A DIXMIER-DOUADY THEORY FOR STRONGLY SELF-ABSORBING 
\textit{C}^*\textit{-ALGEBRAS II: THE BRAUER GROUP}

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Abstract. We have previously shown that the isomorphism classes of orientable locally trivial fields of \textit{C}^*\textit{-algebras over a compact metrizable space }X\textit{ with fiber }D \otimes \mathbb{K} \textit{, where }D \textit{ is a strongly self-absorbing }\textit{C}^*\textit{-algebra, form an abelian group under the operation of tensor product. Moreover this group is isomorphic to the first group }\overline{E}^1_D(X)\textit{ of the (reduced) generalized cohomology theory associated to the unit spectrum of topological K-theory with coefficients in }D\textit{. Here we show that all the torsion elements of the group }\overline{E}^1_D(X)\textit{ arise from locally trivial fields with fiber }D \otimes M_n(\mathbb{C}),\textit{ }n \geq 1,\textit{ for all known examples of strongly self-absorbing }\textit{C}^*\textit{-algebras }D\textit{. Moreover the Brauer group generated by locally trivial fields with fiber }D \otimes M_n(\mathbb{C}),\textit{ }n \geq 1\textit{ is isomorphic to }\text{Tor}(\overline{E}^1_D(X))\textit{.}

1. Introduction

Let }X\textit{ be a compact metrizable space. Let }\mathbb{K}\textit{ denote the }\textit{C}^*\textit{-algebra of compact operators on an infinite dimensional separable Hilbert space. It is well-known that }\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}\textit{ and }M_n(\mathbb{C}) \otimes \mathbb{K} \cong \mathbb{K}\textit{. Dixmier and Douady [7] showed that the isomorphism classes of locally trivial fields of }\textit{C}^*\textit{-algebras over }X\textit{ with fiber }\mathbb{K}\textit{ form an abelian group under the operation of tensor product over }C(X)\textit{ and this group is isomorphic to }H^3(X, \mathbb{Z})\textit{. The torsion subgroup of }H^3(X, \mathbb{Z})\textit{ admits the following description. Each element of }\text{Tor}(H^3(X, \mathbb{Z}))\textit{ arises as the Dixmier-Douady class of a field }A\textit{ which is isomorphic to the stabilization }B \otimes \mathbb{K}\textit{ of some locally trivial field of }\textit{C}^*\textit{-algebras }B\textit{ over }X\textit{ with all fibers isomorphic to }M_n(\mathbb{C})\textit{ for some integer }n \geq 1,\textit{ see [8], [1].}

In this paper we generalize this result to locally trivial fields with fiber }D \otimes \mathbb{K}\textit{ where }D\textit{ is a strongly self-absorbing }\textit{C}^*\textit{-algebra [17]. For a }\textit{C}^*\textit{-algebra }B\textit{, we denote by }\mathcal{C}_B(X)\textit{ the isomorphism classes of locally trivial continuous fields of }\textit{C}^*\textit{-algebras over }X\textit{ with fibers isomorphic to }B\textit{. The isomorphism classes of orientable locally trivial continuous fields is denoted by }\mathcal{C}_B^0(X)\textit{, see Definition 2.1. We have shown in [4] that }\mathcal{C}_D \otimes \mathbb{K}(X)\textit{ is an abelian group under the operation of tensor product over }C(X)\textit{, and moreover, this group is isomorphic to }E^1_D(X)\textit{ of the (reduced) generalized cohomology theory }E^*_D(X)\textit{ which we have proven to be isomorphic to the theory associated to the unit spectrum of topological K-theory with coefficients in }D\textit{, see [5]. Similarly }\mathcal{C}_D(K)(X), \otimes \cong \bar{E}^1_D(X)\textit{ where }\bar{E}^*_D(X)\textit{ is the reduced theory associated to }E^*_D(X)\textit{. For }D = \mathbb{C}\textit{, we have, of course, }E^1_\mathbb{C}(X) \cong H^3(X, \mathbb{Z})\textit{.}

We consider the stabilization map }\sigma : \mathcal{C}_D \otimes M_n(\mathbb{C})(X) \rightarrow (\mathcal{C}_D \otimes \mathbb{K})(X), \otimes \cong E^1_D(X)\textit{ given by }[A] \mapsto [A \otimes \mathbb{K}]\textit{ and show that its image consists entirely of torsion elements. Moreover, if }D\textit{ is any
of the known strongly self-absorbing $C^*$-algebras, we show that the stabilization map
\[ \sigma : \bigcup_{n \geq 1} \mathscr{C}_{D^\otimes M_n(\mathbb{C})}(X) \rightarrow \text{Tor}(\bar{E}_D(X)) \]
is surjective, see Theorem 2.8. In this situation $\mathscr{C}_{D^\otimes M_n(\mathbb{C})}(X) \cong \mathscr{C}_{D^\otimes M_n(\mathbb{C})}(X)$ by Lemma 2.1 and hence the image of the stabilization map is contained in the reduced group $\bar{E}_D(X)$. In analogy with the classic Brauer group generated by continuous fields of complex matrices $M_n(\mathbb{C})$ \cite{Dadarlat}, we introduce a Brauer group $Br_D(X)$ for locally trivial fields of $C^*$-algebras with fibers $M_n(D)$ for $D$ a strongly self-absorbing $C^*$-algebra and establish an isomorphism $Br_D(X) \cong \text{Tor}(\bar{E}_D(X))$, see Theorem 2.10.

Our proof is new even in the classic case $D = \mathbb{C}$ whose original proof relies on an argument of Serre, see \cite[Thm.1.6]{Dadarlat}. In the cases $D = \mathbb{Z}$ or $D = \mathcal{O}_\infty$ the group $\bar{E}_D(X)$ is isomorphic to $H^1(X, BSU_\otimes)$, which appeared in \cite{Winter}, where its equivariant counterpart played a central role.

We introduced in \cite{Dadarlat} characteristic classes
\[ \delta_0 : E^1_D(X) \rightarrow H^1(X, K_0(D))^\infty \quad \text{and} \quad \delta_k : E^k_D(X) \rightarrow H^{2k+1}(X, \mathbb{Q}), \quad k \geq 1. \]
If $X$ is connected, then $\bar{E}^1_D(X) = \ker(\delta_0)$. We show that an element $a$ belongs $\text{Tor}(E^1_D(X))$ if and only if $\delta_0(a)$ is a torsion element and $\delta_k(a) = 0$ for all $k \geq 1$.

In the last part of the paper we show that if $A^{op}$ is the opposite $C^*$-algebra of a locally trivial continuous field $A$ with fiber $D \otimes \mathbb{K}$, then $\delta_k(A^{op}) = (-1)^k \delta_k(A)$ for all $k \geq 0$. This shows that in general $A \otimes A^{op}$ is not isomorphic to a trivial field, unlike what happens in the case $D = \mathbb{C}$. Similar arguments show that in general $[A^{op}]_{Br} \neq -[A]_{Br}$ in $Br_D(X)$ for $A \in \mathscr{C}_{D^\otimes M_n(\mathbb{C})}(X)$, see Example 3.5.

