QUASI-REPRESENTATIONS OF SURFACE GROUPS

JOSÉ R. CARRIÓN AND MARIUS DADARLAT

Abstract. By a quasi-representation of a group G we mean an approximately multiplicative map of G to the unitary group of a unital C^*-algebra. A quasi-representation induces a partially defined map at the level K-theory.

In the early 90s Exel and Loring associated two invariants to almost-commuting pairs of unitary matrices u and v: one a K-theoretic invariant, which may be regarded as the image of the Bott element in $K_0(C(\mathbb{T}^2))$ under a map induced by quasi-representation of $\mathbb{Z}^2$ in $U(n)$; the other is the winding number in $\mathbb{C} \setminus \{0\}$ of the closed path $t \mapsto \det(tvu + (1-t)uv)$. The so-called Exel-Loring formula states that these two invariants coincide if $\|uv - vu\|$ is sufficiently small.

A generalization of the Exel-Loring formula for quasi-representations of a surface group taking values $U(n)$ in was given by the second-named author. Here we further extend this formula for quasi-representations of a surface group taking values in the unitary group of a tracial unital C^*-algebra.

1. Introduction

Let G be a discrete countable group. In [3, 4] the second-named author studied the question of how deformations of the group G (or of the group C^*-algebra $C^*(G)$) into the unitary group of a (unital) C^*-algebra A act on the K-theory of the algebras $\ell^1(G)$ and $C^*(G)$. By a deformation we mean an almost-multiplicative map, a *quasi-representation*, which we will define precisely in a moment. Often, matrix-valued multiplicative maps are inadequate for detecting the K-theory of the aforementioned group algebras. In fact, if a countable, discrete, torsion free group G satisfies the Baum-Connes conjecture, a unital finite dimensional representation $\pi\colon C^*(G) \to M_r(\mathbb{C})$ induces the map $r \cdot \iota_*$ on $K_0(C^*(G))$, where ι is the trivial representation of G (see [3, Proposition 3.2]). It turns out that almost-multiplicative maps detect K-theory quite well for large classes of groups: one can interpolate any group homomorphism of $K_0(C^*(G))$ to $\mathbb{Z}$ on large swaths of $K_0(C^*(G))$ using quasi-representations (see [3, Theorem 3.3]).

Knowing that quasi-representations may be used to detect K-theory, we turn to how it is that they act. An index theorem of Connes, Gromov and Moscovici in [2] is very relevant to this topic, in the following context. Let M be a closed Riemannian manifold with fundamental group G and let D

Date: July 13, 2012. (Draft.)

M.D. was partially supported by NSF grant #DMS–1101305.

be an elliptic pseudo-differential operator on M. The equivariant index of D is an element of $K_0(\ell^1(G))$. Connes, Gromov and Moscovici showed that the push-forward of the equivariant index of D under a quasi-representation of G coming from parallel transport in an almost-flat bundle E over M is equal to the index of D twisted by E.

At around the same time, Exel and Loring studied two invariants associated to pairs of almost-commuting scalar unitary matrices $u, v \in U(r)$. One is a K-theory invariant, which may be regarded as the push-forward of the Bott element β in the K_0-group of $C(\mathbb{T}^2) \cong C^*(\mathbb{Z}^2)$ by a quasi-representation of $\mathbb{Z}^2$ into the unitary group $U(r)$. The Exel-Loring formula proved in [6] states that this invariant equals the winding number in $\mathbb{C} \setminus \{0\}$ of the path $t \mapsto \det\big((1-t)uv + tvu\big)$. An extension of this formula for almost commuting unitaries in a C^*-algebra of tracial rank one is due to to H. Lin and plays an important role in the classification theory of amenable C^*-algebras. In a different direction, the Exel-Loring formula was generalized in [4] to finite dimensional quasi-representations of a surface group using a variant of the index theorem of [2].

In [4], the second-named author used the Mishchenko-Fomenko index theorem to give a new proof and a generalization of the index theorem of Connes, Gromov and Moscovici that allows C^*-algebra coefficients. In this paper we use this generalization to address the question of how a quasi-representation π of a surface group in the unitary group of a tracial C^*-algebra acts at the level of K-theory. We extend the Exel-Loring formula to a surface group Γ_g (with canonical generators α_i, β_i) and coefficients in a unital C^*-algebra A with a trace τ. Briefly, writing $K_0(\ell^1(\Gamma_g)) \cong \mathbb{Z}[\iota] \oplus \mathbb{Z}\mu[\Sigma_g]$ we have

$$\tau\big(\mu[\Sigma_g]\big) = \frac{1}{2\pi i}\tau\Bigg(\log\Bigg(\prod_{i=1}^{g}\big[\pi(\alpha_i), \pi(\beta_i)\big]\Bigg)\Bigg),$$

where $[\Sigma_g]$ is the fundamental class in K-homology of the genus g surface Σ_g and $\mu\colon K_0(\Sigma_g) \to K_0(\ell^1(G))$ is the ℓ^1-version of the assembly map of Lafforgue. For a complete statement see Theorem 2.6. In the proof we make use of Chern-Weil theory for connections on Hilbert A-module bundles as developed by Schick [12] and the de la Harpe-Skandalis determinant [5] to calculate the first Chern class of an almost-flat Hilbert module C^*-bundle associated to a quasi-representation (Theorem 5.4).

The paper is organized as follows. In Section 2 we define quasi-representations and the invariants we are interested in, and state out main result, Theorem 2.6. The invariants make use of the Mishchenko line bundle, which we discuss in Section 3. The push-forward of this bundle by a quasi-representation is considered in Section 4. Section 5 contains our main technical result, Theorem 5.4, which computes one of our invariants in terms of the de la Harpe-Skandalis determinant [5]. To obtain the formula given in the main result, we must work with concrete triangulations of oriented surfaces, and this is contained in Section 6. Assembling these results in Section 7 yields a proof of Theorem 2.6.

2. The main result

In this section we state our main result. It depends on a result in [4] that we revisit. Let us provide some notation and definitions first.

Let G be a discrete countable group and A a unital C^*-algebra.

Definition 2.1. Let $\varepsilon > 0$ and let $\mathcal{F}$ be a finite subset of G. An $(\mathcal{F}, \varepsilon)$-*representation* of G in $U(A)$ is a function $\pi\colon G \to U(A)$ such that for all $s, t \in \mathcal{F}$ we have

- $\pi(1) = 1$,
- $\|\pi(s^{-1}) - \pi(s)^*\| < \varepsilon$, and
- $\|\pi(st) - \pi(s)\pi(t)\| < \varepsilon$.

We refer to the second condition by saying that π is $(\mathcal{F}, \varepsilon)$-multiplicative. Let us note that the second condition follows from the other two if we assume that $\mathcal{F}$ is symmetric, i.e. $\mathcal{F} = \mathcal{F}^{-1}$. A *quasi-representation* is an $(\mathcal{F}, \varepsilon)$-representation where $\mathcal{F}$ and ε are not necessarily specified.

A quasi-representation $\pi\colon G \to U(A)$ induces a map (also denoted π) of the Banach algebra $\ell^1(G)$ to A by $\sum \lambda_s s \mapsto \sum \lambda_s \pi(s)$. This map is a unital linear contraction. We also write π for the extension of π to matrix algebras over $\ell^1(G)$.

2.2. **Pushing-forward via quasi-representations.** A group homomorphism $\pi\colon G \to U(A)$ induces a map $\pi_*\colon K_0(\ell^1(G)) \to K_0(A)$ (via its Banach algebra extension). We think of a quasi-representation π as inducing a partially defined map $\pi_\sharp$ at the level of K-theory, in the following sense. If e is an idempotent in some matrix algebra over $\ell^1(G)$ such that $\|\pi(e) - \pi(e)^2\| < 1/4$, then the spectrum of $\pi(e)$ is disjoint from the line $\{\operatorname{Re} z = 1/2\}$. Writing χ for the characteristic function of $\{\operatorname{Re} z > 1/2\}$, it follows that $\chi(\pi(e))$ is a idempotent and we set

$$\pi_\sharp(e) = \big[\chi(\pi(e))\big] \in K_0(A).$$

For an element x in $K_0(\ell^1(G))$, we make a choice of idempotents e_0 and e_1 in some matrix algebra over $\ell^1(G)$ such that $x = [e_0] - [e_1]$. If $\|\pi(e_i) - \pi(e_i)^2\| < 1/4$ for $i \in \{0, 1\}$, write $\pi_\sharp(x) = \pi_\sharp(e_0) - \pi_\sharp(e_1)$. The choice of idempotents is largely inconsequential: given two choices of representatives one finds that if π is multiplicative enough, then both choices yield the same element of $K_0(A)$.

Of course, the more multiplicative π is, the more elements of $K_0(\ell^1(G))$ we can push-forward into $K_0(A)$.

2.3. **An index theorem.** Fix a closed oriented Riemannian surface M and let G be its fundamental group. Fix also a unital C^*-algebra A with a tracial state τ. Write $K_0(M)$ for $KK(C(M),\mathbb{C})$. Because the assembly map $\mu\colon K_0(M) \to K_0(\ell^1(G))$ is known to be an isomorphism in this case [9], we have

$$K_0(\ell^1(G)) \cong \mathbb{Z}[1] \oplus \mathbb{Z}\mu[M]$$

where $[M]$ is the fundamental class of M in $K_0(M)$ [1, Lemma 7.9]. Since we are interested in how a quasi-representation of G acts on $K_0(\ell^1(G))$, we would like to study push-forward of the generator $\mu[M]$ by a quasi-representation.

2.3.1. Let M be a closed connected orientable manifold with fundamental group G. Consider the universal cover $\widetilde{M} \to M$ and the diagonal action of G on $\widetilde{M} \times \ell^1(G)$ giving rise to the so-called Mishchenko line bundle ℓ, $\widetilde{M} \times_G \ell^1(G) \to G$. We will discuss it in more detail in Section 3, where we will give a description of it as the class of a specific idempotent e in some matrix algebra over $C(M) \otimes \ell^1(G)$.

