A COMPUTATION OF PRISMATIC DIEUDONNÉ MODULE

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Abstract. In this short paper, we establish a natural isomorphism between two fundamental invariants: the second prismatic cohomology of the projective line $\mathbf{P}^1$ and the prismatic Dieudonné module of the p-divisible group μ_{p^∞}, as defined in the work of Anschütz and Le Bras. We call this the "Chern–Dieudonné isomorphism." Our construction of this isomorphism is essentially "motivic", in the sense that it is obtained purely via geometric principles. To achieve this, we use the geometric reconstruction of Dieudonné modules proven by the author and the theory of prismatic Chern classes due to Bhatt–Lurie. As a consequence, we can compute the Dieudonné module of μ_{p^∞} over a general quasi-regular semiperfectoid algebra S (and therefore the associated prismatic Dieudonné crystal) that was left open in the work of Anschütz and Le Bras.

1. Introduction

Let p be a fixed prime. In the paper [AL19], Anschütz and Le Bras use the prismatic cohomology theory developed by Bhatt and Scholze [BS19] to construct the prismatic Dieudonné module of p-divisible groups over fairly general p-adic base rings. They introduce the notion of a filtered prismatic Dieudonné crystal and filtered prismatic Dieudonné module (see [AL19, Section 4]) which can be used to classify p-divisible groups over p-adic base rings, generalizing the earlier results on classical crystalline Dieudonné theory [BBM82] and the relatively recent work of Scholze and Weinstein [SW14] on Dieudonné theory over perfectoid base rings.

In [AL19, Section 4.7], the authors discuss the computation of prismatic Dieudonné crystals of two crucial examples of p-divisible groups, namely $\mathbf{Q}_p/\mathbf{Z}_p$ and μ_{p^∞}. While the prismatic Dieudonné crystal of $\mathbf{Q}_p/\mathbf{Z}_p$ had been described fully, the description of the prismatic Dieudonné crystal of μ_{p^∞} over a general base was left open.

In order to describe the prismatic Dieudonné crystal of μ_{p^∞}, by quasi-syntomic descent, one equivalently needs to be able to compute the prismatic Dieudonné module of μ_{p^∞} denoted as $M_{\mathbb{\Delta}}(\mu_{p^\infty})$ over an arbitrary quasi-regular semiperfectoid algebra S. However, the calculation in [AL19] relied on certain explicit logarithm

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constructions, which gave the computation in the case when S was assumed to be a $\mathbf{Z}_p^{\mathrm{cycl}} := (\varinjlim_n \mathbf{Z}_p[\zeta_{p^n}])_p^\wedge$-algebra, but not in general.

In this paper, we take an entirely different and more geometric perspective, which does not (at least at the surface level) rely on the logarithm and works over general base rings. Let us explain the key idea very briefly. Using the geometric reconstruction of the Dieudonné modules obtained in [Mon21], and assuming a formalism of prismatic Chern classes, one can show that the Dieudonné module of μ_{p^∞} over an *arbitrary* quasi-regular semiperfectoid ring S is isomorphic to $H^2_{\mathbb{\Delta}}(\mathbf{P}^1)$, and therefore should be a *free* $\mathbb{\Delta}_S$-module of rank 1. Very recent work of Bhatt–Lurie [BL22] on syntomic Chern classes (also see [KP21] for a theory of A_{inf}-valued Chern classes) can now be used to justify the necessary use of Chern classes, which yields the desired computation of $M_{\mathbb{\Delta}}(\mu_{p^\infty})$ over an arbitrary quasi-regular semiperfectoid ring S. We point out that even though no logarithm appears in our presentation of the proof, it is implicit in the construction of the syntomic Chern classes. Let us state our results more precisely.