2. Background and main result

The class of strongly self-absorbing $C^*$-algebras was introduced by Toms and Winter \cite{TomsWinter}. They are separable unital $C^*$-algebras $D$ singled out by the property that there exists an isomorphism $D \rightarrow D \otimes D$ which is unitarily homotopic to the map $d \mapsto d \otimes 1_D$ \cite{Dadarlat}, \cite{Winter}.

If $p$ is a prime number we denote by $M_{p^\infty}$ the UHF-algebra $M_{p}(\mathbb{C})^{\otimes \infty}$. If $P$ is a nonempty set of primes, we denote by $M_{P^\infty}$ the UHF-algebra of infinite type $\bigotimes_{p \in P} M_{p^\infty}$. If $P$ is the set of all primes, then $M_{P^\infty}$ is the universal UHF-algebra, which we denote by $M_{\mathbb{Q}}$.

The class $D_{pi}$ of all purely infinite strongly self-absorbing $C^*$-algebras that satisfy the Universal Coefficient Theorem in KK-theory (UCT) was completely described in \cite{TomsWinter}. $D_{pi}$ consists of the Cuntz algebras $O_2$, $O_\infty$ and of all $C^*$-algebras $M_{p^\infty} \otimes O_\infty$ with $P$ an arbitrary set of primes. Let $D_{qd}$ denote the class of strongly self-absorbing $C^*$-algebras which satisfy the UCT and which are quasidiagonal. A complete description of $D_{qd}$ has become possible due to the recent results of Matui and Sato \cite[Cor. 6.2]{MatuiSato} that build on results of Winter \cite{Winter}, and Lin and Niu \cite{LinNiu}. Thus $D_{qd}$ consists of $\mathbb{C}$, the Jiang-Su algebra $\mathcal{Z}$ and all UHF-algebras $M_{P^\infty}$ with $P$ an arbitrary set of primes. The class $D = D_{qd} \cup D_{pi}$ contains all known examples of strongly self-absorbing $C^*$-algebras. It is closed under tensor products. If $D$ is strongly self-absorbing, then $K_0(D)$ is a unital commutative ring. The group of positive invertible elements of $K_0(D)$ is denoted by $K_0(D)_+$. 


Let $B$ be a $C^*$-algebra. We denote by $\text{Aut}_0(B)$ the path component of the identity of $\text{Aut}(B)$ endowed with the point-norm topology. Recall that we denote by $\mathcal{C}_B(X)$ the isomorphism classes of locally trivial continuous fields over $X$ with fibers isomorphic to $B$. The structure group of $A \in \mathcal{C}_B(X)$ is $\text{Aut}(B)$, and $A$ is in fact given by a principal $\text{Aut}(B)$-bundle which is determined up to an isomorphism by an element of the homotopy classes of continuous maps from $X$ to the classifying space of the topological group $\text{Aut}(B)$, denoted by $[X, B\text{Aut}(B)]$.

**Definition 2.1.** A locally trivial continuous field $A$ of $C^*$-algebras with fiber $B$ is **orientable** if its structure group can be reduced to $\text{Aut}_0(B)$, in other words if $A$ is given an element of $[X, B\text{Aut}_0(B)]$.

The corresponding isomorphism classes of orientable and locally trivial fields is denoted by $\mathcal{C}_B^0(X)$.

**Lemma 2.2.** Let $D$ be a strongly self-absorbing $C^*$-algebra satisfying the UCT. Then $\text{Aut}(M_n(D)) = \text{Aut}_0(M_n(D))$ for all $n \geq 1$ and hence $\mathcal{C}_{D \otimes M_n(\mathbb{C})}(X) \cong \mathcal{C}_{D \otimes M_n(\mathbb{C})}^0(X)$.

**Proof.** First we show that for any $\beta \in \text{Aut}(D \otimes M_n(\mathbb{C}))$ there exist $\alpha \in \text{Aut}(D)$ and a unitary $v \in D \otimes M_n(\mathbb{C})$ such that $\beta = u(\alpha \otimes \text{id}_{M_n(\mathbb{C})})u^*$. Let $e_{11} \in M_n(\mathbb{C})$ be the rank-one projection that appears in the canonical matrix units $(e_{ij})$ of $M_n(\mathbb{C})$ and let $1_n$ be the unit of $M_n(\mathbb{C})$. Then

$$n[1_D \otimes e_{11}] = [1_D \otimes 1_n] \text{ in } K_0(D) \text{ and hence } n[\beta(1_D \otimes e_{11})] = n[1_D \otimes e_{11}] \text{ in } K_0(D).$$

Under the assumptions of the lemma, it is known that $K_0(D)$ is torsion free (by [17]) and that $D$ has cancellation of projections by [19] and [15]. It follows that there is a partial isometry $v \in D \otimes M_n(\mathbb{C})$ such that $v^*v = 1_D \otimes e_{11}$ and $vv^* = \beta(1_D \otimes e_{11})$. Then $u = \sum_{i=1}^n \beta(1_D \otimes e_{11})v(1_D \otimes e_{1i}) \in D \otimes M_n(\mathbb{C})$ is a unitary such that the automorphism $u^*u \beta u$ acts identically on $1_D \otimes M_n(\mathbb{C})$. It follows that $u^*u \alpha \otimes \text{id}_{M_n(\mathbb{C})}$ for some $\alpha \in \text{Aut}(D)$. Since both $U(D)$ and $\text{Aut}(D)$ are path connected by [17], [15] and respectively [6] we conclude that $\text{Aut}(D \otimes M_n(\mathbb{C}))$ is path-connected as well.

Let us recall the following results contained in Cor. 3.7, Thm. 3.8 and Cor. 3.9 from [1]. Let $D$ be a strongly self-absorbing $C^*$-algebra.

1. The classifying spaces $B\text{Aut}(D \otimes \mathbb{K})$ and $B\text{Aut}_0(D \otimes \mathbb{K})$ are infinite loop spaces giving rise to generalized cohomology theories $E^*_D(X)$ and respectively $\tilde{E}^*_D(X)$.
2. The monoid $(\mathcal{E}_{D \otimes \mathbb{K}}(X), \otimes)$ is an abelian group isomorphic to $E^1_D(X)$. Similarly, the monoid $(\mathcal{E}_{D \otimes \mathbb{K}}^0(X), \otimes)$ is a group isomorphic to $E^1_D(X)$. In both cases the tensor product is understood to be over $C(X)$.
3. $E^1_{M_Q}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q})$,

$\mathcal{E}^1_{M_Q \otimes \mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q})$,

$E^1_{M_Q \otimes \mathcal{O}_\infty}(X) \cong \tilde{E}^1_{M_Q \otimes \mathcal{O}_\infty}(X) \cong \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q})$.