If π is a quasi-representation of G in $U(A)$, then $\mathrm{id}_{C(M)} \otimes \pi$ is an almost-multiplicative unital linear contraction on $C(M) \otimes \ell^1(G)$ with values in $C(M) \otimes A$. Assuming that π is sufficiently multiplicative, we may define the push-forward of the idempotent e by $\mathrm{id}_{C(M)} \otimes \pi$, just as in 2.2. We set

$$\ell_\pi := (\mathrm{id}_{C(M)} \otimes \pi)_\sharp(\ell) := (\mathrm{id}_{C(M)} \otimes \pi)_\sharp(e) \in K_0(C(M) \otimes A).$$

Let D be an elliptic operator on M^n and let $\mu[D] \in K_0(\ell^1(G))$ be its image under the assembly map. Let q_0 and q_1 be idempotents in some matrix algebra over $\ell^1(G)$ such that $\mu[D] = [q_0] - [q_1]$ and write $\pi_\sharp(\mu[D]) := \pi_\sharp(q_0) - \pi_\sharp(q_1)$. By [4, Corollary 3.8], if $\pi : G \to A$ is sufficiently multiplicative, then

$$\tau(\pi_\sharp(\mu[D]) = (-1)^{n(n+1)/2}\big\langle (p_! \operatorname{ch}(\sigma(D)) \cup \operatorname{Td}(TM \otimes \mathbb{C}) \cup \operatorname{ch}_\tau(\ell_\pi), [M]\big\rangle, \tag{1}$$

where $p\colon TM \to M$ is the canonical projection, $\operatorname{ch}(\sigma(D))$ is the Chern character of the symbol of D, $\operatorname{Td}(T_{\mathbb{C}}M)$ is the Todd class of the complexified tangent bundle, and $[M]$ is the fundamental homology class of M. Set $\alpha = (-1)^{n(n+1)/2}(p_! ch(\sigma(D)) \cup \operatorname{Td}(TM \otimes \mathbb{C})$. Then (1) becomes

$$\tau(\pi_\sharp(\mu[D]) = \big\langle \alpha \cup \operatorname{ch}_\tau(\ell_\pi), [M]\big\rangle = \big\langle \operatorname{ch}_\tau(\ell_\pi), \alpha \cap [M]\big\rangle,$$

On the other hand, it follows from the Atiyah-Singer index theorem that the Chern character in homology $ch : K_0(M) \to H_*(M;\mathbb{Q})$ is given by

$$\operatorname{ch}[D] = \big((-1)^{n(n+1)/2} p_! \operatorname{ch}(\sigma(D)) \cup \operatorname{Td}(T_{\mathbb{C}}M)\big) \cap [M] = \alpha \cap [M].$$

It follows that

$$\tau(\pi_\sharp(\mu[D]) = \big\langle \operatorname{ch}_\tau(\ell_\pi), \operatorname{ch}[D]\big\rangle \tag{2}$$

In the case of surfaces this formula specializes to the following statement.

Theorem 2.4 (cf. [4, Corollary 3.8])**.** *Let M be a closed oriented Riemannian surface of genus g with fundamental group G. Let q_0 and q_1 be idempotents in some matrix algebra over $\ell^1(G)$ such that $\mu[M] = [q_0] - [q_1]$. Then there exist a finite subset $\mathcal{G}$ of G and $\omega > 0$ satisfying the following.*

Let A be a unital C^*-algebra with a tracial state τ and let $\pi\colon G \to U(A)$ be a $(\mathcal{G},\omega)$-representation. Write $\pi_\sharp(\mu[M]) := \pi_\sharp(q_0) - \pi_\sharp(q_1)$. Then

$$\tau\big(\pi_\sharp(\mu[M])\big) = \langle \mathrm{ch}_\tau(\ell_\pi), [M]\rangle.$$

Here $\mathrm{ch}_\tau\colon K_0\big(C(M)\otimes A\big) \to H^2(M,\mathbb{R})$ is a Chern character associated to τ (see Section 5), and $[M] \in H_2(M,\mathbb{R})$ is the fundamental class of M.

Proof. Given another pair of idempotents q_0', q_1' in some matrix algebra over $\ell^1(G)$ such that $\mu[M] = [q_0'] - [q_1']$, there is an $\omega_0 > 0$ such that if $0 < \omega < \omega_0$, then for any $(\mathcal{G},\omega)$-representation π we have $\pi_\sharp(q_0) - \pi_\sharp(q_1) = \pi_\sharp(q_0') - \pi_\sharp(q_1')$. We are therefore free to prove the theorem for a convenient choice of idempotents.

It is known that the fundamental class of M in $K_0(M)$ coincides with $[\bar\partial_g] + (g-1)[\iota]$ where $\bar\partial_g$ is the Dolbeault operator on M and $\iota\colon C(M) \to \mathbb{C}$ is a character (see [1, Lemma 7.9]). Let e_0, e_1, f_0, f_1 be idempotents in some matrix algebra over $\ell^1(G)$ such that

$$\mu[\bar\partial_g] = [e_0] - [e_1] \quad \text{and} \quad \mu[\iota] = [f_0] - [f_1].$$

(This gives an obvious choice of idempotents q_0' and q_1' in some matrix algebra over $\ell^1(G)$ so that $\mu[M] = [q_0'] - [q_1']$.) We want to prove that

$$\tau\big(\pi_\sharp(\mu(z))\big) = \langle \mathrm{ch}_\tau(\ell_\pi), \mathrm{ch}(z)\rangle$$

for $z = [M] \in K_0(M)$. Because of the additivity of this last equation, the fact that $[M] = [\bar\partial_g] + (g-1)[\iota]$, and (2), it is enough to prove that

$$\tau\big(\pi_\sharp(\mu[\iota])\big) = \langle \mathrm{ch}_\tau(\ell_\pi), \mathrm{ch}[\iota]\rangle. \tag{3}$$

By [4, Corollary 3.5]

$$\tau\big(\pi_\sharp(\mu[\iota])\big) = \tau\big(\langle \ell_\pi, [\iota]\otimes 1_A\rangle\big)$$

We can represent ℓ_π by a projection f in matrices over $C(M,A)$. The definition of the Kasparov product implies that

$$\langle [\ell_\pi], [\iota]\otimes 1_A\rangle = \iota_*[f] = [f(x_0)] \in K_0(A).$$

On the other hand, the definition of ch_τ (see [12, Definition 4.1]) implies that $\mathrm{ch}_\tau(f) = \tau(f(x_0))$ + a term in $H^2(M,\mathbb{R})$. Since $\mathrm{ch}[\iota] = 1 \in H_0(M,\mathbb{R})$, we get

$$\langle \mathrm{ch}_\tau(f), \mathrm{ch}[\iota]\rangle = \tau(f(x_0)). \tag{4}$$

□

2.5. **Statement of the main result.** We will often write Σ_g for the closed oriented surface of genus g and Γ_g for its fundamental group. It is well known that Γ_g has a standard presentation

$$\Gamma_g = \left\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \;\middle|\; \prod_{i=1}^{g} [\alpha_i, \beta_i] \right\rangle,$$

where we write $[\alpha, \beta]$ for the multiplicative commutator $\alpha\beta\alpha^{-1}\beta^{-1}$.

Our main result is the following.

Theorem 2.6. *Let $g \geq 1$ be an integer and let q_0 and q_1 be idempotents in some matrix algebra over $\ell^1(\Gamma_g)$ such that $\mu[\Sigma_g] = [q_0] - [q_1] \in K_0(\ell^1(\Gamma_g))$. There exists $\varepsilon_0 > 0$ and a finite subset $\mathcal{F}_0$ of Γ_g such that for every $0 < \varepsilon < \varepsilon_0$ and every finite subset $\mathcal{F} \supseteq \mathcal{F}_0$ of Γ_g the following holds.*

If A is a unital C^*-algebra with a trace τ and $\pi \colon \Gamma_g \to U(A)$ is an $(\mathcal{F}, \varepsilon)$-representation, then

$$\tau\big(\pi_\sharp(\mu[\Sigma_g])\big) = \frac{1}{2\pi i} \tau\left(\log\left(\prod_{i=1}^{g} \big[\pi(\alpha_i), \pi(\beta_i)\big] \right)\right), \tag{5}$$

where $\pi_\sharp(\mu[\Sigma_g]) := \pi_\sharp(q_0) - \pi_\sharp(q_1)$.

The rest of the paper is devoted to the proof.

Remark 2.7. The case $g = 1$ recovers the Exel-Loring formula as well as its extension by H. Lin [10] for C^*-algebras of tracial rank one. Lin's strategy was a reduction to the finite-dimensional case of [6] using approximation techniques.

The following proposition says that we may associate quasi-representations with unitaries that nearly satisfy the group relation. The proof is in Section 7.

Proposition 2.8. *For every $\varepsilon > 0$ and every finite subset $\mathcal{F}$ of Γ_g there is a $\delta > 0$ such that if A is a unital C^*-algebra with a trace τ and $u_1, v_1, ..., u_g, v_g$ are unitaries in A satisfying*

$$\left\| \prod_{i=1}^{g} [u_i, v_i] - 1 \right\| < \delta,$$

then there exists an $(\mathcal{F}, \varepsilon)$-representation $\pi \colon \Gamma_g \to U(A)$ with $\pi(\alpha_i) = u_i$ and $\pi(\beta_i) = v_i$, for all $i \in \{1, \dots, g\}$.

Example. To revisit a classic example, consider the noncommutative 2-torus A_θ, regarded as the universal C^*-algebra generated by unitaries u and v with $[v, u] = e^{2\pi i\theta} \cdot 1$. This is a tracial unital C^*-algebra. If θ is small enough, we may apply Proposition 2.8 and Theorem 2.6 to obtain

$$\tau(\pi_\sharp(\beta)) = \frac{1}{2\pi i} \tau(\log e^{-2\pi i\theta}) = -\theta$$

where $\beta \in K_0(C(\mathbb{T}^2))$ is the Bott element, τ is a unital trace of A_θ, and $\pi \colon \mathbb{Z}^2 \to U(A_\theta)$ is a quasi-representation obtained from Proposition 2.8.

3. The Mishchenko line bundle

Recall our setup: M is a closed oriented surface with fundamental group G and universal cover $p\colon \widetilde{M} \to M$. In this section we give a picture of the Mishchenko line bundle that will enable us to explicitly describe its push-forward by a quasi-representation.

The Mishchenko line bundle is the bundle $\widetilde{M} \times_G \ell^1(G) \to M$, obtained from $\widetilde{M} \times \ell^1(G)$ by passing to the quotient with respect to the diagonal action of G. We write ℓ for its class in $K_0(C(M) \otimes \ell^1(G))$.

3.1. **Triangulations and the edge-path group.** We adapt a construction found in the appendix of [11]. It is convenient to work with a triangulation Λ of M. Let $\Lambda^{(0)} = \{x_0, \dots, x_{N-1}\}$ be the 0-skeleton of Λ and $\Lambda^{(1)}$ be the 1-skeleton. To each edge we assign an element of G as follows. Fix a root vertex x_0 and a maximal (spanning) tree T in Λ. Let γ_i be the unique path along T from x_0 to x_i, and for two adjacent vertices x_i and x_j let $x_i x_j$ be the (directed) edge from x_i to x_j. For two such adjacent vertices, write $s_{ij} \in G$ for the class of the loop $\gamma_i * x_i x_j * \gamma_j^{-1}$.