Theorem 1.1 (Chern–Dieudonné isomorphism, see Corollary 3.11, Corollary 3.12)**.** *Let S be a quasi-regular semiperfectoid ring. We have a natural isomorphism*

$$H^2_{\mathbb{\Delta}}(\mathbf{P}^1) \simeq M_{\mathbb{\Delta}}(\mu_{p^\infty}).$$

Corollary 1.2 (see Proposition 3.13)**.** *Let S be a quasi-regular semiperfectoid ring. The prismatic Dieudonné module of μ_{p^∞} denoted as $M_{\mathbb{\Delta}}(\mu_{p^\infty})$ is free of rank* 1.

In Proposition 3.10, we compute the whole cohomology ring of the classifying stack of the p-divisible group μ_{p^∞} denoted as $B\mu_{p^\infty}$, which extends the above results.

Corollary 1.3 (see Definition 2.8 and Corollary 3.12)**.** *The prismatic Dieudonné crystal of μ_{p^∞} denoted as $\mathcal{M}_{\mathbb{\Delta}}(\mu_{p^\infty})$ is isomorphic to $\mathcal{O}^{\mathrm{pris}}\{-1\}$.*

The above results completely describe the prismatic Dieudonné crystal of μ_{p^∞} that was not addressed in [AL19, Section 4.7]. In fact, the results in [AL19] only showed that over a quasi-regular semiperfectoid algebra S, the module $M_{\mathbb{\Delta}}(\mu_{p^\infty})$ is projective of rank 1. Corollary 1.2 in our paper says that it is actually *free*.

The construction in Theorem 1.1 uses very simple geometric principles. However, it crucially relies on the geometric reconstruction of Dieudonné modules obtained in [Mon21] (see Theorem 2.6) involving *classifying stacks*. Somewhat surprisingly, the construction leading to Theorem 1.1 does not seem to be observed in the literature even in the simplest case of S being a *perfect field* of characteristic p, when prismatic cohomology agrees with the classical theory of crystalline cohomology. We hope that the geometric techniques employed in this paper in computing arithmetic invariants such as Dieudonné modules could be useful in other contexts as well.

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2. Prerequisites

In this section, we will introduce some of the definitions and results that will be used afterwards. Many of the definitions are not introduced in a self-contained manner and are introduced to fix notations. We refer to [BS19], [BMS19], [AL19], [Mon21], [BL22] for more details. In particular, the theory of absolute and relative prismatic cohomology from [BS19] and [BL22] will be freely used in our paper.

Definition 2.1 ([BMS19, Def. 4.10])**.** A ring S is called *quasi-syntomic* if S is p-complete with bounded p^∞-torsion and the cotangent complex $\mathbf{L}_{S/\mathbf{Z}_p}$ has p-complete Tor-amplitude in $[-1,0]$. The category of all quasi-syntomic rings is denoted by QSyn. A map $S \to S'$ of p-complete rings with bounded p^∞-torsion is a *quasi-syntomic morphism* if S' is p-completely flat over S and the cotangent complex $\mathbf{L}_{S'/S}$ has p-complete Tor-amplitude in $[-1,0]$. A quasi-syntomic morphism is called a *quasi-syntomic cover* if the map $S \to S'$ is p-completely faithfully flat.

Definition 2.2 ([BMS19, Def. 4.20])**.** A ring S is called quasi-regular semiperfectoid (QRSP) if $S \in$ QSyn and there exists a perfectoid ring R mapping surjectively onto S.

Definition 2.3. If R is any p-complete ring we will let $(R)_{\mathrm{qsyn}}$ denote the (opposite) full subcategory spanned by all p-complete R-algebras S such that the structure map $R \to S$ is quasi-syntomic. This category can be equipped with a Grothendieck topology generated by quasi-syntomic covers which turns this into a site.

Let S be a QRSP ring. Let $(\mathbb{\Delta}_S, I)$ be the prism associated to S by taking prismatic cohomology. Note that the quasi-regular semiperfectoid algebras form a basis for $(S)_{\mathrm{qsyn}}$. The functor $T \mapsto \mathbb{\Delta}_T$ sending a quasi-regular semiperfectoid algebra T to its associated prism $\mathbb{\Delta}_T$ forms a sheaf on $(S)_{\mathrm{qsyn}}$ [BS19, Construction 7.6 (3)] which we will denote by $\mathcal{O}^{\mathrm{pris}}$. *cf.* [AL19, Definition 4.1.1.] and the proof of [AL19, Proposition 4.1.13].