4. If $D$ satisfies the UCT then $D \otimes M_Q \otimes \mathcal{O}_\infty \cong M_Q \otimes \mathcal{O}_\infty$, by [17]. Therefore the tensor product operation $A \mapsto A \otimes M_Q \otimes \mathcal{O}_\infty$ induces maps $\mathcal{E}_{D \otimes \mathbb{K}}(X) \to \mathcal{E}_{M_Q \otimes \mathcal{O}_\infty \otimes \mathbb{K}}(X)$, $\mathcal{E}_{D \otimes \mathbb{K}}^0(X) \to \mathcal{E}_{M_Q \otimes \mathcal{O}_\infty \otimes \mathbb{K}}^0(X)$ and hence maps

$$E^1_D(X) \xrightarrow{\delta} E^1_{M_Q \otimes \mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}).$$
The invariants $\delta_k(\mathcal{A})$ are called the rational characteristic classes of the continuous field $\mathcal{A}$, see [4, Def.4.6]. The first class $\delta_0^0(\mathcal{A})$ lifts to a map $\delta_0: E_D^1(\mathcal{X}) \rightarrow H^1(\mathcal{X}, K_0(D)_{+}^{\mathbb{Z}})$ induced by the morphism of groups $\text{Aut}(D \otimes \mathbb{K}) \rightarrow \pi_0(\text{Aut}(D \otimes \mathbb{K})) \cong K_0(D)_{+}^{\mathbb{Z}}$. $\delta_0(\mathcal{A})$ represents the obstruction to reducing the structure group of $\mathcal{A}$ to $\text{Aut}_0(D \otimes \mathbb{K})$.

**Proposition 2.3.** A continuous field $\mathcal{A} \in \mathcal{C}_{D \otimes \mathbb{K}}(\mathcal{X})$ is orientable if and only if $\delta_0(\mathcal{A}) = 0$. If $\mathcal{X}$ is connected, then $\bar{E}_D^1(\mathcal{X}) \cong \ker(\delta_0)$.

**Proof.** Let us recall from [4, Cor. 2.19] that there is an exact sequence of topological groups

$$1 \rightarrow \text{Aut}_0(\mathcal{D} \otimes \mathbb{K}) \rightarrow \text{Aut}(\mathcal{D} \otimes \mathbb{K}) \xrightarrow{\pi} K_0(\mathcal{D})_{+}^{\mathbb{Z}} \rightarrow 1.$$ 

The map $\pi$ takes an automorphism $\alpha$ to $[\alpha(1_D \otimes e)]$ where $e \in \mathbb{K}$ is a rank-one projection. If $G$ is a topological group and $H$ is a normal subgroup of $G$ such that $H \rightarrow G \rightarrow G/H$ is a principal $H$-bundle, then there is a homotopy fibre sequence $G/H \rightarrow BH \rightarrow BG \rightarrow B(G/H)$ and hence an exact sequence of pointed sets $[\mathcal{X}, G/H] \rightarrow [\mathcal{X}, BH] \rightarrow [\mathcal{X}, BG] \rightarrow [\mathcal{X}, B(G/H)]$. In particular, in the case of the fibration $[\mathcal{X}, G/H]$ we obtain

$$[\mathcal{X}, K_0(\mathcal{D})_{+}^{\mathbb{Z}}] \rightarrow [\mathcal{X}, \text{BAut}_0(\mathcal{D} \otimes \mathbb{K})] \rightarrow [\mathcal{X}, \text{BAut}(\mathcal{D} \otimes \mathbb{K})] \xrightarrow{\delta_0} H^1(\mathcal{X}, K_0(\mathcal{D})_{+}^{\mathbb{Z}}).$$

A continuous field $\mathcal{A} \in \mathcal{C}_{D \otimes \mathbb{K}}(\mathcal{X})$ is associated to a principal $\text{Aut}(\mathcal{D} \otimes \mathbb{K})$-bundle whose classifying map gives a unique element in $[\mathcal{X}, \text{BAut}(\mathcal{D} \otimes \mathbb{K})]$ whose image in $H^1(\mathcal{X}, K_0(\mathcal{D})_{+}^{\mathbb{Z}})$ is denoted by $\delta_0(\mathcal{A})$. It is clear from (2) that the class $\delta_0(\mathcal{A}) \in H^1(\mathcal{X}, K_0(\mathcal{D})_{+}^{\mathbb{Z}})$ represents the obstruction for reducing this bundle to a principal $\text{Aut}_0(\mathcal{D} \otimes \mathbb{K})$-bundle. If $\mathcal{X}$ is connected, $[\mathcal{X}, K_0(\mathcal{D})_{+}^{\mathbb{Z}}] = \{\ast\}$ and hence $\bar{E}_D^1(\mathcal{X}) \cong \ker(\delta_0)$. \hfill $\square$

**Remark 2.4.** If $D = \mathbb{C}$ or $D = \mathbb{Z}$ then $\mathcal{A}$ is automatically orientable since in those cases $K_0(\mathcal{D})_{+}^{\mathbb{Z}}$ is the trivial group.

**Remark 2.5.** Let $Y$ be a compact metrizable space and let $\mathcal{X} = \Sigma Y$ be the suspension of $Y$. Since the rational Künneth isomorphism and the Chern character on $K^0(\mathcal{X})$ are compatible with the ring structure on $K_0(C(\mathcal{Y}) \otimes \mathcal{D})$, we obtain a ring homomorphism

$$\text{ch}: K_0(C(\mathcal{Y}) \otimes \mathcal{D}) \rightarrow K^0(\mathcal{Y}) \otimes K_0(D) \otimes \mathbb{Q} \rightarrow \prod_{k=0}^{\infty} H^{2k}(\mathcal{Y}, \mathbb{Q}) =: H^{\text{ev}}(\mathcal{Y}, \mathbb{Q}),$$

which restricts to a group homomorphism $\text{ch}: \bar{E}_D^0(\mathcal{Y}) \rightarrow SL_1(H^{\text{ev}}(\mathcal{Y}, \mathbb{Q}))$, where the right hand side denotes the units, which project to $1 \in H^0(\mathcal{Y}, \mathbb{Q})$. If $\mathcal{A}$ is an orientable locally trivial continuous field with fiber $D \otimes \mathbb{K}$ over $\mathcal{X}$, then we have

$$\delta_k(\mathcal{A}) = \log \text{ch}(f_A) \in H^{2k}(\mathcal{Y}, \mathbb{Q}) \cong H^{2k+1}(\mathcal{X}, \mathbb{Q}),$$

where $f_A: Y \rightarrow \Omega \text{BAut}_0(\mathcal{D} \otimes \mathbb{K}) \simeq \text{Aut}_0(\mathcal{D} \otimes \mathbb{K})$ is induced by the transition map of $\mathcal{A}$. The homomorphism $\log: SL_1(H^{\text{ev}}(\mathcal{Y}, \mathbb{Q})) \rightarrow H^{\text{ev}}(\mathcal{Y}, \mathbb{Q})$ is the rational logarithm from [14, Section 2.5].
Lemma 2.6. Let $D$ be a strongly self-absorbing $C^*$-algebra in the class $\mathcal{D}$. If $p \in D \otimes \mathbb{K}$ is a projection such that $[p] \neq 0$ in $K_0(D)$, then there is an integer $n \geq 1$ such that $p(D \otimes \mathbb{K}) p \cong M_n(D)$. Moreover, if $n, m \geq 1$, then $M_n(D) \cong M_m(D)$ if and only if $nK_0(D)_+^\otimes = mK_0(D)_+^\otimes$.