Let $\mathcal{F}$ be the (finite) set $\{s_{ij}\}$. For example, if $M = \mathbb{T}^2$ so that $G = \mathbb{Z}^2 = \langle \alpha, \beta : [\alpha, \beta] = 1 \rangle$, we have $\mathcal{F} = \{1, \alpha^{\pm 1}, \beta^{\pm 1}, (\alpha\beta)^{\pm 1}\}$ for the triangulation and tree pictured in Figure 3 (on page 19)

Definition 3.2. For a vertex x_{i_k} in a 2-simplex $\sigma = \langle x_{i_0}, x_{i_1}, x_{i_2} \rangle$ of Λ, define the *dual cell block* to x_{i_k}

$$U^{\sigma}_{i_k} := \left\{ \sum_{l=0}^{2} t_l x_{i_l} : t_l \geq 0, \sum_{l=0}^{2} t_l = 1, \text{ and } t_{i_k} \geq t_l \text{ for all } l \right\}.$$

Define the *dual cell* to the vertex $x_i \in \Lambda^{(0)}$ by

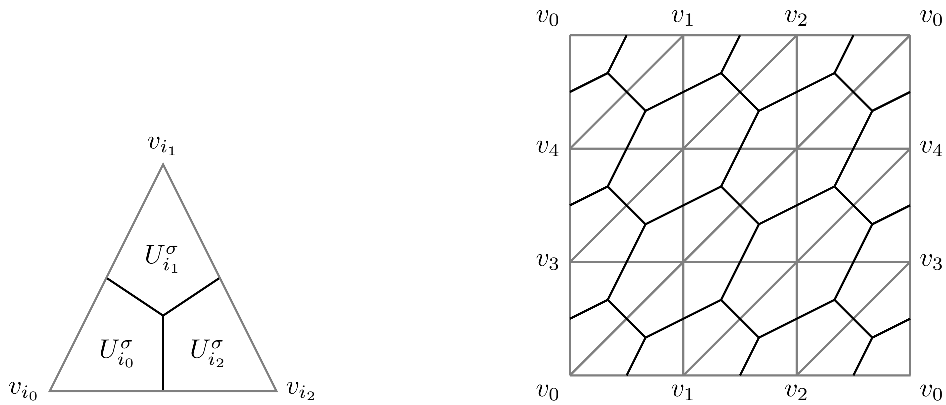
$$U_i = \cup \{ U^{\sigma}_i : x_i \in \sigma \}.$$

Let $U^{\sigma}_{ij} = U^{\sigma}_i \cap U^{\sigma}_j$ etc. (See Figure 1.)

Since $p\colon \widetilde{M} \to M$ is a covering space of M, we may fix an open cover of M such that for every element V of this cover, $p^{-1}(V)$ is a disjoint union of open subsets of $\widetilde{M}$, each of which is mapped homeomorphically onto V by p. We require that Λ be fine enough so that every dual cell U_i is contained in some element of this cover.

Lemma 3.3. *The Mishchenko line bundle $\widetilde{M} \times_G \ell^1(G) \to M$ is isomorphic to the bundle E obtained from the disjoint union $\bigsqcup U_i \times \ell^1(G)$ by identifying (x, a) with $(x, s_{ij} a)$ whenever $x \in U_i \cap U_j$.*

Proof. Lift x_0 to a vertex $\widetilde{x}_0$ in $\widetilde{M}$. By the unique path-lifting property, every path γ_i lifts (uniquely) to a path $\widetilde{\gamma}_i$ from $\widetilde{x}_0$ to a lift $\widetilde{x}_i$ of x_i. In this way lift T to a tree $\widetilde{T}$ in $\widetilde{M}$. Each U_i also lifts to a dual cell to $\widetilde{x}_i$, denoted $\widetilde{U}_i$, which p maps homeomorphically onto U_i.



(A) (B)

FIGURE 1. (A) Dual cell blocks in a simplex $\sigma = \langle v_{i_0}, v_{i_1}, v_{i_2}\rangle$. (B) A triangulation of $\mathbb{T}^2$ with the dual cell structure highlighted.

We first describe the cocycle (transition functions) for the Mishchenko line bundle. Identify the fundamental group G of M with the group of deck transformations of $\widetilde{M}$; see for example [7, Proposition 1.39]. Use this to write $p^{-1}(U_i)$ as the disjoint union $\sqcup\{s\widetilde{U}_i : s \in G\}$. Consider the isomorphism $\Phi_i \colon p^{-1}(U_i) \times_G \ell^1(G) \to U_i \times \ell^1(G)$ described by the following diagram:

$$\begin{array}{ccc} & & (s\tilde{x}, a) \longmapsto (p(\tilde{x}), s^{-1}a) \\ p^{-1}(U_i) \times \ell^1(G) & = & \bigsqcup_{s\in G} s\tilde{U}_i \times \ell^1(G) \longrightarrow U_i \times \ell^1(G) \\ \downarrow & & \nearrow \\ p^{-1}(U_i) \times_G \ell^1(G) & \overset{\Phi_i}{\dashrightarrow} & \end{array}$$

If $U_{ij} := U_i \cap U_j \neq \emptyset$, we obtain the cocycle $\phi_{ij} \colon U_{ij} \to \mathrm{Aut}(\ell^1(G))$:

$$U_{ij} \times \ell^1(G) \xrightarrow{\Phi_j^{-1}} p^{-1}(U_{ij}) \times_G \ell^1(G) \xrightarrow{\Phi_i} U_{ij} \times \ell^1(G)$$
$$(x, a) \longmapsto (x, \phi_{ij}(x)a)$$

Observe that $\widetilde{M} \times_G \ell^1(G)$ is isomorphic to the bundle obtained from the disjoint union $\bigsqcup U_i \times \ell^1(G)$ by identifying (x, a) with $(x, \phi_{ij}(x)a)$ whenever $x \in U_{ij}$. We only need to prove that ϕ_{ij} is constantly equal to s_{ij}.

Let $x \in U_{ij}$ and let $\widetilde{x} \in \widetilde{U}_j$ be a lift of x. Then $\Phi_j\big([\widetilde{x}, a]\big) = (x, a)$. Because $p(\widetilde{x}) \in U_{ij}$ there is a (unique) $s \in G$ such that $\widetilde{x} \in s\widetilde{U}_i \cap \widetilde{U}_j \neq \emptyset$. Thus $\Phi_i\big([\widetilde{x}, a]\big) = (x, s^{-1}a)$. Now, the path $s\widetilde{\gamma}_i * s\widetilde{x}_i\,\widetilde{x}_j * \widetilde{\gamma}_j^{-1}$, starts at $s\widetilde{x}_0$

and ends at $\widetilde{x}_0$. Its projection in M is the loop defining s_{ij}, so $s^{-1} = s_{ij}$ (see [7, Proposition 1.39], for example). Thus $\phi_{ij}(x) = s_{ij}$. □

3.4. **The push-forward of the line bundle.** We will need an open cover of M, so we dilate the dual cells U_i to obtain one. Let $0 < \delta < 1/2$ and define V_i^σ to be the δ-neighborhood of U_i^σ intersected with σ. As before, set $V_i = \bigcup_\sigma V_i^\sigma$. Let $\{\chi_i\}$ be a partition of unity subordinate to $\{V_i\}$.

By Lemma 3.3 the class of the Mishchenko line bundle in $K_0(C(M) \otimes \ell^1(G))$, denoted earlier by ℓ, corresponds to the class of the projection

$$e := \sum_{i,j} e_{ij} \otimes \chi_i^{1/2} \chi_j^{1/2} \otimes s_{ij} \in M_N(\mathbb{C}) \otimes C(M) \otimes \ell^1(G),$$

where $\{e_{ij}\}$ are the canonical matrix units of $M_N(\mathbb{C})$ and N is the number of vertices in Λ.

We may fix a pair of idempotents q_0 and q_1 in some matrix algebra over $\ell^1(G)$ satisfying $[q_0] - [q_1] = \mu[M] \in K_0(\ell^1(G))$. Let $\omega > 0$ be given by Theorem 2.4. (We may assume that $\omega < 1/4$.)

Fix $0 < \varepsilon < \omega$ and an $(\mathcal{F}, \varepsilon)$-representation $\pi \colon G \to U(A)$. We recall the following notation from the introduction.

Notation 3.5. For an $(\mathcal{F}, \varepsilon)$-representation $\pi \colon G \to U(A)$ as above, let

$$\ell_\pi := (\mathrm{id}_{C(M)} \otimes \pi)_\sharp(e).$$

4. Hilbert-module bundles and quasi-representations

As mentioned in the introduction, in [2] a quasi-representation (with scalar values) of the fundamental group of a manifold is associated to an "almost-flat" bundle over the manifold. In this section we instead define a canonical bundle E_π over M associated with quasi-representation π. Its class in $K_0(C(M) \otimes A)$ will be the class ℓ_π of the push-forward of the Mishchenko line bundle by π. Our construction will be explicit enough so that we can use Chern-Weil theory for such bundles to analyze $\mathrm{ch}_\tau(\ell_\pi)$, see [12].

Recall that A is a C^*-algebra with trace τ.

Definition. Let X be a locally compact Hausdorff space. A *Hilbert A-module bundle* W over X is a topological space W with a projection $W \to X$ such that the fiber over each point has the structure of a Hilbert A-module V, and with local trivializations $W|_U \xrightarrow{\sim} U \times V$ which are fiberwise Hilbert A-module isomorphisms.

We should point out that the K_0-group of the C^*-algebra $C(M) \otimes A$ is isomorphic to the Grothendieck group of isomorphism classes of finitely generated projective Hilbert A-module bundles over M. We identify the two groups.

4.1. **Constructing bundles.** We adapt a construction found in [11].

First we define a family of maps $\{u_{ij}\colon U_{ij} \to \mathrm{GL}(A)\}$ satisfying

$$\begin{aligned} u_{ji}(x) &= u_{ij}^{-1}(x), \qquad x \in U_{ij}, \\ u_{ik}(x) &= u_{ij}(x)u_{jk}(x), \qquad x \in U_{ijk}. \end{aligned}$$

These maps will be then extended to a cocycle defined on the collection $\{V_{ij}\}$.

Following [11] we will find it convenient to fix a partial order **o** on the vertices of Λ such that the vertices of each simplex form a totally ordered subset. We then call Λ a *locally ordered* simplicial complex. One may always assume such an order exists by passing to the first barycentric subdivision of Λ: if $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are the barycenters of simplices σ_1 and σ_2 of Λ, define $\hat{\sigma}_1 < \hat{\sigma}_2$ if σ_1 is a face of σ_2.