Definition 2.4 ([AL19, Def. 4.2.8])**.** Let G be a p-divisible group over S. We define

$$M_{\mathbb{\Delta}}(G) := \mathrm{Ext}^1_{(S)_{\mathrm{qsyn}}}(\underline{G}, \mathcal{O}^{\mathrm{pris}}),$$

$$\mathrm{Fil}\, M_{\mathbb{\Delta}}(G) := \mathrm{Ext}^1_{(S)_{\mathrm{qsyn}}}(\underline{G}, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}),$$

and $\varphi_{M_{\mathbb{\Delta}}(G)}$ as the endomorphism induced by φ on $\mathcal{O}^{\mathrm{pris}}$. Then

$$\underline{M}_{\mathbb{\Delta}}(G) := (M_{\mathbb{\Delta}}(G), \mathrm{Fil}\, M_{\mathbb{\Delta}}(G), \varphi_{M_{\mathbb{\Delta}}(G)})$$

is called the *filtered prismatic Dieudonné module* of G.

Example 2.5. For the étale p-divisible group $\mathbf{Q}_p/\mathbf{Z}_p$ over S, we have $\underline{M}_{\Delta}(\mathbf{Q}_p/\mathbf{Z}_p) \simeq (\Delta_R, \mathcal{N}^{\geq 1}\Delta_R, \varphi)$. We refer to [AL19, §4.7] and Remark 4.9.6 loc. cit. for more discussions.

In [Mon21, Definition 4.27], we defined the classifying stack of a p-divisible group and proved the following

Theorem 2.6. *Let G be a p-divisible group over a quasi-regular semiperfectoid ring S. Then the prismatic cohomology $H^2_{\Delta}(BG)$ is naturally isomorphic to the prismatic Dieudonné module $M_{\Delta}(G)$. This isomorphism identifies the natural Frobenius on $H^2_{\Delta}(BG)$ with the endomorphism $\varphi_{M_{\Delta}(G)}$ on $M_{\Delta}(G)$. Further, the Nygaard filtration $\mathcal{N}^{\geq 1}H^2_{\Delta}(BG)$ on prismatic cohomology of the stack BG is isomorphic to* $\operatorname{Fil} M_{\Delta}(G)$. *See* [Mon21, Theorem 1.6].

The above result gives a geometric reconstruction of the prismatic Dieudonné module in terms of prismatic cohomology of classifying stacks which will be useful for us later on.

Notation 2.7. Let (A, I) be a prism. The Breuil–Kisin twist $A\{1\}$ as defined in [BL22, Definition 2.5.2.] is an invertible A-module. For every integer $n \in \mathbf{Z}$, one uses $A\{n\}$ to denote the nth tensor power of $A\{1\}$ with respect to A.

The construction of the Breuil–Kisin twist is functorial on the prism (A, I), i.e., for a map $(A, I) \to (B, J)$ of prisms, one has $A\{1\} \otimes_A B \simeq B\{1\}$. As explained in [BL22, Remark 2.5.9], one also has a Frobenius map

$$\varphi_{A\{n\}} : A\{n\} \to I^{-n} \otimes_A A\{n\}$$

for each $n \in \mathbf{Z}$. The map $\varphi_{A\{n\}}$ is φ_A-semilinear, where φ_A is the Frobenius on A. Over the q-de Rham prism $(\mathbf{Z}_p[[q-1]], [p]_q)$, one can explicitly describe the Frobenius map by using the logarithm. We refer to [BL22, Notation 2.6.3] for these formulas and more details.