Proof. Recall that $K_0(D)$ is an ordered unital ring with unit $[1_D]$ and with positive elements $K_0(D)_+$ corresponding to classes of projections in $D \otimes \mathbb{K}$. The group of invertible elements is denoted by $K_0(D)_+$ and $K_0(D)_+^\otimes$ consists of classes $[p]$ of projections $p \in D \otimes \mathbb{K}$ such that $[p] \in K_0(D)_+^\otimes$. It was shown in [4] Lemma 2.14 that if $p \in D \otimes \mathbb{K}$ is a projection, then $[p] \in K_0(D)_+^\otimes$ if and only if $p(D \otimes \mathbb{K})p \cong D$. The ring $K_0(D)$ and the group $K_0(D)_+^\otimes$ are known for all $D \in \mathcal{D}$, [17].

To prove the second part of the lemma, suppose now that $\alpha : D \otimes M_n(\mathbb{C}) \to D \otimes M_m(\mathbb{C})$ is an isomorphism. Let $e \in M_n(\mathbb{C})$ be a rank one projection. Then $\alpha(1_D \otimes e)(D \otimes M_m(\mathbb{C})) \alpha(1_D \otimes e) \cong D$. By [4] Lemma 2.14 it follows that $\alpha([1_D]) = [\alpha(1_D \otimes e)] \in K_0(D)_+^\otimes$. Since $\alpha$ is unital, $\alpha_*([1_D]) = m[1_D]$ and hence $m[1_D] \in nK_0(D)_+^\otimes$. This is equivalent to $nK_0(D)_+^\otimes = mK_0(D)_+^\otimes$.

Conversely, suppose that $m[1_D] = nu$ for some $u \in K_0(D)_+^\otimes$. Let $\alpha \in \text{Aut}(D \otimes \mathbb{K})$ be such that $[\alpha(1_D \otimes e)] = u$. Then $\alpha_*([1_D]) = nu = m[1_D]$. This implies that $\alpha$ maps a corner of $D \otimes \mathbb{K}$ that is isomorphic to $M_n(D)$ to a corner that is isomorphic to $M_m(D)$. □

Corollary 2.7. Let $D \in \mathcal{D}$ and let $A \in \mathcal{C}_{D \otimes \mathbb{K}}(X)$ with $X$ a connected compact metrizable space. If $p \in A$ is a projection such that $[p(x_0)] \in K_0(D) \setminus \{0\}$ for some point $x_0$, then there is an integer $n \geq 1$ such that $(pAp)(x) \cong M_n(D)$ for all $x \in X$ and hence $pAp \in \mathcal{C}_{D \otimes M_n(\mathbb{C})}(X)$.

Proof. Let $V_1, \ldots, V_k$ be a finite cover of $X$ by compact sets such that there are bundle isomorphisms $\phi_i : A(V_i) \cong C(V_i) \otimes D \otimes \mathbb{K}$. Let $p_i$ be the image of the restriction of $p$ to $V_i$ under $\phi_i$. After refining the cover $(V_i)$, if necessary, we may assume that $\|p_i(x) - p_i(y)\| < 1$ for all $x, y \in V_i$. This allows us to find a unitary $u_i$ in the multiplier algebra of $C(V_i) \otimes D \otimes \mathbb{K}$ such that after replacing $\phi_i$ by $u_i \phi_i u_i^*$ and $p_i$ by $u_i p_i u_i^*$, we may assume that $p_i$ are constant projections. Since $X$ is connected and $[p(x_0)] \neq 0$, it follows that $[p_i(x)] \neq 0$ for $x \in V_i$. By Lemma 2.6 there are integers $n_i \geq 1$ such that $(pAp)(V_i) \cong C(V_i) \otimes M_{n_i}(D)$. Since $X$ is connected, we must have $M_{n_i}(D) \cong M_{n_j}(D)$ for all $1 \leq i, j \leq k$ and so $n := n_1$ has the desired properties. □
We study the image of the stabilization map
\[ \mathcal{E}_{D \otimes M_n(C)}(X) \to \mathcal{E}_{D \otimes K}(X) \]
induced by the map \( A \mapsto A \otimes K \), or equivalently by the map
\[ \text{Aut}(D \otimes M_n(C)) \to \text{Aut}(D \otimes M_n(C) \otimes K) \cong \text{Aut}(D \otimes K). \]

Let us recall that \( D \) denotes the class of strongly self-absorbing \( C^* \)-algebras which satisfy the UCT and which are either quasidiagonal or purely infinite.

**Theorem 2.8.** Let \( D \) be a strongly self-absorbing \( C^* \)-algebra in the class \( \mathcal{D} \). Let \( A \) be a locally trivial continuous field of \( C^* \)-algebras over a connected compact metrizable space \( X \) such that \( A(x) \cong D \otimes K \) for all \( x \in X \). The following assertions are equivalent:

1. \( \delta_k(A) = 0 \) for all \( k \geq 0 \).
2. The field \( A \otimes M_Q \) is trivial.
3. There is an integer \( n \geq 1 \) and a unital locally trivial continuous field \( \mathcal{B} \) over \( X \) with all fibers isomorphic to \( M_n(D) \) such that \( A \cong \mathcal{B} \otimes K \).
4. \( A \) is orientable and \( A \otimes \mathcal{O}_m \cong C(X) \otimes D \otimes K \) for some \( m \in \mathbb{N} \).

**Proof.** The statement is immediately verified if \( D \cong \mathcal{O}_2 \). Indeed all locally trivial fields with fiber \( \mathcal{O}_2 \otimes K \) are trivial since \( \text{Aut}(\mathcal{O}_2 \otimes K) \) is contractible by [4, Cor. 17 & Thm. 2.17]. For the remainder of the proof we may therefore assume that \( D \not\cong \mathcal{O}_2 \).