Consider a simplex $\sigma = \langle x_{i_0}, x_{i_1}, x_{i_2}\rangle$ (with vertices written in increasing **o**-order). Observe that in this case $U^\sigma_{i_0} \cap U^\sigma_{i_2} = U^\sigma_{i_0 i_2}$ may be described using a single parameter t_1:

$$U^\sigma_{i_0 i_2} = \left\{ \sum_{l=0}^{2} t_l x_{i_l} : t_0 = t_2 = \frac{1-t_1}{2} : 0 \le t_1 \le 1/3 \right\}.$$

Define

$$\begin{aligned} u^\sigma_{i_0 i_1} &= \text{ the constant function on } U^\sigma_{i_0 i_1} \text{ equal to } \pi(s_{i_0 i_1}) \\ u^\sigma_{i_1 i_2} &= \text{ the constant function on } U^\sigma_{i_1 i_2} \text{ equal to } \pi(s_{i_1 i_2}) \\ u^\sigma_{i_0 i_2}(t_1) &= (1-3t_1)\pi(s_{i_0 i_2}) + 3t_1 \pi(s_{i_0 i_1})\pi(s_{i_1 i_2}), \qquad 0 \le t_1 \le 1/3. \end{aligned}$$

Define $u^\sigma_{i_2 i_0}$ etc. to be the pointwise inverse of $u^\sigma_{i_0 i_2}$. For fixed i and j, the maps $u^\sigma_{ij}\colon U^\sigma_{ij} \to \mathrm{GL}(A)$ define a map $u_{ij}\colon U_{ij} \to \mathrm{GL}(A)$. Indeed, if $x_i x_j$ is a common edge of two simplices σ and σ', then $U^\sigma_{ij} \cap U^{\sigma'}_{ij}$ is the barycenter of $\langle x_i, x_j\rangle$, where by definition both u^σ_{ij} and $u^{\sigma'}_{ij}$ take the value $\pi(s_{ij})$. By construction the family $\{u_{ij}\}$ has the desired properties. (Note that U_{ijk} is the barycenter of a 2-simplex.)

4.1.1. Recall the sets V_i etc. from 3.4. To define the smooth transition function $v^\sigma_{i_0 i_2}\colon V^\sigma_{i_0 i_2} \to \mathrm{GL}(A)$ that will replace $u^\sigma_{i_0 i_2}$, let us assume for simplicity that the simplex σ is the triangle with vertices $v_{i_0} = (-1/2, 0)$, $v_{i_1} = (0,1)$, and $v_{i_2} = (1/2, 0)$. (It may be helpful to consider Figure 1A.)

Define $v^\sigma_{i_0 i_2}$ as follows:

$$v^\sigma_{i_0 i_2}(x,y) = \begin{cases} \pi(s_{i_0 i_1})\pi(s_{i_1 i_2}), & 1/3 - \delta \le y \le 1/3 + \delta \\ (1 - \frac{y}{1/3-\delta})\pi(s_{i_0 i_2}) + \\ \quad + \frac{y}{1/3-\delta}\pi(s_{i_0 i_1})\pi(s_{i_1 i_2}), & 0 \le y \le 1/3 - \delta \end{cases}$$

(so $v^\sigma_{i_0 i_2}$ is constant along the horizontal segments in $V_{i_0 i_2}$). The remaining two transition functions remain constant:

$$v^\sigma_{i_0 i_1} = \pi(s_{i_0 i_1})$$
$$v^\sigma_{i_1 i_2} = \pi(s_{i_1 i_2})$$

Again, for fixed i and j the maps $v^\sigma_{ij} \colon V^\sigma_{ij} \to \mathrm{GL}(A)$ define a map $v_{ij} \colon V_{ij} \to \mathrm{GL}(A)$. Since $v^\sigma_{i_0 i_2}$ is constant and equal to $\pi(s_{i_0 i_1})\pi(s_{i_1 i_2})$ in $V_{i_0} \cap V_{i_2} \cap V_{i_2}$, we indeed obtain a family $\{v_{ij}\}$ of transition functions.

Definition 4.2. The Hilbert A-module bundle E_π is constructed from the disjoint union $\bigsqcup V_i \times A$ by identifying (x, a) with $(x, v_{ij}(x)a)$ for x in V_{ij}.

Proposition 4.3. *The class of E_π in $K_0(C(M) \otimes A)$ coincides with ℓ_π, the class of the push-forward of e by $\mathrm{id}_{C(M)} \otimes \pi$ (see 3.4).*

Proof. The bundle E_π is a quotient of $\bigsqcup V_i \times A$ and from its definition it is clear that for each i the quotient map is injective on $V_i \times A$. The restriction of the quotient map to $V_i \times A$ has an inverse, call it ψ_i, and ψ_i is a trivialization of $E_\pi|_{V_i}$. Recalling that N is the number of vertices in Λ (which is the same as the number of sets V_i in the cover), we define an isometric embedding

$$\theta \colon E_\pi \to M \times A^N$$
$$[x, a] \mapsto \left(\chi_i^{1/2}(x)\psi_i([x, a])\right)_{i=0}^{N-1}.$$

Let $e_\pi \colon M \to M_N(A)$ be the function

$$x \mapsto \sum_{i,j} e_{ij} \otimes \chi_i^{1/2}(x)\chi_j^{1/2}(x)v_{ij}(x).$$

Because $\psi_i\psi_j^{-1}(x, a) = (x, v_{ij}(x)a)$ for $x \in V_{ij}$, it is easy to check that $e_\pi(x)$ is the matrix representing the orthogonal projection of A^N onto $\theta(E_\pi|_x)$. In this way we see that $[E_\pi] = [e_\pi] \in K_0(C(M) \otimes A)$.

Since $\mathcal{F} = \{s_{ij}\}$ and π is an $(\mathcal{F}, \varepsilon)$-representation, it follows immediately that the transition functions v_{ij} satisfy $\|v_{ij}(x) - \pi(s_{ij})\| < \varepsilon$ for all $x \in V_{ij}$. Thus

$$\|e_\pi - (1 \otimes \pi)(e)\| = \left\| e_\pi - \sum_{i,j} e_{ij} \otimes \chi_i^{1/2}\chi_j^{1/2}\pi(s_{ij}) \right\| < \varepsilon$$

as well. Recall that ℓ_π is obtained by perturbing $(1 \otimes \pi)(e)$ to a projection using functional calculus and then taking its K_0-class (see 2.2). The previous estimate shows that this class must be $[e_\pi]$. □

Remark 4.4. The previous proposition shows that the class $[E_\pi]$ is independent of the order $\mathbf{o}$ on the vertices of Λ_0.

4.5. **Connections arising from transition functions.** We now define a canonical connection on E_π associated with the family $\{v_{ij}\}$ of transition functions. This connection will be used in the proof of Theorem 5.4.

4.5.1. The smooth sections $\Gamma(E_\pi)$ of E_π may be identified with

$$\Big\{(s_i) \in \bigoplus_i \Omega^0(V_i, E_\pi) : s_j = v_{ji} s_i \text{ on } V_{ij}\Big\}.$$

Let $\nabla_i \colon \Omega^0(V_i, A) \to \Omega^1(V_i, A)$ be given by

$$\nabla_i(s) = ds + \omega_i s \qquad \forall s \in \Omega^0(V_i, A),$$

where

$$\omega_i = \sum_k \chi_k v_{ki}^{-1} dv_{ki}.$$

Notice that $v_{ki} \in \Omega^0(V_{ik}, \mathrm{GL}(A))$ and so ω_i may be regarded as an A-valued 1-form on V_i, which can be multiplied fiberwise by the values of the section s.

We define a connection ∇ on E_π by

$$\nabla(s_i) = (\nabla_i s_i).$$

That ∇ takes values in $\Omega^1(M, E_\pi)$ follows from a straightforward computation verifying

$$\nabla_j s_j = v_{ji} \nabla_i s_i.$$

It is just as straightforward to verify that ∇ is A-linear and satisfies the Leibniz rule.

4.5.2. Define $\Omega_i = d\omega_i + \omega_i \wedge \omega_i \in \Omega^2(V_i, A)$. One checks that $\Omega_i = v_{ji}^{-1} \Omega_j v_{ji}$ and so (Ω_i) defines an element Ω of $\Omega^2(M, \mathrm{End}_A(E_\pi))$. This is nothing but the curvature of ∇ (see [12, Proposition 3.8]).

5. The Chern character

In this section we prove our main technical result, Theorem 2.6. It computes the trace of the push-forward of $\mu[M]$ in terms of the de la Harpe-Skandalis determinant by using that the cocycle conditions almost hold for the elements $\pi(s_{ij})$,

5.1. **The de la Harpe-Skandalis determinant.** The de la Harpe-Skandalis determinant [5] appears in our formula below. Let us recall the definition. Write $\mathrm{GL}_\infty(A)$ for the (algebraic) inductive limit of $(\mathrm{GL}_n(A))_{n\geq 1}$ with standard inclusions. For a piecewise smooth path $\xi \colon [t_1, t_2] \to \mathrm{GL}_\infty(A)$, define

$$\widetilde{\Delta}_\tau(\xi) = \frac{1}{2\pi i} \tau\left(\int_{t_1}^{t_2} \xi'(t)\xi(t)^{-1} dt \right) = \frac{1}{2\pi i} \int_{t_1}^{t_2} \tau(\xi'(t)\xi(t)^{-1}) dt.$$

We will make use of some of the properties of $\widetilde{\Delta}_\tau$ stated below.

Lemma 5.2 (cf. Lemme 1 of [5])**.**

(1) *Let $\xi_1, \xi_2 \colon [t_1, t_2] \to \mathrm{GL}^0_\infty(A)$ be two paths and ξ be their pointwise product. Then $\widetilde{\Delta}_\tau(\xi) = \widetilde{\Delta}_\tau(\xi_1) + \widetilde{\Delta}_\tau(\xi_2)$.*

(2) *Let* $\xi\colon [t_1,t_2] \to \mathrm{GL}^0_\infty(A)$ *be a path with* $\|\xi(t)-1\| < 1$ *for all* t. *Then*

$$2\pi i \cdot \widetilde{\Delta}_\tau(\xi) = \tau\big(\log \xi(t_2)\big) - \tau\big(\log \xi(t_1)\big).$$

(3) *The integral* $\widetilde{\Delta}_\tau(\xi)$ *is left invariant under a fixed-end-point homotopy of* ξ.

5.3. **The Chern character on** $K_0(C(M)\otimes A)$**.** Assume τ is a trace on A. Then τ induces a map on $\Omega^2(V_i, \mathrm{End}_A(E_\pi|_{V_i}))$ and by the trace property $\tau(\Omega_i) = \tau(\Omega_j)$ on V_{ij}. We obtain in this way a globally defined form $\tau(\Omega) \in \Omega^2(M,\mathbb{C})$.