Definition 2.8. We define $\mathcal{O}^{\text{pris}}\{-1\}$ to be the locally free sheaf of $\mathcal{O}^{\text{pris}}$-modules on $(S)_{\text{qsyn}}$ that is determined by the functor that sends every quasi-regular semiperfectoid algebra $T \mapsto \Delta_T\{-1\}$.

3. Computation of the prismatic Dieudonné module

In this section, we will construct the Chern–Dieudonné isomorphism and prove Theorem 1.1. To do so, we fix a QRSP algebra S. Let $\widehat{\mathbf{G}}_m$ denote the p-completion of $\mathbf{G}_m$ and let us consider $B\widehat{\mathbf{G}}_m$ as a p-adic formal stack over S. We prove the following proposition.

Proposition 3.1. *We have an isomorphism $R\Gamma_{\Delta}(B\mu_{p^\infty}) \simeq R\Gamma_{\Delta}(B\widehat{\mathbf{G}}_m)$. (Here we consider absolute prismatic cohomology.)*

Proof. Let us first fix some notations. The sheaves on the site $(S)_{\mathrm{qsyn}}$ forms a topos which we denote by $\mathcal{X}$. Then $\widehat{\mathbf{G}}_m$ and μ_{p^∞} both defines objects in $\mathcal{X}$ by considering their functor of points which we will still denote as $\widehat{\mathbf{G}}_m$ and μ_{p^∞}. Let us use $\mathbf{Z}[\widehat{\mathbf{G}}_m]$ and $\mathbf{Z}[\mu_{p^\infty}]$ to denote the associated free abelian group objects in $\mathcal{X}$. We note that $\mathcal{X}$ is a replete topos [BS15, Section 3]. In particular, there is a good theory of derived completion in the topos $\mathcal{X}$, which we will use. For the background material on derived completions on a replete topos we refer to [BS15, Section 3.4]. We note that $\mathcal{O}^{\mathrm{pris}}$ as defined in Definition 2.3 is a derived p-complete object of $\mathcal{X}$. Further, we note that the objects of the site $(S)_{\mathrm{qsyn}}$ are affine formal schemes; therefore the topos $\mathcal{X}$ is (locally) coherent and by Deligne's theorem [AGV72, exposé VI, p.336], it has enough points.

For any group object $G \in \mathcal{X}$, the Čech nerve of the map $* \to BG$ is given by the following simplicial object

$$\cdots G \times G \times G \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} G \times G \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} G \rightrightarrows * .$$

The associated simplicial abelian group object in $\mathcal{X}$ is

$$\cdots \mathbf{Z}[G \times G \times G] \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \mathbf{Z}[G \times G] \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \mathbf{Z}[G] \rightrightarrows \mathbf{Z} .$$

With this simplicial object we can associate an object in the derived category of abelian sheaves on $\mathcal{X}$ which we will denote simply by $\mathbf{Z}[BG]$. When, $G = \mu_{p^\infty}$ or $\widehat{\mathbf{G}}_m$, the maps $* \to BG$ is effective quasi-syntomic epimorphism and therefore, we can compute $R\Gamma_{\Delta}(BG)$ by quasi-syntomic descent. More precisely, we have

$$R\Gamma_{\Delta}(BG) \simeq R\varprojlim \big(R\Gamma_{\Delta}(*) \rightrightarrows R\Gamma_{\Delta}(G) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} R\Gamma_{\Delta}(G \times G) \cdots\big),$$

which can be rewritten as

$$\simeq R\varprojlim \big(R\mathrm{Hom}(\mathbf{Z}, \mathcal{O}^{\mathrm{pris}}) \rightrightarrows R\mathrm{Hom}(\mathbf{Z}[G], \mathcal{O}^{\mathrm{pris}}) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} R\mathrm{Hom}(\mathbf{Z}[G \times G], \mathcal{O}^{\mathrm{pris}}) \cdots\big).$$

We can take the $R\varprojlim$ inside as a homotopy colimit, which gives us

$$R\Gamma_{\Delta}(BG) \simeq R\mathrm{Hom}(\mathbf{Z}[BG], \mathcal{O}^{\mathrm{pris}}).$$