1. \( \leftrightarrow \) (2) If \( D \in \mathcal{D}_{qd} \), then it is known that \( D \otimes M_Q \cong M_Q \). Similarly, if \( D \in \mathcal{D}_{pi} \) and \( D \not\cong \mathcal{O}_2 \) then \( D \otimes M_Q \cong \mathcal{O}_\infty \otimes M_Q \). If \( A \) is as in the statement, then \( A \otimes M_Q \) is a locally trivial field whose fibers are all isomorphic to either \( M_Q \otimes K \) or to \( \mathcal{O}_\infty \otimes M_Q \otimes K \). In either case, it was shown in [4, Cor. 4.5] that such a field is trivial if and only if \( \delta_k(A) = 0 \) for all \( k \geq 0 \). As reviewed earlier in this section, this follows from the explicit computation of \( E_{M_Q}^1(X) \) and \( E_{M_Q^\infty \otimes \mathcal{O}_\infty}^1(X) \).

2. \( \Rightarrow \) (3) Assume now that \( A \otimes M_Q \) is trivial, i.e. \( A \otimes M_Q \cong C(X) \otimes D \otimes M_Q \otimes K \). Let \( p \in A \otimes M_Q \) be the projection that corresponds under this isomorphism to the projection \( 1 \otimes e \in C(X) \otimes D \otimes M_Q \otimes K \) where \( 1 \) is the unit of the \( C^* \)-algebra \( C(X) \otimes D \otimes M_Q \) and \( e \in K \) is a rank-one projection. Then \( [p(x)] \neq 0 \) in \( K_0(A(x) \otimes M_Q) \) for all \( x \in X \) (recall that \( D \not\cong \mathcal{O}_2 \)). Let us write \( M_Q \) as the direct limit of an increasing sequence of its subalgebras \( M_k(D) \). Then \( A \otimes M_Q \) is the direct limit of the sequence \( A_i = A \otimes M_k(D) \). It follows that there exist \( i \geq 1 \) and a projection \( p_i \in A_i \) such that \( \|p - p_i\| < 1 \). Then \( \|p(x) - p_i(x)\| < 1 \) and so \( [p_i(x)] \neq 0 \) in \( K_0(A_i(x)) \) for each \( x \in X \), since its image in \( K_0(A(x) \otimes M_Q) \) is equal to \( [p(x)] \neq 0 \). Let us consider the locally trivial unital field \( \mathcal{B} := p_i(A \otimes M_k(D))p_i \). Since the fibers of \( A \otimes M_k(D) \) are isomorphic to \( D \otimes K \otimes M_k(D) \), it follows by Corollary 2.7 that there is \( n \geq 1 \) such that all fibers of \( \mathcal{B} \) are isomorphic to \( M_n(D) \). Since \( \mathcal{B} \) is isomorphic to a full corner of \( A \otimes K \), it follows by [3] that \( A \otimes K \cong \mathcal{B} \otimes K \). We conclude by noting that since \( A \) is locally trivial and each fiber is stable, then \( A \cong A \otimes K \) by [9] and so \( A \cong \mathcal{B} \otimes K \).

3. \( \Rightarrow \) (2) This implication holds for any strongly self-absorbing \( C^* \)-algebra \( D \). Let \( A \) and \( \mathcal{B} \) be as in (3). Let us note that \( \mathcal{B} \otimes M_Q \) is a unital locally trivial field with all fibers isomorphic to the strongly self-absorbing \( C^* \)-algebra \( D \otimes M_Q \). Since \( \text{Aut}(D \otimes M_Q) \) is contractible by [4, Thm. 2.3], it follows that \( \mathcal{B} \otimes M_Q \) is trivial. We conclude that \( A \otimes M_Q \cong (\mathcal{B} \otimes M_Q) \otimes K \cong C(X) \otimes D \otimes M_Q \otimes K \).
Theorem 2.10. Let $D$ be a strongly self-absorbing $C^*$-algebra if $A$ is orientable. In particular we do not need to assume that $D$ satisfies the UCT. In the UCT case we note that since the map $K_0(D) \to K_0(D \otimes M_\mathbb{Q})$ is injective, it follows that $A$ is orientable if and only if $A \otimes M_\mathbb{Q}$ is orientable, i.e. $\delta_0(A) = 0$ if and only if $\delta_0^0(A) = 0$. Since $\delta_0(A) = 0$, $A$ is determined up to isomorphism by its class $[A] \in \tilde{E}_D^1(X)$. To complete the proof it suffices to show that the kernel of the map $\tau : \tilde{E}_D^1(X) \to \tilde{E}_D^{1}(\mathbb{Q})(X)$, $[A] = [A \otimes M_\mathbb{Q}]$, consists entirely of torsion elements. Consider the natural transformation of cohomology theories:

$$\tau \otimes \text{id}_\mathbb{Q} : \tilde{E}_D^1(X) \otimes \mathbb{Q} \to \tilde{E}_D^{1}(\mathbb{Q})(X) \otimes \mathbb{Q} \cong \tilde{E}_D^{1}(\mathbb{Q})(X).$$

If $D \not= \mathbb{C}$, it induces an isomorphism on coefficients since $\tilde{E}_D^{1}(pt) = \pi_1(\text{Aut}_0(D \otimes \mathbb{K})) \cong K_1(D)$ by [1] Thm.2.18 and since the map $K_1(D) \otimes \mathbb{Q} \to K_1(D \otimes M_\mathbb{Q})$ is bijective. We conclude that the kernel of $\tau$ is a torsion group. The same property holds for $D = \mathbb{C}$ since $\tilde{E}_D^{1}(X)$ is a direct summand of $\tilde{E}_D^{1}(\mathbb{Q})(X)$ by [1] Cor.3.8.

\[\square\]

**Definition 2.9.** Let $D$ be a strongly self-absorbing $C^*$-algebra. If $X$ is connected compact metrizable space we define the Brauer group $Br_D(X)$ as equivalence classes of continuous fields $A \in \bigcup_{n \geq 1} \mathscr{C}_{M_n(D)}(X)$. Two continuous fields $A_i \in \mathscr{C}_{M_n(D)}(X)$, $i = 1, 2$ are equivalent, if

$$A_1 \otimes p_1 C(X, M_{N_1}(D))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}(D))p_2,$$

for some full projections $p_i \in C(X, M_{N_i}(D))$. We denote by $[A]_{Br}$ the class of $A$ in $Br_D(X)$. The multiplication on $Br_D(X)$ is induced by the tensor product operation, after fixing an isomorphism $D \otimes D \cong D$. We will show in a moment that the monoid $Br_D(X)$ is a group.

One has the following generalization of a result of Serre, [5] Thm.1.6.

**Theorem 2.10.** Let $D$ be a strongly self-absorbing $C^*$-algebra in $D$.

(i) $\text{Tor}(\tilde{E}_D^1(X)) = \text{ker} \left( \tilde{E}_D^1(X) \xrightarrow{\delta_2} \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}) \right)$

(ii) The map $\theta : Br_D(X) \to \text{Tor}(\tilde{E}_D^1(X))$, $[A]_{Br} \mapsto [A \otimes \mathbb{K}]$ is an isomorphism of groups.

