} \cdot (2\lambda^{-1}m^{-1}F + \delta) \leqslant \varepsilon.$$

Then for $t \in T_q$ setting $\eta = \operatorname{signum}(t - t_1)$ we may write

$$\|S\boldsymbol{x}(t) - S\boldsymbol{y}(t)\| =$$

$$= \left\| \boldsymbol{B}_1(t) \left[\int_{t_1}^{t_1 + \eta/m} (\boldsymbol{x}(s) - \boldsymbol{y}(s))\, \mathrm{d}s + \int_{t_1 + \eta/m}^{t} (\boldsymbol{x}(s) - \boldsymbol{y}(s))\, \mathrm{d}s \right] \right\| \leqslant$$

$$\leqslant \sup\{\|\boldsymbol{B}_1(t)\|: t \in T_q\} \cdot (m^{-1} 2\lambda^{-1}F + \|\boldsymbol{x} - \boldsymbol{y}\|_m) \leqslant \varepsilon.$$

We finish the investigation of this operator by proving that the image $S(W)$ is relatively compact. In virtue of the *Ascoli-Arzela theorem* it suffices to prove that $S(W)$ is equibounded and equicontinuous. The equiboundedness of $S(W)$ is obvious since $S(W) \subset Z_1$ and Z_1 is uniformly bounded. To show that $S(W)$ is equicontinuous, for given $t_0 \in T$ and $\varepsilon > 0$ we find $\delta > 0$ such that $\|S\boldsymbol{x}(t) - S\boldsymbol{x}(t_0)\| \leqslant \varepsilon$ for all $\boldsymbol{x}$ in W and $t \in T$ with $|t - t_0| < \delta$. To do this we choose $\delta > 0$ small enough so that $\delta\lambda^{-1}F\|\boldsymbol{B}_1(t)\| < \varepsilon/2$ and $\|\boldsymbol{B}_1(t) - \boldsymbol{B}_1(t_0)\|\, \lambda^{-1}F < \varepsilon/2$ for any $t \in T$ with $|t - t_0| < \delta$. This choice is possible since $\boldsymbol{B}_1$ is continuous in t_0. Now using the identity

$$S\boldsymbol{x}(t) - S\boldsymbol{x}(t_0) = \boldsymbol{B}_1(t) \int_{t_0}^{t} \boldsymbol{x}(s)\, \mathrm{d}s + (\boldsymbol{B}_1(t) - \boldsymbol{B}_1(t_0)) \int_{t_1}^{t_0} \boldsymbol{x}(s)\, \mathrm{d}s$$

we get the desired inequality. To complete the proof we return to the operator U_1 and observe that $U_1 = SR$. Therefore U_1 is continuous and $U_1(Z_{k+p}) \subset S(W)$ is relatively compact.

Lemma 2.2. *The operators $(U_i)_{2 \leqslant i \leqslant k}$ and $(S_j)_{1 \leqslant j \leqslant p}$ are well defined and continuous.*

Proof. Since the operators (S_j) are very simple the only thing to prove is the continuity of the operators (U_i).

Set $2 \leqslant i \leqslant k$ and consider $U_i\colon Z_{i-1} \to Z_i$. Let $\varepsilon > 0$ and $q \geqslant 1$. We shall find $\delta > 0$ and $m \in \mathbf{N}$ such that $\| U_i \boldsymbol{x} - U_i \boldsymbol{y} \|_{i-1,q} \leqslant \varepsilon$ whenever $\boldsymbol{x}, \boldsymbol{y} \in Z_{i-1}$ and $\| \boldsymbol{x} - \boldsymbol{y} \|_{i-2,m} \leqslant \delta$. Let

$$M = \sup \left\{ \sup \left\{ \left\| \frac{\mathrm{d}^k \boldsymbol{B}_i(t)}{\mathrm{d}t^k} \right\| : 0 \leqslant k \leqslant i-1 \right\} : t \in T_q \right\}$$

and choose $m > q$ large enough and $\delta > 0$ small enough such that

$$M(2\lambda_1 \dots \lambda_{i-1} \lambda^{-1} F m^{-1} + 2^{i-1}\delta) \leqslant \varepsilon$$