Since the fibers of our bundle are all equal to A, and our manifold is 2-dimensional, the definition of the Chern character associated with τ (from [12, Definition 4.1], but we have included a normalization coefficient) reduces to

$$\text{(6)}\quad \mathrm{ch}_\tau(\ell_\pi) = \tau\left(\exp\left(\frac{i\Omega}{2\pi}\right)\right) = \tau\left(\sum_{k=0}^{\infty} \frac{i\Omega/2\pi \wedge\cdots\wedge i\Omega/2\pi}{k!}\right) = \\ = \tau\left(\frac{i\Omega}{2\pi}\right) \in \Omega^2(M,\mathbb{C}).$$

This is a closed form whose cohomology class does not depend on the choice of the connection ∇ (see [12, Lemma 4.2]).

A few remarks are in order before stating the next result.

Because Λ is a locally ordered simplicial complex (recall the partial order **o** from 4.1), every 2-simplex σ may be written uniquely as $\langle x_i, x_j, x_k\rangle$ with the vertices written in increasing **o**-order. Whenever we write a simplex in this way it is implicit that the vertices are written in increasing **o**-order. We may write σ for σ along with this order.

The orientation $[M]$ induces an orientation of the boundary of the dual cell U_i and in particular of the segment U^σ_{ik}. Let $s(\sigma) = 0$ if the initial endpoint of U^σ_{ik} under this orientation is the barycenter of σ, and let $s(\sigma) = 1$ otherwise.

Theorem 5.4. *For a simplex* $\sigma = \langle x_i, x_j, x_k\rangle$ *of* Λ, *let* ξ_σ *be the linear path*

$$\xi_\sigma(t) = (1-t)\pi(s_{ik}) + t\pi(s_{ij})\pi(s_{jk}), \quad t\in[0,1]$$

in $\mathrm{GL}(A)$. *Then*

$$\tau\big(\pi_\sharp(\mu[M])\big) = \sum_\sigma (-1)^{s(\sigma)}\widetilde{\Delta}_\tau(\xi_\sigma),$$

where the sum ranges over all 2-simplices σ *of* Λ.

Proof. The path ξ_σ lies entirely in $\mathrm{GL}(A)$ because $\|\pi(s_{ik}) - \pi(s_{ij})\pi(s_{jk})\| < \varepsilon$. It follows from Theorem 2.4 (on page 5) and Equation (6) above that

$$\tau\big(\pi_\sharp(\mu[M])\big) = \langle \mathrm{ch}_\tau(\ell_\pi), [M]\rangle = -\frac{1}{2\pi i}\int_M \tau(\Omega).$$

We compute this integral.

First observe that by the trace property of τ we have $\tau(\omega_l \wedge \omega_l) = 0$ for every l. Thus

$$\int_M \tau(\Omega) = \sum_l \int_{U_l} \tau(\Omega_l) = \sum_l \int_{U_l} \tau(d\omega_l + \omega_l \wedge \omega_l) = \\ = \sum_l \int_{U_l} \tau(d\omega_l) = \sum_l \int_{U_l} d\tau(\omega_l) = \sum_l \int_{\partial U_l} \tau(\omega_l),$$

where we used Green's theorem for the last equality and ∂U_l has the orientation induced from $[M]$. Recall that U_l is the dual cell to v_l. Write this as a sum over the 2-simplices of Λ:

$$\sum_l \int_{\partial U_l} \tau(\omega_l) = \sum_l \sum_\sigma \int_{(\partial U_l)\cap\sigma} \tau(\omega_l) = \sum_\sigma \sum_l \int_{(\partial U_l)\cap\sigma} \tau(\omega_l).$$

Exactly three dual cells meet a 2-simplex $\sigma = \langle x_i, x_j, x_k\rangle$—$U_i$, U_j, and U_k—so for each simplex there are three integrals we need to account for. Let us treat each of these in turn.

The definition of the connection forms (see 4.5.1) implies that ω_i restricted to σ equals

$$\omega_i = \chi_k v_{ki}^{-1} dv_{ki} + \chi_j v_{ji}^{-1} dv_{ji} = \chi_k v_{ki}^{-1} dv_{ki},$$

where the last equality follows from the fact that v_{ji} is constant. Now, $(\partial U_i)\cap\sigma$ is the union of the two segments U_{ij}^σ and U_{ik}^σ. Observe that v_{ik} is constantly equal to $\pi(s_{ij})\pi(s_{jk})$ on $V_i \cap V_j \cap V_k$ (see 4.1.1). Since $U_{ij}^\sigma \cap V_k \subseteq V_i \cap V_j \cap V_k$ and χ_k vanishes outside V_k, we get

$$\int_{(\partial U_i)\cap\sigma} \tau(\omega_i) = \int_{U_{ij}^\sigma} \tau(\chi_k v_{ki}^{-1} dv_{ki}) + \int_{U_{ik}^\sigma} \tau(\chi_k v_{ki}^{-1} dv_{ki}) = \int_{U_{ik}^\sigma} \tau(\chi_k v_{ki}^{-1} dv_{ki}).$$

The second integral $\int_{(\partial U_j)\cap\sigma} \tau(\omega_j)$ vanishes. This is because v_{ij} and v_{jk} are constant and so

$$\omega_j = \chi_i v_{ij}^{-1} dv_{ij} + \chi_k v_{kj}^{-1} dv_{kj} = 0.$$

The third integral may be calculated just as the first, with the roles of i and k reversed. We obtain

$$\int_{(\partial U_k)\cap\sigma} \tau(\omega_k) = \int_{U_{ki}^\sigma} \tau(\chi_i v_{ik}^{-1} dv_{ik}).$$

Combining the three integrals we get

$$\sum_{\sigma}\sum_{l}\int\limits_{(\partial U_l)\cap\sigma}\tau(\omega_l)=\sum_{\sigma}\Bigg(\int\limits_{U_{ik}^{\sigma}}\tau(\chi_k v_{ki}^{-1}dv_{ki})+\int\limits_{U_{ki}^{\sigma}}\tau(\chi_i v_{ik}^{-1}dv_{ik})\Bigg)=$$

$$=\sum_{\sigma}\int\limits_{U_{ik}^{\sigma}}\tau(\chi_k v_{ki}^{-1}dv_{ki}-\chi_i v_{ik}^{-1}dv_{ik}),$$

where the last equality is due to the opposite orientations of the segment U_{ik}^{σ} in the preceding two integrals.

It follows from $v_{ik}v_{ki}=1$ that $dv_{ik}\,v_{ik}^{-1}+v_{ki}^{-1}dv_{ki}=0$. Therefore, the last line in the equation above is equal to

$$\sum_{\sigma}\int\limits_{U_{ik}^{\sigma}}\tau(\chi_k v_{ki}^{-1}dv_{ki}+\chi_i v_{ki}^{-1}dv_{ki})=\sum_{\sigma}\int\limits_{U_{ik}^{\sigma}}\tau(v_{ki}^{-1}dv_{ki})=-\sum_{\sigma}\int\limits_{U_{ik}^{\sigma}}\tau(v_{ik}^{-1}dv_{ik}).$$

To arrive at the conclusion of the theorem, consider the restriction of v_{ik} to the segment U_{ik}^{σ}. This is the segment between the barycenter of σ, where v_{ik} takes the value $\pi(s_{ij})\pi(s_{jk})$, and the barycenter of $\langle x_i,x_k\rangle$, where v_{ik} takes the value $\pi(s_{ik})$ (see 4.1.1). Then

$$\int\limits_{U_{ik}^{\sigma}}\tau(v_{ki}^{-1}dv_{ki})=(-1)^{s(\sigma)}2\pi i\cdot\widetilde{\Delta}_{\tau}(\xi_{\sigma}).$$

This concludes the proof. □

6. Oriented surfaces

For the proof of Theorem 2.6, we will use a convenient triangulation Λ_g of the orientable genus g surface Σ_g that we proceed to describe. The covering space of Σ_g is the open disc and we may take as a fundamental domain a regular $4g$-gon, call it $\widetilde{\Sigma}_g$, drawn in the hyperbolic plane.

Figure 2 depicts a procedure to obtain $\widetilde{\Sigma}_2$ by gluing together two copies of $\widetilde{\Sigma}_1$. (We will give a more explicit description of $\widetilde{\Sigma}_g$ in a moment). It also illustrates the labeling we use for the (oriented) sides of $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$. To obtain Σ_1, for example, we identify the side a with $*\mathsf{a}$ and the side b with $*\mathsf{b}$. To obtain the double torus Σ_2, we identify a_k with $*\mathsf{a}_k$ and b_k with $*\mathsf{b}_k$ for $k\in\{1,2\}$.

6.1. **Triangulations.** Let us first define a triangulation $\widetilde{\Lambda}_g$ of the fundamental domain $\widetilde{\Sigma}_g$. We do this by gluing g triangulated copies of $\widetilde{\Sigma}_1$ together. Figure 4A on page 20 shows the triangulation for the kth copy of $\widetilde{\Sigma}_1$ (with a hole), call it $\widetilde{\Lambda}_1^k$. Ignore the labels on the edges and the highlighted edges for now. The vertex labeling also indicates how to glue $\widetilde{\Lambda}_1^k$ to $\widetilde{\Lambda}_1^{k-1}$ and $\widetilde{\Lambda}_1^{k+1}$, with addition modulo g. Figure 4B illustrates the result of this gluing, the end-result being $\widetilde{\Lambda}_g$ by definition.

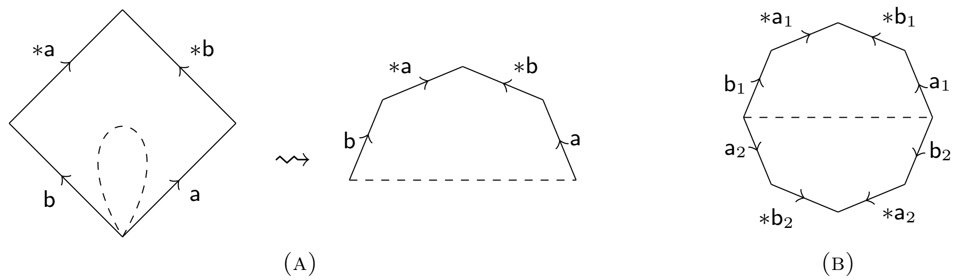


FIGURE 2. (A) The fundamental domain $\widetilde{\Sigma}_1$ (with a hole). (B) The fundamental domain $\widetilde{\Sigma}_2$.

The underlying space of $\widetilde{\Lambda}_g$ is $\widetilde{\Sigma}_g$. Identifying all the vertices v_i^k, as well as identifying a_i^k with $*a_i^k$ and b_i^k with $*b_i^k$, for each $i \in \{1,2\}$ and $k \in \{1,\dots,n\}$, yields a triangulation Λ_g of Σ_g.