**Proof.** (i) was established in the last part of the proof of Theorem 2.8

(ii) We denote by $L_p$ the continuous field $p C(X, M_N(D))p$. Since $L_p \otimes \mathbb{K} \cong C(X, D \otimes \mathbb{K})$ it follows that the map $\theta$ is a well-defined morphism of monoids.

We use the following observation. Let $\theta : S \to G$ be a unital surjective morphism of commutative monoids with units denoted by 1. Suppose that $G$ is a group and that $\{s \in S : \theta(s) = 1\} = \{1\}$. Then $S$ is a group and $\theta$ is an isomorphism. Indeed if $s \in S$, there is $t \in S$ such that $\theta(t) = \theta(s)^{-1}$ by surjectivity of $\theta$. Then $\theta(st) = \theta(s)\theta(t) = 1$ and so $st = 1$. It follows that $S$ is a group and that $\theta$ is injective.

We are going to apply this observation to the map $\theta : Br_D(X) \to \text{Tor}(\tilde{E}_D^1(X))$. By condition (3) of Theorem 2.8 we see that $\theta$ is surjective. Let us determine the set $\theta^{-1}([0])$. We are going to show that if $B \in \mathscr{C}_{D \otimes M_n(C)}(X)$, then $[B \otimes \mathbb{K}] = 0$ in $\tilde{E}_D^1(X)$ if and only if

$$B \cong p(C(X) \otimes D \otimes M_N(C))p \cong \mathcal{L}_{C(X,D)}(p C(X, D)^N)$$

for some selfadjoint projection $p \in C(X) \otimes D \otimes M_N(C) \cong M_N(C(X, D))$. Let $B \in \mathscr{C}_{D \otimes M_n(C)}(X)$ be such that $[B \otimes \mathbb{K}] = 0$ in $\tilde{E}_D^1(X)$. Then there is an isomorphism of continuous fields $\phi :
$B \otimes K \xrightarrow{\cong} C(X) \otimes D \otimes K$. After conjugating $\phi$ by a unitary we may assume that $p := \phi(1_B \otimes e_{11}) \in C(X) \otimes D \otimes M_N(\mathbb{C})$ for some integer $N \geq 1$. It follows immediately that the projection $p$ has the desired properties. Conversely, if $B \cong p(C(X) \otimes D \otimes M_N(\mathbb{C}))$ $p$ then there is an isomorphism of continuous fields $B \otimes K \cong C(X) \otimes D \otimes K$ by \cite{3}. We have thus shown that that $\theta([B]_{Br}) = 0$ iff and only if $[B]_{Br} = 0$.

We are now able to conclude that $Br_D(X)$ is a group and that $\theta$ is injective by the general observation made earlier. \hfill $\square$

**Definition 2.11.** Let $D$ be a strongly self-absorbing $C^*$-algebra. Let $A$ be a locally trivial continuous field of $C^*$-algebras with fiber $D \otimes K$. We say that $A$ is a **torsion continuous field** if $A^\otimes k$ is isomorphic to a trivial field for some integer $k \geq 1$.

**Corollary 2.12.** Let $A$ be as in Theorem 2.8. Then $A$ is a torsion continuous field if and only if $\delta_0(A) \in H^1(X, K_0(D)_{\langle i \rangle})$ is a torsion element and $\delta_k(A) = 0 \in H^{2k+1}(X, \mathbb{Q})$ for all $k \geq 1$.

**Proof.** Let $m \geq 1$ be an integer such that $m\delta_0(A) = 0$. Then $\delta_0(A^\otimes m) = 0$. We conclude by applying Theorem 2.8 to the orientable continuous field $A^\otimes m$. \hfill $\square$

### 3. Characteristic classes of the opposite continuous field

Given a $C^*$-algebra $B$ denote by $B^{op}$ the **opposite $C^*$-algebra** with the same underlying Banach space and norm, but with multiplication given by $b^{op} \cdot a^{op} = (a \cdot b)^{op}$. The **conjugate $C^*$-algebra** $\overline{B}$ has the conjugate Banach space as its underlying vector space, but the same multiplicative structure. The map $a \mapsto a^*$ provides an isomorphism $B^{op} \to \overline{B}$. Any automorphism $\alpha \in \text{Aut}(B)$ yields in a canonical way automorphisms $\overline{\alpha} : \overline{B} \to \overline{B}$ and $\alpha^{op} : B^{op} \to B^{op}$ compatible with $\ast : B^{op} \to B$. Therefore we have group isomorphisms $\theta : \text{Aut}(B) \to \text{Aut}(\overline{B})$ and $\text{Aut}(B) \to \text{Aut}(B^{op})$. Note that $\alpha \in \text{Aut}(B)$ is equal to $\theta(\alpha)$ when regarded as set-theoretic maps $B \to B$. Given a locally trivial continuous field $A$ with fiber $B$, we can apply these operations fiberwise to obtain the locally trivial fields $A^{op}$ and $\overline{A}$, which we will call the **opposite** and the **conjugate field**. They are isomorphic to each other and isomorphic to the conjugate and the opposite $C^*$-algebras of $A$.

A **real form** of a complex $C^*$-algebra $A$ is a real $C^*$-algebra $A^R$ such that $A \cong A^R \otimes \mathbb{C}$. A real form is not necessarily unique \cite{2} and not all $C^*$-algebras admit real forms \cite{16}. If two $C^*$-algebras $A$ and $B$ admit real forms $A^R$ and $B^R$, then $A^R \otimes_{\mathbb{R}} B^R$ is a real form of $A \otimes B$.

**Example 3.1.** All known strongly self-absorbing $C^*$-algebras $D \in D$ admit a real form.

Indeed, the real Cuntz algebras $O_2^R$ and $O_\infty^R$ are defined by the same generators and relations as their complex versions. Alternatively $O_\infty^R$ can be realized as follows. Let $H_\mathbb{R}$ be a separable infinite dimensional real Hilbert space and let $F^R(H_\mathbb{R}) = \bigoplus_{n=0}^{\infty} H^\otimes_n$ be the real Fock space associated to it. Every $\xi \in H_\mathbb{R}$ defines a shift operator $s_\xi(\eta) = \xi \otimes \eta$ and we denote the algebra spanned by the $s_\xi$ and their adjoints $s_\xi^* \in O_\infty^R$. If $F(H_\mathbb{R} \otimes \mathbb{C})$ denotes the Fock space associated to the complex Hilbert space $H = H_\mathbb{R} \otimes \mathbb{C}$, then we have $F^R \otimes \mathbb{C} \cong F(H)$. If we represent $O_\infty$ on $F(H)$ using the above construction, then the map $s_\xi + i s_\xi^* \mapsto s_{\xi + i \xi'}$ induces an isomorphism $O_\infty^R \otimes \mathbb{C} \to O_\infty^R$. Likewise define $M^R_\infty$ to be the infinite tensor product $M_2(\mathbb{R}) \otimes M_3(\mathbb{R}) \otimes M_4(\mathbb{R}) \otimes \ldots$. Since $M_n(\mathbb{C}) \cong M_n(\mathbb{R}) \otimes \mathbb{C}$, we obtain an isomorphism $M^R_\infty \otimes \mathbb{C} \cong M^R_\infty$ on the inductive limit. Let $K^R$
be the compact operators on $H_R$ and $K$ those on $H$, then we have $K^R \otimes \mathbb{C} \cong K$. Thus, $M_Q \otimes O_\infty \otimes K$ is the complexification of the real $C^*$-algebra $M^R_Q \otimes O^R_\infty \otimes K^R$.