Now for any $0 \leqslant j \leqslant i-1$ we show that

$$\left\| \frac{\mathrm{d}^j}{\mathrm{d}t^j} (U_i \boldsymbol{x} - U_i \boldsymbol{y}) \right\|_q \leqslant \varepsilon$$

and this will complete the proof. Let $t \in T_q$ and $\eta = \operatorname{signum}(t - t_i)$. Then we may write

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j} (U_i \boldsymbol{x} - U_i \boldsymbol{y})(t) = \frac{\mathrm{d}^j \boldsymbol{B}_i(t)}{\mathrm{d}t^j} \left[\int_{t_i}^{t_i + \eta/m} (\boldsymbol{x}(s) - \boldsymbol{y}(s)) \, \mathrm{d}s + \right.$$

$$\left. + \int_{t_i + \eta/m}^{t} (\boldsymbol{x}(s) - \boldsymbol{y}(s)) \, \mathrm{d}s \right] + \sum_{k=1}^{j} \binom{j}{k} \frac{\mathrm{d}^{j-k} \boldsymbol{B}_i(t)}{\mathrm{d}t^{j-k}} \cdot \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} (\boldsymbol{x} - \boldsymbol{y})(t)$$

whence

$$\left\| \frac{\mathrm{d}^j}{\mathrm{d}t^j} (U_i \boldsymbol{x} - U_i \boldsymbol{y})(t) \right\| \leqslant$$

$$\leqslant M(2\lambda_1 \dots \lambda_{i-1} \lambda^{-1} F m^{-1} + \| \boldsymbol{x} + \boldsymbol{y} \|_m) + \sum_{k=1}^{j} \binom{j}{k} M \left\| \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} (\boldsymbol{x} - \boldsymbol{y}) \right\|_m \leqslant$$

$$\leqslant M(2\lambda_1 \dots \lambda_{i-1} \lambda^{-1} F m^{-1} + 2^j \| \boldsymbol{x} - \boldsymbol{y} \|_{j-1,m}) \leqslant \varepsilon$$

since $j \leqslant i-1$ and $\| \boldsymbol{x} - \boldsymbol{y} \|_{j-1,m} \leqslant \| \boldsymbol{x} - \boldsymbol{y} \|_{i-2,m} \leqslant \delta$.
Taking advantage of lemmas 2.1 and 2.2 the main theorem of this section follows straighforwardly:

Theorem 2.3. *The integral operator* $Q\colon Z_{k+p} \to Z_{k+p}$, $Q = S_p \dots S_1 U_k \dots U_1$ *has at least one fixed point.*

Proof. The set Z_{k+p} is nonvoid, convex and closed. The operator Q is continuous, being a product of continuous operators, and the image $Q(Z_{k+p})$ is relatively compact since it is equal to $S_p \dots S_1 U_k \dots U_1(Z_{k+p})$ and we have seen that $U_1(Z_{k+p})$ is relatively compact. Now we apply the Tyhonov fixed point theorem to obtain the desired conclusion.

3. Bounded solutions

With the same notation as in the introduction we prove now the main result of the paper. Let $\boldsymbol{P}\colon T \times \mathbf{R}^{n(p+1)} \to \mathbf{R}^n$, $(t, \boldsymbol{u}) \mapsto \boldsymbol{P}(t, \boldsymbol{u}) = \boldsymbol{C}_0(t)u^p + \ldots + \boldsymbol{C}_p(t)u^0$ be the function canonically associated with the differential operator $\boldsymbol{Px}$. Then we have the following.

Theorem 3.1. *Suppose thast the coefficients $\boldsymbol{A}_i \in C^i(T, \mathbf{R}^{n\times n})$ and the continuous functions $\boldsymbol{P}(t, \boldsymbol{u})$ and $\boldsymbol{f}(t, \boldsymbol{u}, \boldsymbol{v})$ satisfy the following conditions (derived from* (2.1—2.3)).