6.2. **Surface groups.** We identify the fundamental group Γ_g of Σ_g with the group of deck transformations of the universal covering space of Σ_g. We give a more concrete description of this group now.

The fundamental domain $\widetilde{\Sigma}_g$ is a regular $4g$-gon. We write a_k, b_k $*\mathsf{a}_k$ and $*\mathsf{b}_k$, $k \in \{1,\dots,n\}$, for its (oriented) sides. The triangulation $\widetilde{\Lambda}_g$ gives a subdivision of the side a_k into the three edges in the path $(v_0^k, a_1^k, a_2^k, v_1^k)$ (with orientation given by the directed edge (a_1^k, a_2^k)). The subdivision of the sides b_k, $*\mathsf{a}_k$ and $*\mathsf{b}_k$ is similar. See Figure 4A.

The group of deck transformations Γ_g is generated by the hyperbolic isometries α_k and β_k, $k \in \{1,\dots,g\}$, defined as follows: α_k maps $*\mathsf{a}_k$ to a_k in such a way that, locally, the half-plane bounded by $*\mathsf{a}_k$ containing $\widetilde{\Sigma}_g$ is mapped to the half-plane bounded by a_k but opposite $\widetilde{\Sigma}_g$. The transformation β_k is defined analogously, mapping $*\mathsf{b}_k$ to b_k. We refer the reader to [8, Chapter VII] for more details. When $g = 1$, for example, the transformations α_1 and β_1 are just translations. See Figure 3, where we have omitted the sub- and superscripts corresponding to $k = 1$, since $g = 1$.

For $k \in \{1,\dots,g\}$, let

$$\kappa_k = \prod_{j=1}^{k} [\alpha_k, \beta_k]$$

and let $\kappa_0 = 1$. We have that $\kappa_g = 1$.

6.3. **Local orders and trees.** We need Λ_g to be locally ordered, so we proceed to fix a partial order on the vertices of Λ_g such that the vertices of every simplex form a totally ordered set. Let us define an order on the vertices of $\widetilde{\Lambda}_g$ that drops down to the order we need. On the kth copy $\widetilde{\Lambda}_1^k$, the corresponding order is indicated in Figure 4A by arrows on the edges, always pointing from a smaller vertex to a larger one. It is defined as follows:

- for the "inner" vertices we go "counter-clockwise": for fixed $k \in \{1, \dots, g\}$, $w_i^k < w_j^k$ if $i < j$, except when $k = g$ and $j = 4$ (in which case $w_4^g = w_0^1$ and we already have $w_0^1 < w_k^i$);
- the "inner" vertices are larger than the "outer" ones: $w_i^k > v_j^l$, a_j^l, b_j^l, $*a_j^l$, $*b_j^l$ for all i, j, k and l;
- for the "outer" vertices: $v_i^k < a_j^l, b_j^l, *a_j^l, *b_j^l$ for all i, j, k and l; for every k, $a_1^k < a_2^k$, $*a_1^k < *a_2^k$, and similarly for the b_j^k.

Finally, we will need a spanning tree T_g of Λ_g, and a lift $\widetilde{T}_g$ to the triangulation $\widetilde{\Lambda}_g$ of the fundamental domain $\widetilde{\Sigma}_g$. Again, we define $\widetilde{T}_g$ first. It is obtained as the union of the edge between w_0^1 and v_0^1 (including those two vertices) and trees in each copy Σ_1^k. The tree in Σ_1^k is depicted in Figure 4A by highlighted (heavier) edges. This drops to a spanning tree T_g of Σ_g. we regard these trees as "rooted" at the vertex v_0^k.

7. Proof of the main result

This section contains the proof of Theorem 2.6. The proof is split into a number of lemmas.

To apply Theorem 5.4 we will first compute the group element s_{ij} corresponding to each edge $x_i x_j$ of Λ_g , in the sense discussed in 3.1. Equivalently, we compute group elements corresponding to edges in the cover $\widetilde{\Lambda}_g$, keeping in mind that the lifts of any edge of Λ_g will all correspond to the same group element.

A concise way of stating the result of these computations is to label each edge in Figure 4A with the corresponding group element.

Lemma 7.1. *The labels in Figure 4A are correct.*

Proof. We carry out the computations in three separate claims.

Claim. An edge of the form $a_i^k w_j^k$ corresponds to $\alpha_k^{-1} \in \Gamma_g$. Similarly, an edge of the form $b_i^k w_j^k$ corresponds to $\beta_k^{-1} \in \Gamma_g$.

Consider $a_i^k w_j^k$ first. When we add this edge to the forest that is the union of all the lifts of T_g (that is, translates of $\widetilde{T}_g$), we obtain a unique path P between v_0^1, our root vertex, and some translate sv_0^1, where $s \in \Gamma_g$. We regard P as directed in the direction of the edge $a_i^k w_j^k$ that we started with, so it is a path from sv_0^1 to v_0^1. It therefore drops down to a loop in Σ_g whose class is s^{-1}, the group element we want to compute (see [7, Proposition 1.39], for example). Now notice that because $*a_i^k$ belongs to $\widetilde{T}_g$, its translate $\alpha_k(*a_i^k) = a_i^k$ belongs to the translate $\alpha_k \widetilde{T}_g$ of $\widetilde{T}_g$. Thus P is a path between v_0^k and $\alpha_k v_0^k$. The corresponding group element is therefore α_k^{-1}. An entirely similar argument applies to the edge $b_i^k w_j^k$.

Claim. Any edge between inner vertices (vertices of the form w_i^k) corresponds to $1 \in \Gamma_g$. The edges $a_1^k a_2^k$, $b_1^k b_2^k$, $*a_1^k *a_2^k$, and $*b_1^k *b_2^k$ all correspond to $1 \in \Gamma_g$.

We proceed as in the previous claim. Any edge between inner vertices is either in $\widetilde{T}_g$ or between two vertices that are in $\widetilde{T}_g$. The associated path we get is therefore from v_0^k to itself. The same is true of the edges $b_1^k b_2^k$ and $*a_1^k *a_2^k$. It follows that the corresponding group element is 1. Since $a_1^k a_2^k$ and $*a_1^k *a_2^k$ are both lifts of the same edge, they correspond to the same element. Similarly, $*b_1^k *b_2^k$ corresponds to 1.

Claim. An edge that is incident to v_i^k and to a vertex z in the tree $\widetilde{T}_g$ corresponds to the element $s \in \Gamma_g$ such that $v_0^1 = s v_i^k$. (The edge is given the orientation induced by the order on the vertices, as usual.) For $k \in \{1, \ldots, g\}$,

$$\begin{aligned} v_0^1 &= \kappa_{k-1} \cdot v_0^k \\ v_0^1 &= \kappa_{k-1} \alpha_k \beta_k \alpha_k^{-1} \cdot v_1^k \\ v_0^1 &= \kappa_{k-1} \alpha_k \beta_k \cdot v_2^k \\ v_0^1 &= \kappa_{k-1} \alpha_k \cdot v_3^k \end{aligned}$$

(Recall that κ_k is the product of commutators $[\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_k, \beta_k]$ for $k \in \{1, \ldots, g\}$, and that $\kappa_0 = 1$.)

Observe that, because of how the order was defined, $v_i^k < z$ always holds. When we add the edge $v_i^k z$ to the tree $\widetilde{T}_g$ we obtain a path from v_i^k to v_0^1. (See Figure 4, but keep in mind that in the case $k = 1$ the edge $v_0^1 w_0^1$ belongs to the tree.) It follows that the corresponding element is the $s \in \Gamma_g$ such that $v_0^1 = s v_i^k$.

To compute these elements s we argue by induction on k. Assume $k = 1$. We observe that

$$v_4^1 \xmapsto{\beta_1^{-1}} v_1^1 \xmapsto{\alpha_1^{-1}} v_2^1 \xmapsto{\beta_1} v_3^1 \xmapsto{\alpha_1} v_0^1.$$

Indeed, from the definition (see 6.2) we see that the transformation α_1 takes v_3^1 to v_0^1—think of the side $*\mathsf{a}_1 = (v_3^1, *a_1^1, *a_2^1, v_2^1)$ being mapped to the side $\mathsf{a}_1 = (v_0^1, a_1^1, a_2^1, v_1^1)$: the vertex $*a_1^1$ is mapped to a_1^1 and so v_3^1 is mapped to v_0^1. We also see from 6.2 and Figure 4A that β_1 maps v_2^1 to v_3^1, and so $v_0^1 = \alpha_1 \beta_1 \cdot v_2^1$. A similar argument shows that $v_0^1 = \alpha_1 \beta_1 \alpha_1^{-1} \cdot v_1^1$ and that

$$v_0^1 = \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdot v_4^1 = \kappa_1 \cdot v_4^1.$$

Assuming the computations hold for $k - 1$, we prove them for k. In fact, most of the work is already done. The same argument we used for the case $k = 1$ shows that

$$v_4^k \xmapsto{\beta_k^{-1}} v_1^k \xmapsto{\alpha_k^{-1}} v_2^k \xmapsto{\beta_k} v_3^k \xmapsto{\alpha_k} v_0^k.$$

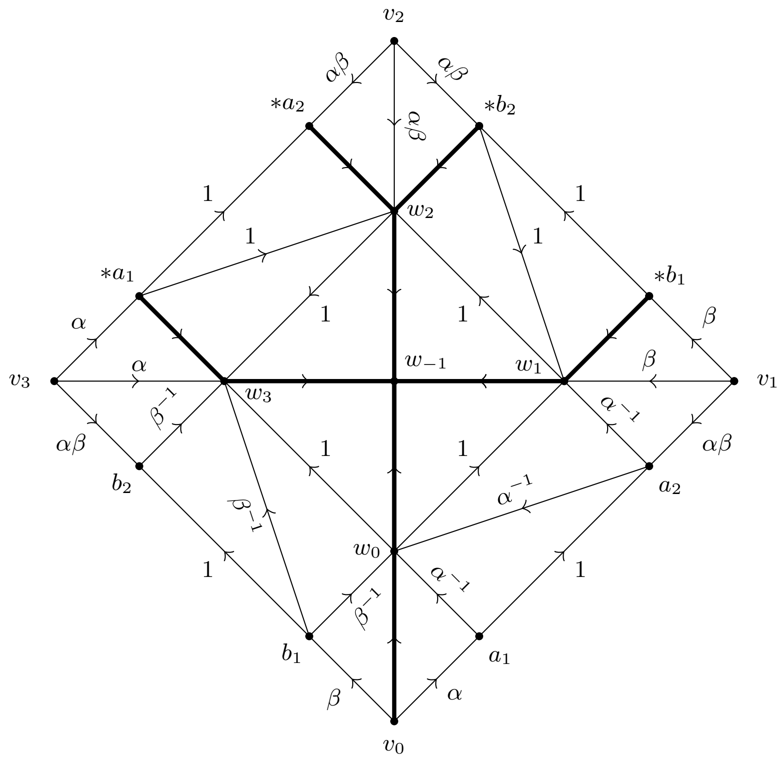


FIGURE 3. The triangulation $\widetilde{\Lambda}_1$ of $\widetilde{\Sigma}_1$. Edges are labeled with the group element associated with the loop they induce.