The Jiang-Su algebra $Z$ admits a real form $Z^\mathbb{R}$ which can be constructed in the same way as $Z$. Indeed, one constructs $Z^\mathbb{R}$ as the inductive limit of a system

$$\cdots \to C([0,1], M_{p_nq_n}(\mathbb{R})) \overset{\phi_n}{\to} C([0,1], M_{p_{n+1}q_{n+1}}(\mathbb{R})) \to \cdots$$

where the connecting maps $\phi_n$ are defined just as in the proof of [11] Prop. 2.5 with only one modification. Specifically, one can choose the matrices $u_0$ and $u_1$ to be in special orthogonal group $SO(p_nq_n)$ and this will ensure the existence of a continuous path $u_t$ in $O(p_nq_n)$ from $u_0$ to $u_1$ as required.

If $B$ is the complexification of a real $C^*$-algebra $B^\mathbb{R}$, then a choice of isomorphism $B \cong B^\mathbb{R} \otimes \mathbb{C}$ provides an isomorphism $c: B \to \overline{B}$ via complex conjugation on $\mathbb{C}$. On automorphisms we have $\text{Ad}_{c^{-1}}: \text{Aut}(\overline{B}) \to \text{Aut}(B)$. Let $\eta = \text{Ad}_{c^{-1}} \circ \theta: \text{Aut}(B) \to \text{Aut}(B)$. Now we specialize to the case $B = D \otimes K$ with $D \in \mathcal{D}$ and study the effect of $\eta$ on homotopy groups, i.e. $\eta_\ast: \pi_{2k}(\text{Aut}(B)) \to \pi_{2k}(\text{Aut}(B))$. By [3] Theorem 2.18 the groups $\pi_{2k+1}(\text{Aut}(B))$ vanish.

Let $R$ be a commutative ring and denote by $[K^0(S^{2k}) \otimes R]^\times$ be the group of units of the ring $K^0(S^{2k}) \otimes R$. Let $[K^0(S^{2k}) \otimes R]_1^\times$ be the kernel of the morphism of multiplicative groups $[K^0(S^{2k}) \otimes R]^\times \to R^\times$. This is the group of virtual rank 1 vector bundles with coefficients in $R$ over $S^{2k}$. Let $c_S: K^0(S^{2k}) \to K^0(S^{2k})$ and $c_R: K_0(D) \to K_0(D)$ be the ring automorphisms induced by complex conjugation.

**Lemma 3.2.** Let $D$ be a strongly self-absorbing $C^*$-algebra in the class $\mathcal{D}$, let $R = K_0(D)$ and let $k > 0$. There is an isomorphism $\pi_{2k}(\text{Aut}(D \otimes \mathbb{K})) \to [K^0(S^{2k}) \otimes R]_1^\times$ $(k > 0)$ such that the following diagram commutes

$$\begin{array}{ccc}
\pi_{2k}(\text{Aut}(D \otimes \mathbb{K})) & \xrightarrow{\eta_\ast} & \pi_{2k}(\text{Aut}(D \otimes \mathbb{K})) \\
\downarrow & & \downarrow \\
[K^0(S^{2k}) \otimes R]_1^\times & \xrightarrow{c_S \otimes c_R} & [K^0(S^{2k}) \otimes R]_1^\times
\end{array}$$

**Proof.** Observe that $\pi_{2k}(\text{Aut}(D \otimes \mathbb{K})) = \pi_{2k}(\text{Aut}_0(D \otimes \mathbb{K}))$ (for $k > 0$) and $\text{Aut}_0(D \otimes \mathbb{K})$ is a path connected group, therefore $\pi_{2k}(\text{Aut}(D \otimes \mathbb{K})) = [S^{2k}, \text{Aut}_0(D \otimes \mathbb{K})]$. Let $e \in \mathbb{K}$ be a rank 1 projection such that $c(1_D \otimes e) = 1_D \otimes e$. It follows from the proof of [3] Theorem 2.22 that the map $\alpha \mapsto \alpha(1 \otimes e)$ induces an isomorphism $[S^{2k}, \text{Aut}_0(D \otimes \mathbb{K})] \to K_0(C(S^{2k}) \otimes D)_{11}^\times = 1 + K_0(C_0(S^{2k} \setminus x_0) \otimes D)$. We have $\eta(\alpha)(1 \otimes e) = c^{-1}(\alpha(c(1 \otimes e))) = c^{-1}(\alpha(1 \otimes e))$, i.e. the isomorphism intertwines $\eta$ and $c^{-1}$.

Consider the following diagram of rings:

$$\begin{array}{ccc}
K^0(S^{2k}) \otimes R & \xrightarrow{c_S \otimes c_R} & K^0(S^{2k}) \otimes R \\
\downarrow & & \downarrow \\
K_0(C(S^{2k}) \otimes D) & \xrightarrow{p \mapsto c^{-1}(p)} & K_0(C(S^{2k}) \otimes D)
\end{array}$$
The vertical maps arise from the Künneth theorem. Since \( K_1(D) = 0 \), these are isomorphisms. Since \( c_S \) corresponds to the operation induced on \( K_0(C(S^{2k})) \) by complex conjugation on \( K \), the above diagram commutes.

**Remark 3.3.** (i) If \( D \in D \) then \( R = K_0(D) \subset \mathbb{Q} \) with \([1_D] = [1_{D^R}] = 1\). Thus \( c^{-1}(1_D) = 1_D \) and this shows that the above automorphism \( c_R \) is trivial. The \( K^0 \)-ring of the sphere is given by \( K^0(S^{2k}) \cong \mathbb{Z}[X_k]/(X_k^2) \). The element \( X_k \) is the \( k \)-fold reduced exterior tensor power of \( H-1 \), where \( H \) is the tautological line bundle over \( S^2 \cong \mathbb{C}P^1 \). Since \( c_S \) maps \( H-1 \) to \( 1-H \), it follows that \( X_k \) is mapped to \( -X_k \) if \( k \) is odd and to \( X_k \) if \( k \) is even. We have \([K^0(S^2) \otimes R]_1 = \{1+tX_k \mid t \in R\} \subset R[X_k]/(X_k^2) \). Thus, \( c_S \) maps \( 1+tX_k \) to its inverse \( 1-tX_k \) if \( k \) is odd and acts trivially if \( k \) is even.