Therefore, Proposition 3.1 follows from Lemma 3.2 by taking $G = \widehat{\mathbf{G}}_m$ and recalling that $\mathcal{O}^{\mathrm{pris}}$ is derived p-complete. This finishes the proof. $\square$

Lemma 3.2. *Let $\mathcal{X}$ be a replete topos with enough points and let G be a group object such that multiplication by p map induces a surjection. Let us define $G[p^\infty] := \varinjlim G[p^n]$. Then we have an isomorphism*

$$\mathbf{Z}[BG[p^\infty]]^{\wedge p} \to \mathbf{Z}[BG]^{\wedge p}$$

in the derived category of p-complete objects in $\mathcal{X}$.

Proof of Lemma 3.2. Since the map $\mathbf{Z}[BG[p^\infty]]^{\wedge p} \to \mathbf{Z}[BG]^{\wedge p}$ is a map between derived p-complete objects, it is enough to check isomorphism by going derived modulo p, i.e., we need to prove that $\mathbf{F}_p[BG[p^\infty]] \to \mathbf{F}_p[BG]$ is an isomorphism.

Since $\mathcal{X}$ has enough points, this can be checked after taking stalks. Thus, the proposition follows from an application of Lemma 3.3. $\square$

Lemma 3.3. *Let G be an ordinary group on which multiplication by p is surjective. Then we have an isomorphism $\mathbf{F}_p[BG[p^\infty]] \simeq \mathbf{F}_p[BG]$.*

Proof. It is enough to show that $H^i(\mathbf{F}_p[BG[p^\infty]]) \to H^i(\mathbf{F}_p[BG])$ is an isomorphism for all i. Equivalently, we need to show that the maps on group homology with coefficients in $\mathbf{F}_p$ denoted as

$$H_i(G[p^\infty], \mathbf{F}_p) \to H_i(G, \mathbf{F}_p)$$

induce isomorphisms. Note that we have an exact sequence

$$0 \to G[p^\infty] \to G \to \varinjlim_p G \to 0.$$

It follows that the group $Q := \varinjlim_p G$ is *uniquely* p-divisible. By an application of the Hochschild–Serre spectral sequence, it would be enough to prove that $H_i(Q, \mathbf{F}_p) = 0$ for $i > 0$. This follows from [Mon21, Lemma 3.30]. We give a less elementary, but quicker argument to see the latter as pointed out by the referee: the homology groups $H_i(Q, \mathbf{F}_p)$ are computed as $\mathrm{Tor}_i^{\mathbf{F}_p[Q]}(\mathbf{F}_p, \mathbf{F}_p)$, where $\mathbf{F}_p[Q]$ is the group algebra of Q. Since Q is uniquely p-divisible, $\mathbf{F}_p[Q]$ is a perfect ring. The statement now follows from noting that $\mathbf{F}_p \otimes^{\mathbf{L}}_{\mathbf{F}_p[Q]} \mathbf{F}_p$ is a perfect simplicial commutative ring and therefore is discrete by [BS17, Proposition 11.6]. $\square$

Now we turn to the main construction of our paper.

Construction 3.4. Note that we have a natural map $\mathbf{P}^1 \to B\mathbf{G}_m$ corresponding to the line bundle $\mathcal{O}(1)$ on $\mathbf{P}^1$. This induces a map

$$H^2_{\Delta}(B\widehat{\mathbf{G}}_m) \to H^2_{\Delta}(\mathbf{P}^1).$$

By Theorem 2.6 and Proposition 3.1, we have a sequence of isomorphisms and a natural map

$$M_{\Delta}(\mu_{p^\infty}) \simeq H^2_{\Delta}(B\mu_{p^\infty}) \simeq H^2_{\Delta}(B\widehat{\mathbf{G}}_m) \to H^2_{\Delta}(\mathbf{P}^1).$$