(3.1) *For every $1 \leqslant i \leqslant k$ there is $\lambda_i > 0$ such that $\|(t - t_i)\boldsymbol{A}_i(t)^{-1}\| \leqslant \lambda_i$ for all t in T. Let $\lambda = \lambda_1\lambda_2 \ldots \lambda_k$.*

(3.2) *There is $F > 0$ such that $\lambda\|\boldsymbol{f}(t, \boldsymbol{u}, \boldsymbol{v}) - \boldsymbol{P}(t, \boldsymbol{u})\| \leqslant F$, whenever $t \in T$, $\|u^0\| \leqslant F, \ldots, \|u^p\| \leqslant F$ and $\boldsymbol{v} \in \mathbf{R}^{n(q-1)}$.*

(3.3) *$\boldsymbol{f}$ is uniformly continuous on $K \times \mathbf{R}^{n(q-1)}$ for every compact subset K of $T \times \mathbf{R}^{n(p+1)}$.*

Then the singular boundary value problem (1.3)—(1.4) *has a solution $\boldsymbol{x} \in C^{p+k}(T, \mathbf{R}^n)$ which can be extended to a function belonging to $C^{p-1}([0, 1], \mathbf{R}^n)$. Moreover $\|\boldsymbol{x}(t)\| \leqslant F, \ldots, \|\boldsymbol{x}^{(p)}(t)\| \leqslant F$ for all t in T.*

Proof. Consider the operator Q defined in the second section with $\boldsymbol{A}_i^{-1}$ instead of $\boldsymbol{B}_i$ and $\boldsymbol{f}(t, \boldsymbol{u}, \boldsymbol{v}) - \boldsymbol{P}(t, \boldsymbol{u})$ instead of $\boldsymbol{g}(t, \boldsymbol{u}, \boldsymbol{v})$. By Theorem 2.3 the operator Q has a fixed point $\boldsymbol{x} \in Z_{k+p}$. One will differentiate the equality $Q\boldsymbol{x} = \boldsymbol{x}$ to see that $\boldsymbol{x}$ satisfies equation (1.3) and $\boldsymbol{x} \in C^{p+k}(T, \mathbf{R}^n)$. Now since $\boldsymbol{x} \in Z_{k+p}$, it is clear that $\boldsymbol{x}, \ldots, \boldsymbol{x}^{(p)}$ are bounded by the constant F. To end the proof observe that for $p \geqslant 1$ the equality $\boldsymbol{x} = S_p \ldots S_1 U_k \ldots U_1 \boldsymbol{x}$ implies

$$\boldsymbol{x}^{(j-1)} = (S_{p-j+1} \ldots U_k \ldots U_1)(\boldsymbol{x})(t) = \int_{l_{j-1}}^{t} (S_{p-j} \ldots U_k \ldots U_1)(\boldsymbol{x})(s)\,\mathrm{d}s,$$
$$1 \leqslant j \leqslant p$$

and since $S_{p-j} \ldots U_k \ldots U_1\boldsymbol{x}$ is an element of Z_{k+p-j}, which is a set consisting of continuous bounded functions, the previous formulae can be used to extend $\boldsymbol{x}$ to a function in $C^{p-1}([0, 1], \mathbf{R}^n)$. It is also clear that $\boldsymbol{x}^{(j-1)}(l_{j-1}) = 0$.

Corollary 3.2. *Suppose that the function $\boldsymbol{f}$ is bounded and satisfies the condition* (3.3) *of Theorem* 3.1. *If in addition*

$$\text{(3.4)} \qquad \sup_{t\in T}\left(\sum_{j=0}^{p} \|\boldsymbol{C}_j(t)\|\right) \cdot \prod_{i=1}^{k} \sup_{t\in T} \|(t - t_i)\boldsymbol{A}_i(t)^{-1}\| < 1,$$

each member of the product being finite, then the singular boundary problem (1.3)—(1.4) *has a solution* $\boldsymbol{x} \in C^{p+k}(T, \mathbf{R}^n)$ *such that the derivative* $\boldsymbol{x}^{(p)}$ *is bounded on T. Therefore* $\boldsymbol{x}$ *can be extended to a function be'onging to* $C^{p-1}([0, 1], \mathbf{R}^n)$.