(ii) By [4, Theorem 2.18] there is an isomorphism \( \pi_0(\text{Aut}(D \otimes \mathbb{K})) \cong K_0(D)^\times = R \) given by \([\alpha] \mapsto [\alpha(1 \otimes e)]\). Arguing as in Lemma 3.2 we see that the action of \( \eta \) on this groups is given by \( c_R \). id.

**Theorem 3.4.** Let \( X \) be a compact metrizable space and let \( A \) be a locally trivial continuous field with fiber \( D \otimes \mathbb{K} \) for a strongly self-absorbing \( C^* \)-algebra \( D \in D \). Then we have for \( k \geq 0 \):

\[
\delta_k(A^{op}) = \delta_k(\overline{A}) = (-1)^k \delta_k(A) \in H^{2k+1}(X, \mathbb{Q})
\]

**Proof.** Let \( D^R \) be a real form of \( D \). The group isomorphism \( \eta \) : \( \text{Aut}(D \otimes \mathbb{K}) \rightarrow \text{Aut}(D \otimes \mathbb{K}) \) induces an infinite loop map \( B\eta \) : \( \text{BAut}(D \otimes \mathbb{K}) \rightarrow \text{BAut}(D \otimes \mathbb{K}) \), where the infinite loop space structure is the one described in [4, Section 3]. If \( f : X \rightarrow \text{BAut}(D \otimes \mathbb{K}) \) is the classifying map of a locally trivial field \( A \), then \( B\eta \circ f \) classifies \( \overline{A} \). Thus the induced map \( \eta_* : E^1_{D}(X) \rightarrow E^1_{D}(X) \) has the property that \( \eta_*[A] = [\overline{A}] \).

The unital inclusion \( D^R \rightarrow B^R := D^R \otimes \mathcal{O}_\infty \otimes M^R \) induces a commutative diagram

\[
\begin{array}{ccc}
\text{Aut}(D \otimes \mathbb{K}) & \xrightarrow{\eta} & \text{Aut}(D \otimes \mathbb{K}) \\
\downarrow & & \downarrow \\
\text{Aut}(B \otimes \mathbb{K}) & \xrightarrow{\eta} & \text{Aut}(B \otimes \mathbb{K})
\end{array}
\]

with \( B := B^R \otimes \mathbb{C} \). From this we obtain a commutative diagram

\[
\begin{array}{ccc}
E^1_{D}(X) & \xrightarrow{\eta_*} & E^1_{D}(X) \\
\delta \downarrow & & \delta \downarrow \\
E^1_{B}(X) & \xrightarrow{\eta_*} & E^1_{B}(X)
\end{array}
\]

As explained earlier, \( B \cong M_S \otimes \mathcal{O}_\infty \). Recall that \( E^1_{M_S \otimes \mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q}) \). By Lemma 3.2 and Remark 3.3(i) the effect of \( \eta \) on \( H^{2k+1}(X, \pi_{2k}(\text{Aut}(B))) \cong H^{2k+1}(X, \mathbb{Q}) \) is given by multiplication with \((-1)^k\) for \( k > 0 \). By Remark 3.3(ii) \( \eta \) acts trivially on \( H^1(X, \pi_0(\text{Aut}(B))) = H^1(X, \mathbb{Q}^\times) \).

**Example 3.5.** Let \( Z \) be the Jiang-Su algebra. We will show that in general the inverse of an element in the Brauer group \( Br_Z(X) \) is not represented by the class of the opposite algebra. Let
$Y$ be the space obtained by attaching a disk to a circle by a degree three map and let $X_n = S^n \wedge Y$ be $n^{th}$ reduced suspension of $Y$. Then $E^2_2(X_3) \cong K^0(X_2)^+ \cong 1 + \tilde{K}^0(X_2)$ by [1] Thm.2.22. Since this is a torsion group, $Br_{Z}(X_3) \cong E^2_1(X_3)$ by Theorem 2.10. Using the Künneth formula, $Br_{Z}(X_3) \cong 1 + \tilde{K}^0(S^2) \otimes \tilde{K}^0(Y) \cong 1 + \mathbb{Z}/3$. Reasoning as in Lemma 3.2 with $X_3$ in place of $S^{2k}$, we identify the map $\eta_\ast : E^2_2(X_3) \to E^2_2(X_3)$ with the map $K^0(X_2)^+ \to K^0(X_2)^+$ that sends the class $x = [V_1] - [V_2]$ to $\bar{x} = [\bar{V}_1] - [\bar{V}_2]$, where $\bar{V}_i$ is the complex conjugate bundle of $V_i$. If $V$ is complex vector bundle, and $c_1$ is the first Chern class, $c_1(\bar{V}) = -c_1(V)$ by [10] p.206. Since conjugation is compatible with the Künneth formula, we deduce that $x = \bar{x}$ for $x \in K^0(X_2)^+$. Indeed, if $\beta \in \tilde{K}^0(S^2)$, $y \in \tilde{K}^0(Y)$ and $x = 1 + \beta y$, then $\bar{x} = 1 + (-\beta)(-y) = x$. Let $A$ be a continuous field over $X_3$ with fibers $M_N(\mathbb{Z})$ such that $[A]_{Br} = 1 + \beta y$ in $Br_{Z}(X_3) \cong 1 + \tilde{K}^0(S^2) \otimes \tilde{K}^0(Y) \cong 1 + \mathbb{Z}/3$, where $\beta$ a generator of $\tilde{K}^0(S^2)$ and $y$ a generator of $\tilde{K}^0(Y)$. Then $[A]_{Br} = 1 + (-\beta)(-y) = [A]_{Br}$ and hence $[\bar{A} \otimes_{C(X_3)} A]_{Br} = (1 + \beta y)^2 = 1 + 2\beta y \neq 1$.

Corollary 3.6. Let $X$ be a compact metrizable space and let $A$ be a locally trivial continuous field with fiber $D \otimes \mathbb{K}$ with $D$ in the class $D$. If $H^{4k+1}(X, \mathbb{Q}) = 0$ for all $k \geq 0$, then there is an $N \in \mathbb{N}$ such that

$$(A \otimes_{C(X)} A^{op})^N \cong C(X, D \otimes \mathbb{K}).$$

Proof. If $H^{4k+1}(X, \mathbb{Q}) = 0$, then $\delta_{2k}(A \otimes_{C(X)} A^{op}) = 0$ for all $k \geq 0$. Moreover, $\delta_{2k+1}(A \otimes_{C(X)} A^{op}) = \delta_{2k+1}(A) - \delta_{2k+1}(A) = 0$. The statement follows from Corollary 2.12.

References


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