Having constructed the maps, now we proceed onto proving that the map $H^2_{\Delta}(B\widehat{\mathbf{G}}_m) \to H^2_{\Delta}(\mathbf{P}^1)$ is an isomorphism. In order to do so, we will make use of their construction of *syntomic Chern classes.* We will briefly mention the necessary definitions from [BL22]. Let X denote a scheme, formal scheme or an algebraic stack. Then one can consider the *absolute* prismatic cohomology of X (with a twist) denoted as $R\Gamma_{\Delta}(X)\{n\}$ for $n \in \mathbf{Z}$ as in [BL22, Construction 4.4.19]. This is equipped with the Nygaard filtration which gives rise to the objects $\mathrm{Fil}^m_N R\Gamma_{\Delta}(X)\{n\}$ for every $m \in \mathbf{Z}$ [BL22, Notation 5.5.23]. For every $i \in \mathbf{Z}$, one also has the syntomic cohomology $R\Gamma_{\mathrm{syn}}(X, \mathbf{Z}_p(i))$, as constructed in [BL22, Section 8.4.7]. For a vector bundle $\mathscr{E}$ of rank r on X, they have constructed its syntomic Chern classes

$$c_i^{\mathrm{syn}}(\mathscr{E}) \in H^{2i}_{\mathrm{syn}}(X, \mathbf{Z}_p(i))$$

with the property that $c_0^{\mathrm{syn}}(\mathscr{E}) = 1$ and $c_i^{\mathrm{syn}}(\mathscr{E}) = 0$ for $i > r$. The construction of these classes are functorial and satisfies the expected additivity formula [BL22, Theorem 9.2.7].

Proposition 3.5. *Let us work over a QRSP algebra S. The map $\Delta_S\{-1\} \to H^2_{\Delta}(\mathbf{P}^1_S)$ induced by the syntomic Chern class $c_1^{\mathrm{syn}}(\mathcal{O}(1))$ is an isomorphism.*

Proof. The proof appears in [BL22, Lemma 9.1.4. (4)]. We point out that their proof makes use of the classical computation of coherent cohomology groups of the sheaf of differential forms on projective space. □

Proposition 3.6. *Let us work over a QRSP algebra S. There is a natural isomorphism*

$$H^*_{\Delta}(B\widehat{\mathbf{G}}_m) \simeq \mathrm{Sym}^*(\Delta_S\{-1\}[-2]).$$

Here $\Delta_S\{-1\}[-2]$ denotes the convention that $\Delta_S\{-1\}$ is being considered in degree 2 with respect to the grading on both sides.

Proof. Let $\mathscr{E}_{\mathrm{univ}}$ be the tautological line bundle on $B\mathbf{G}_m$. There is a well-defined syntomic Chern class $c_1^{\mathrm{syn}}(\mathscr{E}_{\mathrm{univ}}) \in H^2(B\widehat{\mathbf{G}}_m, \mathbf{Z}_p(1))$ whose powers $c_1^{\mathrm{syn}}(\mathscr{E}_{\mathrm{univ}})^i$ are classes in $H^{2i}_{\mathrm{syn}}(B\widehat{\mathbf{G}}_m, \mathbf{Z}_p(i))$. By construction of syntomic cohomology, this induces a natural map

$$\bigoplus_{i\geq 0} \mathrm{Sym}^i(\Delta_S\{-1\})[-2i] \to R\Gamma_{\Delta}(B\widehat{\mathbf{G}}_m) \tag{3.1}$$