The inductive hypothesis implies that

$$\kappa_{k-1} v_0^k = \kappa_{k-1} v_4^{k-1} = v_0^1,$$

This ends the proof of the claim.

These three claims prove that the labels in Figure 4A are correct.

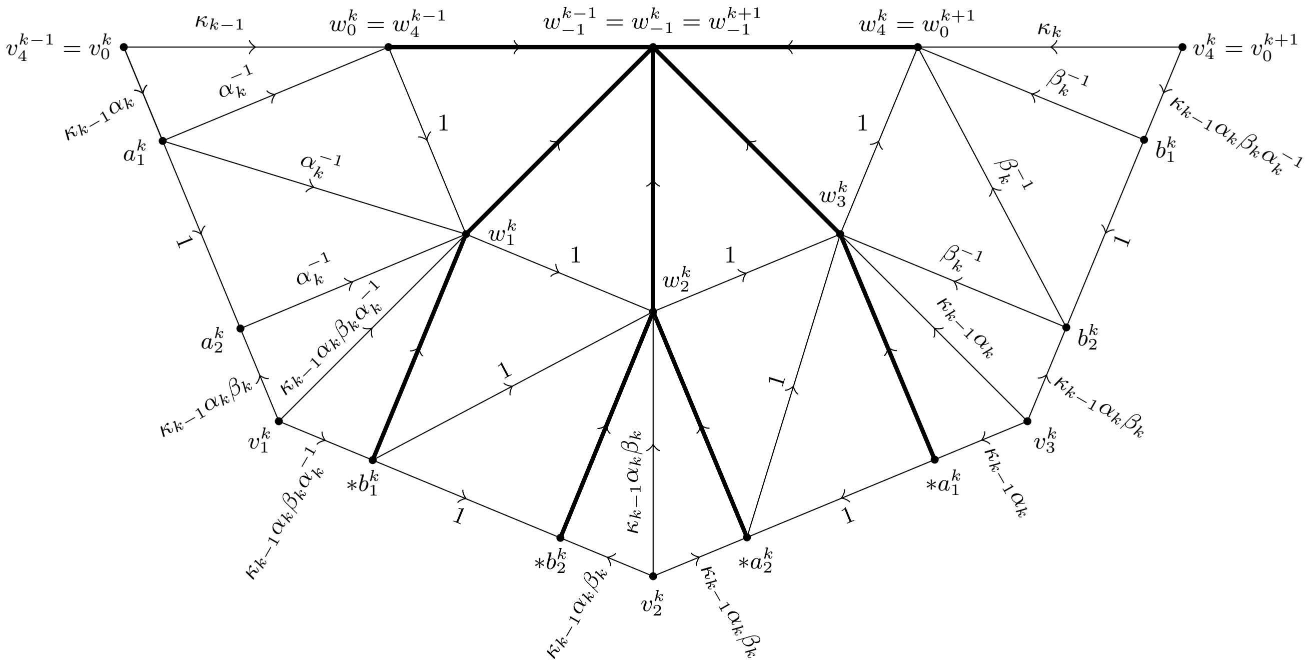
(The labels in Figure 3 also follow from these calculations, but may be obtained by more straightforward arguments because the generators of $\Gamma_1 \cong \mathbb{Z}^2$ may be regarded as shifts in the plane.) □

Notation 7.2. For $k \in \{1, \ldots, g\}$, let

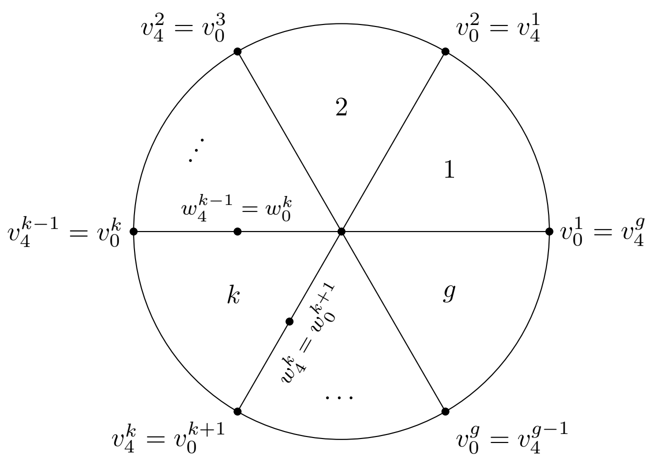
$$\mathcal{F}_k = \{\alpha_k^{-1},\ \beta_k^{-1},\ \kappa_{k-1},\ \kappa_{k-1}\alpha_k,\ \kappa_{k-1}\alpha_k\beta_k,\ \kappa_{k-1}\alpha_k\beta_k\alpha_k^{-1}\},$$

and notice that the set $\mathcal{F} = \{s_{ij}\}$ considered in section 3.1 is equal to the union $\mathcal{F}_1 \cup \mathcal{F}_1^{-1} \cup \cdots \cup \mathcal{F}_g \cup \mathcal{F}_g^{-1}$ by Lemma 7.1.

7.3. **Choosing quasi-representations.** We want to apply Theorem 5.4 using the labels obtained in Lemma 7.1 and some convenient choice of a quasi-representation of G in $U(A)$. We begin by proving a slightly stronger version Proposition 2.8, which guarantees the existence of quasi-representations (under certain conditions). Let us set up some notation first.



(A)



(B)

FIGURE 4. (A) The triangulation we use for $\widetilde{\Sigma}_1^k$, the kth copy of $\widetilde{\Sigma}_1$ (with a hole). Every edge is labeled with the element of Γ_g corresponding to the loop it induces. (B) How the simplicial complex $\widetilde{\Sigma}_n$ is defined. The kth "wedge" is pictured in (A).

For certain unitaries $u_1, v_1, \ldots, u_g, v_g$ in A we will need to produce a quasi-representation π satisfying

$$\pi(\alpha_k) = u_k, \text{ and } \pi(\beta_k) = v_k \quad \forall k \in \{1, \ldots, g\}. \tag{7}$$

Write $\mathbb{F}_{2g} = \langle \hat{\alpha}_1, \hat{\beta}_1, \ldots, \hat{\alpha}_g, \hat{\beta}_g \rangle$ for the free group on $2g$ generators. Let $q\colon \mathbb{F}_{2g} \to \Gamma_g$ and $\hat{\pi}\colon \mathbb{F}_{2g} \to U(A)$ be the homomorphisms given by

$$q(\hat{\alpha}_k) = \alpha_k, \quad q(\hat{\beta}_k) = \beta_k$$

and

$$\hat{\pi}(\hat{\alpha}_k) = u_k, \quad \hat{\pi}(\hat{\beta}_k) = v_k$$

for all $k \in \{1, \ldots, g\}$. Notice that the kernel of q is the normal subgroup generated by

$$\hat{\kappa}_g := \prod_{k=1}^{g} [\hat{\alpha}_k, \hat{\beta}_k]$$

and therefore consists of products of elements of the form $\hat{\gamma}\hat{\kappa}_g^{\pm 1}\hat{\gamma}^{-1}$ where $\hat{\gamma} \in \mathbb{F}_{2g}$.

Choose a set-theoretic section $s\colon \Gamma_g \to \mathbb{F}_{2g}$ of q such that $s(1) = 1$,

$$s(\alpha_k) = \hat{\alpha}_k, \quad \text{and} \quad s(\beta_k) = \hat{\beta}_k \quad \forall k \in \{1, \ldots, g\}.$$

Lemma 7.4. *For all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if A is a unital C^*-algebra and $u_1, v_1, \ldots, u_g, v_g \in U(A)$ satisfy*

$$\left\| \prod_{i=1}^{g} [u_i, v_i] - 1 \right\| < \delta(\varepsilon), \tag{8}$$

then $\pi = \hat{\pi} \circ s$ (with s as constructed above) is an $(\mathcal{F}, \varepsilon)$-representation satisfying (7).

This lemma obviously implies Proposition 2.8.

Proof. We only need to check that π is $(\mathcal{F}, \varepsilon)$-multiplicative. Assume that (8) holds for some δ in place of $\delta(\varepsilon)$.

Because $\hat{\pi}$ is a homomorphism, for all $\gamma, \gamma' \in \Gamma_g$ we have

$$\begin{aligned} \|\pi(\gamma)\pi(\gamma') - \pi(\gamma\gamma')\| = \|\pi(\gamma)\pi(\gamma')\pi(\gamma\gamma')^* - 1\| = \\ = \left\| \hat{\pi}\big(s(\gamma)s(\gamma')s(\gamma\gamma')^{-1}\big) - 1 \right\|. \end{aligned}$$

Now, $s(\gamma)s(\gamma')s(\gamma\gamma')^{-1}$ is in the kernel of q and is therefore a product of the form

$$\prod_{i=1}^{m} \hat{\gamma}_i \hat{\kappa}_g^{\varepsilon_i} \hat{\gamma}_i^{-1}$$

where m depends on γ and γ' and $\varepsilon_i \in \{1, -1\}$. Thus

$$\begin{aligned}\|\pi(\gamma)\pi(\gamma') - \pi(\gamma\gamma')\| &= \left\|\hat{\pi}\left(\prod_{i=1}^{m} \hat{\gamma}_i \hat{\kappa}_g^{\varepsilon_i} \hat{\gamma}_i^{-1}\right) - 1\right\| \\ &\le \sum_{i=1}^{m} \|\hat{\pi}(\hat{\gamma}_i)\hat{\pi}(\hat{\kappa}_g)^{\varepsilon_i}\hat{\pi}(\hat{\gamma}_i)^* - 1\| \\ &\le m\left\|\prod_{i=1}^{g}[u_i, v_i] - 1\right\| < m\delta.\end{aligned}$$

Since $\mathcal{F}$ is a finite set, there is a positive integer M such that if $\gamma, \gamma' \in \mathcal{F}$, then $s(\gamma)s(\gamma')s(\gamma\gamma')^{-1}$ is a product of at most M elements of the form $\hat{\gamma}_i \hat{\kappa}_g^{\varepsilon_i} \hat{\gamma}_i^{-1}$ as above. It follows that π is an $(\mathcal{F}, M\delta)$-representation. Choose $\delta(\varepsilon) = \varepsilon/M$. □

Notation 7.5. Recall the set $\mathcal{F}_k$ defined in 7.2. Let $s_0\colon \Gamma_g \to \mathbb{F}_{2g}$ be a set-theoretic section of q such that

$$\begin{gathered} s_0(\alpha_k^{\pm 1}) = \hat{\alpha}_k^{\pm 1}, \quad s_0(\beta_k^{\pm 1}) = \hat{\beta}_k^{\pm 1}, \quad s_0(\kappa_{k-1}) = \hat{\kappa}_{k-1}, \\ s_0(\kappa_{k-1}\alpha_k) = \hat{\kappa}_{k-1}\hat{\alpha}_k, \quad s_0(\kappa_{k-1}\alpha_k\beta_k) = \hat{\kappa}_{k-1}\hat{\alpha}_k\hat{\beta}_k \end{gathered}$$

for all $k \in \{1, \dots, g\}$, and

$$s_0(\kappa_{k-1}\alpha_k\beta_k\alpha_k^{-1}) = \hat{\kappa}_{k-1}\hat{\alpha}_k\hat{\beta}_k\hat{\alpha}_k^{-1}$$

for all $k \in \{1, \dots, g-1\}$. That such a section exists follows from the fact that all the words in the list $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_g \cup \{\alpha_1, \beta_1, \dots \alpha_g, \beta_g\}$ are distinct, with two exceptions: $\alpha_1 = \kappa_0\alpha_1 \in \mathcal{F}_1$ appears twice, as does $\beta_g = \kappa_{g-1}\alpha_g\beta_g\alpha_g^{-1} \in \mathcal{F}_g$.