Proof. Set $\lambda_i = \sup \|(t - t_i)\boldsymbol{A}_i(t)^{-1}\| < \infty$, $\lambda = \lambda_1 \dots \lambda_k$ and

$$\eta = \lambda \sup \left(\sum_{i=1}^{p} \|\boldsymbol{C}_j(t)\| \right) < 1.$$

If $\|\boldsymbol{f}(t, \boldsymbol{u}, \boldsymbol{v})\| \leqslant M$, then we may choose $F = \lambda M(1 - \eta)^{-1}$ to verify the condition (3.2). Indeed if $\|u^0\| \leqslant F, \dots, \|u^p\| \leqslant F$, then

$$\lambda \|\boldsymbol{f}(t, \boldsymbol{u}, \boldsymbol{v}) - \boldsymbol{P}(t, \boldsymbol{u})\| \leqslant \lambda M + \lambda \left(\sum_{j=1}^{p} \|\boldsymbol{C}_j(t)\| \cdot \|u^j\| \right) \leqslant \lambda M + \eta F = F$$

and so we may apply Theorem 3.1.

Corollary 3.3. *Consider the singular boundary value problem*:

$$a(t)x''(t) + b(t)x'(t) + c(t)x(t) = f(t, x(t), x'(t)) \tag{3.5}$$

$$x(1) = 0 \quad t \in]0, 1] \tag{3.6}$$

where $a \in C^1(]0, 1])$, $b, c \in C(]0, 1])$ *and* $f \in C(]0, 1] \times \mathbf{R}^2)$ *is bounded. Assume that*

$$\inf_{0 < t \leqslant 1} |t^{-1}a(t)| > \sup_{0 < t \leqslant 1} (|a'(t) - b(t)| + |c(t)|) \tag{3.7}$$

Then the problem (3.5)—(3.6) *has a solution* $x \in C^2(]0, 1])$ *such that x and x' are bounded.*

Proof. Apply the previous corollary with $n = 1$, $T =]0, 1]$

$$Lx = \frac{\mathrm{d}}{\mathrm{d}t}\, a(t) \frac{\mathrm{d}}{\mathrm{d}t}\, x(t), \quad Px = (-a'(t) + b(t))x'(t) + c(t)x(t).$$

Since $q = 1$ the condition (3.3) is trivial.

Corollary 3.4. *Let* $f \in C(]0, 1] \times \mathbf{R}^3)$ *be a bounded function which is uniformly continuous on* $K \times \mathbf{R}$ *whenever K is a compact subset of* $]0, 1] \times \mathbf{R}^2$. *If* $c_0, c_1 \in C(]0, 1])$ *are such that*

$$\sup (|c_0(t)| + |c_1(t)|) < 1 \quad (\textit{over } t \textit{ in }]0, 1])$$

then the following singular boundary value problem

$$\begin{aligned} t^2x'''(t) + 3tx''(t) + (1 + c_0(t))x'(t) + c_1(t)x(t) = \\ = f(t, x(t), x'(t), x''(t)), \quad t \in]0, 1] \end{aligned} \tag{3.8}$$

$$x(1) = 0 \tag{3.9}$$

has a solution $x \in C^3(]0, 1])$ *such that* x *and* x' *are bounded.*

Proof. Apply the Corollary 3.2 with $n = 1$, $k = 2$, $t_1 = t_2 = 0$, $p = 1$,

$$Lx = \frac{d}{dt} t \frac{d}{dt} t \frac{d}{dt} x(t) \quad \text{and} \quad Px = c_0(t)x'(t) + c_1(t)x(t).$$

Next we revise under weakened hypotheses the main result of D. Andrica [1].