in the derived category of Δ_S-modules. Further, since S is a QRSP algebra, by [BS19, Proposition 7.10], the prism (Δ_S, I) defines a final object in the absolute prismatic site of S. Therefore, the absolute prismatic cohomology $R\Gamma_{\Delta}(B\widehat{\mathbf{G}}_m)$ is naturally isomorphic to the prismatic cohomology of $B\widehat{\mathbf{G}}_m$ (thought of as a stack over $\mathrm{Spf}\,\Delta_S/I$ by base change) relative to the prism (Δ_S, I). From now onward in this proof, we will assume $B\widehat{\mathbf{G}}_m$ to be over $\mathrm{Spf}\,\Delta_S/I$ (by base changing) and use the aforementioned identification of absolute and relative prismatic cohomology. Since the ring Δ_S is derived I-complete, to prove that the map in (3.1) is an isomorphism, we can reduce derived modulo I and use the Hodge–Tate comparison theorem in prismatic cohomology as we will explain. In order to prove that $\bigoplus_{i\geq 0}\mathrm{Sym}^i(\Delta_S\{-1\})[-2i] \to R\Gamma_{\Delta}(B\widehat{\mathbf{G}}_m)$ is an isomorphism, it is enough to prove the same after applying $(\bullet) \otimes^{\mathbf{L}}_{\Delta_S} \Delta_S/I$. By the Hodge–Tate comparison in (relative) prismatic cohomology, $R\Gamma_{\Delta}(B\widehat{\mathbf{G}}_m) \otimes^{\mathbf{L}}_{\Delta_S} \Delta_S/I$ has an exhaustive increasing filtration whose ith graded piece is computed by

$$R\Gamma(B\widehat{\mathbf{G}}_m, \bigwedge^i \mathbf{L}_{B\widehat{\mathbf{G}}_m/(\Delta_S/I)}[-i])\{-i\} \simeq R\Gamma(B\widehat{\mathbf{G}}_m, \mathcal{O}[-2i])\{-i\};$$

here the equivalence follows from Lemma 3.7. Note that for a $\mathbb{\Delta}_S/I$-module M, we use $M\{i\}$ to denote $M \otimes_{\mathbb{\Delta}_S/I} (I/I^2)^{\otimes i}$. Finally, to show that (3.1) is an isomorphism, it would now be enough to show that

$$R\Gamma(B\widehat{\mathbf{G}}_m, \mathcal{O}[-2i]) \simeq (\mathbb{\Delta}_S/I)[-2i].$$

But that follows from Lemma 3.8. This finishes the proof of the proposition. □

Lemma 3.7. *Let us take an arbitrary commutative ring R and consider the stack $B\mathbf{G}_m$ over R. Then*

$$\bigwedge^i \mathbf{L}_{B\mathbf{G}_m/R} \simeq \mathcal{O}_{B\mathbf{G}_m}[-i]$$

in the derived category of quasi-coherent sheaves on $B\mathbf{G}_m$; here $\mathbf{L}_{B\mathbf{G}_m/R}$ denotes the cotangent complex of the stack $B\mathbf{G}_m$ relative to R. (The wedge power is also taken in the derived sense.)

Proof. To see this, we note that since $\mathbf{G}_m$ is a smooth group scheme over R, the co-Lie complex $\mathrm{coLie}(\mathbf{G}_m)$ is dual to the adjoint representation of $\mathbf{G}_m$. Further, since $\mathbf{G}_m$ is a commutative group scheme of dimension 1, it follows that the adjoint representation must be trivial. So as an object in the derived category, $\mathrm{coLie}(\mathbf{G}_m) \simeq \mathcal{O}_{B\mathbf{G}_m}$. This implies that $\mathbf{L}_{B\mathbf{G}_m/R} \simeq \mathcal{O}_{B\mathbf{G}_m}[-1]$. Now the desired formula follows since

$$\bigwedge^i \mathcal{O}_{B\mathbf{G}_m}[-1] \simeq \mathrm{Sym}^i \mathcal{O}_{B\mathbf{G}_m}[-i],$$

by the décalage formula. We refer to [Ill71, Proposition 4.3.2.1] for more on these formulas. □

Lemma 3.8. *Let us take an arbitrary commutative ring R and consider the stack $B\mathbf{G}_m$ over R. Then $R\Gamma(B\mathbf{G}_m, \mathcal{O}) \simeq R$.*