Define $\pi_0 = \hat{\pi} \circ s_0\colon \Gamma_g \to U(A)$.

Lemma 7.6. *If $\langle x_i, x_j, x_k\rangle$ is any 2-simplex in Λ_g different from $\langle v_1^g, a_2^g, w_1^g\rangle$, then $\pi_0(s_{ik}) = \pi_0(s_{ij})\pi_0(s_{jk})$.*

If $\langle x_i, x_j, x_k\rangle = \langle v_1^g, a_2^g, w_1^g\rangle$, then $\pi_0(s_{ik}) = v_g$ and

$$\pi_0(s_{ij})\pi_0(s_{jk}) = \left(\prod_{i=1}^{g}[u_i, v_i]\right)v_g.$$

Proof. The definition of s_0 implies that the image under s_0 of any "word" in the list $\mathcal{F}_k$ is the word obtained by replacing $\alpha_k^{\pm 1}$ by $\hat{\alpha}_k^{\pm 1}$ and $\beta_k^{\pm 1}$ by $\hat{\beta}_k^{\pm 1}$, with one exception: the image of $\kappa_{g-1}\alpha_g\beta_g\alpha_g^{-1} = \beta_g$ under s_0 is $\hat{\beta}_g$.

This observation along with inspection of Figure 4a shows that $s_0(s_{ik}) = s_0(s_{ij})s_0(s_{jk})$ for every 2-simplex in Λ_g different from $\langle v_1^g, a_2^g, w_1^g\rangle$. For instance, let $l \in \{1, \dots, g\}$ and consider the simplex

$$\langle v_0^l, a_1^l, w_0^l\rangle = \langle x_i, x_j, x_k\rangle.$$

The corresponding group elements are

$$s_{ij} = \kappa_{l-1}\alpha_l$$
$$s_{jk} = \alpha_l^{-1}, \quad \text{and}$$
$$s_{ik} = \kappa_{l-1}.$$

Then

$$s_0(s_{ik}) = \hat{\kappa}_{l-1} = \hat{\kappa}_{l-1}\hat{\alpha}_l \cdot \hat{\alpha}_l^{-1} = s_0(\kappa_{l-1}\alpha_l) \cdot s_0(\alpha_l^{-1}) = s_0(s_{ij}) \cdot s_0(s_{jk}).$$

The computations in all other 2-simplices but $\langle v_1^g, a_2^g, w_1^g\rangle$ are very similar. For this exceptional simplex we get

$$s_0(s_{ik}) = s_0(\kappa_{g-1}\alpha_g\beta_g\alpha_g^{-1}) = s_0(\beta_g) = \hat{\beta}_g$$

but

$$s_0(s_{ij})s_0(s_{jk}) = s_0(\kappa_{g-1}\alpha_g\beta_g)s_0(\alpha_g^{-1}) = \hat{\kappa}_{g-1}\hat{\alpha}_g\hat{\beta}_g\hat{\alpha}_g^{-1} = \hat{\kappa}_g\hat{\beta}_g$$

Since $\pi_0 = \hat{\pi} \circ s_0$ and $\hat{\pi}$ is a homomorphism, the lemma follows. □

Recall that $\omega > 0$ is given in Theorem 2.4.

Lemma 7.7. *If $0 < \varepsilon < \omega$ and (8) holds (so that π_0 is an $(\mathcal{F}, \varepsilon)$-representation), then*

$$\tau\big(\pi_{0\sharp}(\mu[\Sigma_g])\big) = \frac{1}{2\pi i}\tau\left(\log\left(\prod_{i=1}^{g}[u_i, v_i]\right)\right)$$

Proof. We apply Theorem 5.4. For each simplex $\langle x_i, x_j, x_k\rangle$ we compute $\widetilde{\Delta}_\tau(\xi)$ where ξ_σ is the path

$$\xi_\sigma(t) = (1-t)\pi(s_{ik}) + t\pi(s_{ij})\pi(s_{jk}), \quad t \in [0,1]$$

Observe that the value of $\widetilde{\Delta}_\tau$ on a constant path is 0. Lemma 7.6 implies that there is only one 2-simplex σ such that ξ_σ is not constant: $\sigma_0 = \langle v_1^g, a_2^g, w_1^g\rangle$. By Lemma 7.6 it yields the linear path ξ_{σ_0} from v_g to

$$\left(\prod_{i=1}^{g}[u_i, v_i]\right)v_g.$$

Using Lemma 5.2 we obtain

$$\widetilde{\Delta}_\tau(\xi_{\sigma_0}) = \frac{1}{2\pi i}\tau\left(\log\left(\prod_{i=1}^{g}[u_i, v_i]\right)\right).$$

Finally, Theorem 5.4 implies

$$\tau\big(\pi_{0\sharp}(\mu[\Sigma_g])\big) = (-1)^{s(\sigma_0)}\frac{1}{2\pi i}\tau\left(\log\left(\prod_{i=1}^{g}[u_i, v_i]\right)\right)$$

where the sign $(-1)^{s(\sigma_0)}$ depends on the the orientation $[\Sigma_g]$. The standard orientation on Σ_g gives $s(\sigma_0) = 1$. □

By putting these lemmas together we can prove Theorem 2.6.

Proof of Theorem 2.6. Recall that the statement of the theorem fixes a positive integer g and idempotents q_0 and q_1 in some matrix algebra over $\ell^1(\Gamma_g)$ such that $\mu[\Sigma_g] = [q_0] - [q_1] \in K_0(\ell^1(\Gamma_g))$.

Let $\mathcal{F}_0$ be the finite set $\{s_{ij}\}$ defined in 3.1 and described explicitly in 7.2. Theorem 2.4 provides an $\omega > 0$ so small that if $\pi\colon \Gamma_g \to U(A)$ is an $(\mathcal{F}_0, \omega)$-representation with, then $\pi_\sharp(\mu[\Sigma_g]) := \pi_\sharp(q_0) - \pi_\sharp(q_1)$ is defined and

$$\tau\big(\pi_\sharp(\mu[\Sigma_g])\big) = \langle \mathrm{ch}_\tau(\ell_\pi), [\Sigma_g]\rangle.$$

By setting $u_i := \pi(\alpha_i)$ and $v_i := \pi(\beta_i)$ for all $i \in \{1, \dots, g\}$, we see that such a quasi-representation π may be used to define a quasi-representation π_0 as in Section 7.5. The more multiplicative π is on $\mathcal{F}_0$, the smaller the quantity

$$\left\| \prod_{i=1}^{g} [u_i, v_i] - 1 \right\|$$

is. Lemma 7.4 shows that by making this quantity smaller we can make π_0 more multiplicative on $\mathcal{F}_0$. Therefore, because π and π_0 agree on the generators of Γ_g, there exists an $0 < \varepsilon_0 < \omega$ so small that if π is an $(\mathcal{F}_0, \varepsilon_0)$-representation, then $\pi_\sharp$ and $\pi_{0\,\sharp}$ agree on $\{q_0, q_1\} \subset K_0(\ell^1(\Gamma_g))$.

Finally,

$$\tau\big(\pi_\sharp(\mu[\Sigma_g])\big) = \tau\big(\pi_{0\,\sharp}(\mu[\Sigma_g])\big) = \frac{1}{2\pi i}\tau\left(\log\left(\prod_{i=1}^{g}[u_i, v_i]\right)\right)$$

by Lemma 7.7. □

References

[1] H. Bettaieb, M. Matthey, and A. Valette. Unbounded symmetric operators in K-homology and the Baum-Connes conjecture. *J. Funct. Anal.*, 229(1):184–237, 2005.

[2] A. Connes, M. Gromov, and H. Moscovici. Conjecture de Novikov et fibrés presque plats. *C. R. Acad. Sci. Paris Sér. I Math.*, 310(5):273–277, 1990.

[3] M. Dadarlat. Group quasi-representations and almost flat bundles. To appear in J. of Noncomut. Geom.

[4] M. Dadarlat. Group quasi-representations and index theory. To appear in J. of Topology and Analysis.

[5] P. de la Harpe and G. Skandalis. Déterminant associé à une trace sur une algébre de Banach. *Ann. Inst. Fourier (Grenoble)*, 34(1):241–260, 1984.

[6] R. Exel and T. A. Loring. Invariants of almost commuting unitaries. *J. Funct. Anal.*, 95(2):364–376, 1991.

[7] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[8] B. Iversen. *Hyperbolic geometry*, volume 25 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1992.

[9] V. Lafforgue. K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes. *Invent. Math.*, 149(1):1–95, 2002.

[10] H. Lin. Asymptotic unitary equivalence and classification of simple amenable C^*-algebras *Invent. Math.*, 183(2), 385–450, 2011.

[11] A. V. Phillips and D. A. Stone. The computation of characteristic classes of lattice gauge fields. *Comm. Math. Phys.*, 131(2):255–282, 1990.

[12] T. Schick. L^2-index theorems, KK-theory, and connections. *New York J. Math.*, 11:387–443 (electronic), 2005.

E-mail address: jcarrion@math.purdue.edu

E-mail address: mdd@math.purdue.edu

Department of Mathematics, Purdue University, West Lafayette, IN, 47907, United States