Theorem 3.5. *Consider the boundary value problem*

$$a(t)x''(t) + b(t)x'(t) = a(t)h'(t)f(t, x(t), x'(t)) \tag{3.10}$$

$$x(1) = 0 \quad t \in]0, 1] \tag{3.11}$$

where $a, b \in C(]0, 1])$, $f \in C(]0, 1] \times R^2)$ *is a bounded function, and* $h \in C^2(]0, 1])$ *is a solution of the associated homogeneous equation* (i.e., $a(t)h''(t) + b(t)h'(t) = 0$). *If in addition*

$$\sup t|h'(t)| < \infty, \quad (\textit{over } t \textit{ in }]0, 1]),$$

then the problem (3.10)—(3.11) *has a solution* $x \in C^2(]0, 1])$ *such that* x *and* x' *are bounded.*

Proof. Let $n = 1$, $k = 1$, $p = 1$, $q = 1$, $t_1 = 0$, $T =]0, 1]$. Then the functions $B_1(t) = h'(t)$ and $g(t, u) = f(t, u)$ satisfy the conditions (2.1)—(2.3) with $\lambda = \lambda_1 = \sup(|th'(t)|)$ and $F = \lambda \sup |f(t, u)|$, hence by Theorem 2.3 the operator $Q: Z_2 \to Z_2$

$$Qx(t) = \int_1^t h'(s) \left(\int_0^s f(r, x(r), x'(r))\, dr \right) ds$$

has a fixed point x. One will differentiate the equality $x(t) = Qx(t)$ to obtain

$$x'(t) = h'(t) \int_0^t f(r, x(r), x'(r))\, dr$$

and

$$x''(t) = h''(t) \int_0^t f(r, x(r), x'(r))\, dr + h'(t)f(t, x(t), x'(t)).$$

It follows that $a(t)x''(t) + b(t)x'(t) = a(t)h'(t)f(t, x(t), x'(t))$, which means that x verifies the equation (3.10). Also $x(1) = Qx(1) = 0$. Since $x \in Z_2$, x and x' are bounded by F.

4. Examples and concluding remarks

Consider the singular equation

$$\frac{d}{dt} t \frac{d}{dt} tx(t) + c_0(t)x'(t) + c_1(t)x(t) = 1, \quad t \in]0, 1] \tag{4.1}$$

In virtue of Corollary 3.2 we know that this equation has a bounded solution if $\sup(|c_0(t)| + |c_1(t)|) < 1$.

From the point of view of our approach this restriction seems to be essential, since if we take $c_0(t) = 0$ and $c_1(t) = -1$, then the corresponding equation has the general solution $x(t) = 2^{-1}\ln(t) + at^{-2} + b$ with $a, b \in \mathbf{R}$ and no bounded solution is available. However, other choices of the coefficients show that this restriction is not necessary.

Concerning the boundary conditions (1.3) they are maximal, since if one more condition is added the problem may have no bounded solutions, despite the fact that all the hypotheses of Theorem 3.1 are satisfied. In this respect, the following example is conclusive: the problem $tx'(t) + x(t) = 1$, $x(1) = 0$, $t \in]0, 1]$ has the unique solution $x(t) = 1 - t^{-1}$ which is not bounded.

To explain the relation between the number of the singularities (counted with multiplicities) and the number of the boundary conditions as it appears in our approach, we can offer only a not exact argument which is, however, pertinent in some cases. Generally speaking the solutions of a $(k + p)$-order differential equation depend on $(k + p)$ parameters. To obtain bounded solutions there are needed suitable choices for k parameters that annihilate the undesired behaviour of the general solution around the singularities $t_1, \ldots, t_k$. The rest of the p parameters are available for the boundary conditions.

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ОГРАНИЧЕННЫЕ РЕШЕНИЯ СИНГУЛЯРНЫХ КРАЕВЫХ ЗАДАЧ

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Резюме

В этой работе изложены нелинейные граничные задачи с более многими сингулярностями.

Исползуя теорему Тихонова о неподвижной точки в случае адекватных интегральных операторов доказывается сущестование некоторых решении которые ограничены вместе со свими определенными производными.