Proof. The lemma will be proven if we show that the global section functor defined in the abelian category of quasi-coherent sheaves on $B\mathbf{G}_m$ is exact. But the category of quasi-coherent sheaves on $B\mathbf{G}_m$ is equivalent to the category of $\mathbf{G}_m$-representations which can further be identified with $\mathbf{Z}$-graded R-modules. Under this equivalence of categories, taking the global section of a quasi-coherent sheaf on $B\mathbf{G}_m$ corresponds to considering the degree zero part of a $\mathbf{Z}$-graded R-module, which is clearly an exact functor. This proves the lemma. □

Corollary 3.9. *The map $H^2_{\mathbb{\Delta}}(B\widehat{\mathbf{G}}_m) \to H^2_{\mathbb{\Delta}}(\mathbf{P}^1)$ constructed in Construction 3.4 is an isomorphism.*

Proof. Follows from Proposition 3.5, Proposition 3.6 and the functoriality of the syntomic Chern class construction. □

Proposition 3.10. *We have an isomorphism*

$$H^*_{\mathbb{\Delta}}(B\mu_{p^\infty}) \simeq \mathrm{Sym}^*(\mathbb{\Delta}_S\{-1\}[-2]).$$

Here $\mathbb{\Delta}_S\{-1\}[-2]$ denotes the convention that $\mathbb{\Delta}_S\{-1\}$ is being considered in degree 2 with respect to the grading on both sides.

Proof. Follows from Proposition 3.1 and Proposition 3.6. ☐

Corollary 3.11 (Chern–Dieudonné isomorphism)**.** *Let S be a quasi-regular semiperfectoid ring. We have a natural isomorphism*

$$H^2_{\mathbb{\Delta}}(\mathbf{P}^1) \simeq M_{\mathbb{\Delta}}(\mu_{p^\infty}).$$

Proof. Follows from Construction 3.4 and Corollary 3.9. ☐

Corollary 3.12. *We have a sequence of isomorphisms*

$$M_{\mathbb{\Delta}}(\mu_{p^\infty}) \simeq H^2_{\mathbb{\Delta}}(B\mu_{p^\infty}) \simeq H^2_{\mathbb{\Delta}}(B\widehat{\mathbf{G}}_m) \to H^2_{\mathbb{\Delta}}(\mathbf{P}^1) \simeq \mathbb{\Delta}_S\{-1\}$$

Proof. Follows from Construction 3.4, Proposition 3.5, and Corollary 3.9. ☐

Proposition 3.13. *Let S be a quasi-regular semiperfectoid ring. The prismatic Dieudonné module of μ_{p^∞} denoted as $M_{\mathbb{\Delta}}(\mu_{p^\infty})$ is free of rank* 1.

Proof. By Proposition 3.5 and Construction 3.4, we know that $M_{\mathbb{\Delta}}(\mu_{p^\infty}) \simeq \mathbb{\Delta}_S\{-1\}$. It is thus enough to show that $\mathbb{\Delta}_S\{-1\}$ is free of rank 1. By [BL22, Remark 2.5.7, Remark 2.5.8], $\mathbb{\Delta}_S\{-1\}/I\mathbb{\Delta}_S\{-1\} \simeq (I/I^2)^\vee$. Since $\mathbb{\Delta}_S$ is I-adically complete, it is enough to show that I/I^2 is free of rank 1 as a $\mathbb{\Delta}_S/I$-module. Therefore, we would be done if we show that I is principal. For that, we note that since S is a quasi-regular semiperfectoid algebra, there is a perfctoid ring R mapping onto S. This gives a natural map

$$(A_{\text{inf}}(R), \text{Ker}(\theta)) \to (\mathbb{\Delta}_S, I)$$

or prisms. Note that $\text{Ker}(\theta)$ is a principal ideal ([BS19, Theorem 3.19]). Now [BS19, Lemma 3.5] implies that I is generated by image of $\text{Ker}(\theta)$ under the above map of prisms, and is therefore principal. This finishes the proof. ☐

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