AUTOMORPHISMS OF FROBENIUS TWISTED DE RHAM COHOMOLOGY

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ABSTRACT. In this short paper, we prove that the moduli of automorphisms of Frobenius twisted de Rham cohomology functor is given by $\mathbb{G}_m$. Our method is to use the notion of $\mathbb{G}_a^{\mathrm{perf}}$-modules and its connection to the de Rham cohomology functor introduced in [Mon22]. As an application of the induced $\mathbb{G}_m$-action, we reprove a result of Bhatt, Petrov, and Vologodsky on the decomposition of the Frobenius twisted de Rham complex.

1. Introduction

In [LM24], we described the moduli of endomorphisms of de Rham cohomology in a variety of scenarios and deduced Drinfeld's refinement [Dri24, § 5.12.1] to the Deligne–Illusie decomposition theorem [DI87] as an application (the latter conclusion is also due to Bhatt–Lurie, which uses "prismatization"; see [BL22a, Rmk. 4.7.18] and [BL22b, Rmk. 5.16]). In this paper, inspired by the results in [Bha23, Rmk 2.7.4, 2.7.5] and [Pet25], we consider the following moduli problem (Definition 1.2).

Definition 1.1. Let k be a perfect field. We use $F^*F_*\mathrm{dR}$ to denote the functor $(\mathrm{Sch}_k^{\mathrm{sm}})^{\mathrm{op}} \to \mathrm{CAlg}(k)$, defined by the following assignment

$$(\mathrm{Sch}_k^{\mathrm{sm}}) \ni X \mapsto F_{X/k}^* F_{X/k,*}\Omega_{X/k}^*,$$

where $\mathrm{CAlg}(k)$ denotes the ∞-category of E_∞-algebras over k.

Definition 1.2 (Moduli of automorphisms)**.** Let k be a perfect field. Let Alg_k denote the category of ordinary commutative k-algebras. We define a moduli functor

$$\mathcal{M}^{\mathrm{Aut}} \colon \mathrm{Alg}_k \to \infty\text{−Groupoids}$$

by the assignment

$$\mathrm{Alg}_k \ni B \mapsto \mathrm{Aut}(F^*F_*\mathrm{dR} \otimes_k B),$$

where $F^*F_*\mathrm{dR} \otimes_k B \colon (\mathrm{Sch}_k^{\mathrm{sm}})^{\mathrm{op}} \to \mathrm{CAlg}(B)$ is the functor base changed from Definition 1.1.

Our main new theorem is the following.

Theorem 1.3. *In the above set up, we have an isomorphism*

$$\mathcal{M}^{\mathrm{Aut}} \simeq \mathbb{G}_m,$$

compatible with the group structures.

To prove our result, we use quasi-syntomic descent, and general techniques for understanding certain "cohomological" functors on quasi-regular semiperfect algebras (which includes e.g., de Rham cohomology) as in [Mon22], which is renamed "semiperfect transmutation" in this paper. The key idea in the proof of Theorem 1.3 is to use this transmutation construction to reduce the calculation of automorphisms of $F^*F_*\,\mathrm{dR}$ to the calculation of automorphisms of a simpler, more tractable object, called an *augmented* $\mathbb{G}_a^{\mathrm{perf}}$*-module* (see Section 3). The results of this paper, combined with that of [LM24] are presented in the table below.

Cohomology theory	**Moduli of automorphisms**
$\mathrm{dR}\colon (\mathrm{Sch}_k^{\mathrm{sm}})^{\mathrm{op}} \to \mathrm{CAlg}(k)$	$\{\mathrm{id}\}$
$\mathrm{dR}^n\colon (\mathrm{Sch}_{W_n(k)}^{\mathrm{sm}})^{\mathrm{op}} \to \mathrm{CAlg}(k), n \geq 2$	$\mathbb{G}_m^\sharp$
$F^*F_*\mathrm{dR}\colon (\mathrm{Sch}_k^{\mathrm{sm}})^{\mathrm{op}} \to \mathrm{CAlg}(k)$	$\mathbb{G}_m$

As a consequence of Theorem 1.3, we obtain a $\mathbb{G}_m$-action on the functor $F^*F_*\mathrm{dR}$. The following result describes how this $\mathbb{G}_m$-action interacts with the conjugate filtration induced on $F^*F_*\,\mathrm{dR}$ (see Notation 2.5).

Theorem 1.4. *The $\mathbb{G}_m$-action preserves the conjugate filtration on $F^*F_*\,\mathrm{dR}$, and $\mathrm{gr}_i^{\mathrm{conj}} F^*F_*\,\mathrm{dR}[i]$ is a $\mathbb{G}_m$-representation purely of weight i.*

As a consequence of Theorem 1.4, we obtain

Corollary 1.5 (c.f. [Bha23, Rmk. 2.7.5] and [Pet25, Thm. 1.2])**.** *For a smooth k-scheme X, there is a functorial decomposition in $D(X, \mathcal{O}_{X^{(1)}})$*

$$F^*F_*\mathrm{dR}_X \cong \bigoplus_{i\geq 0} F^*\Omega^i_{X^{(1)}/k}[-i].$$

Remark 1.6. As observed in [Pet25], Corollary 1.5 implies that the de Rham complex decomposes for F-split varieties. We point out that in [Pet25, Thm. 1.2], Petrov proves a decomposition statement after pullback to $F_*W(\mathcal{O}_X)/p$, which proves the decomposition statement for quasi-F-split varieties as in [Pet25, Thm. 1.1]. The main difference in our approach is to deduce Corollary 1.5 from the moduli description in Theorem 1.3, which is proven by using the notion of augmented $\mathbb{G}_a^{\mathrm{perf}}$-modules.

2. Preliminary reductions

Let us fix $B \in \mathrm{Alg}_k$ and consider the functor $F^*F_*\mathrm{dR} \otimes_k B$. By left Kan extension, we can extend it to a functor still denoted as

$$F^*F_*\mathrm{dR} \otimes_k B \colon (\mathrm{Sch}_k^{\mathrm{qsyn}})^{\mathrm{op}} \to \mathrm{CAlg}(D(B)).$$

Proposition 2.1 (Quasisyntomic descent)**.** *The above functor $F^*F_*\mathrm{dR} \otimes_k B$* satisfies quasisyntomic descent*.*

Proof. By using the conjugate filtration on derived de Rham cohomology, and the fact that for a quasi-syntomic k-algebra S, the contangent complex $\mathbb{L}_{S/k}$ as an S-module has Tor amplitude in cohomological degrees $[-1, 0]$, we deduce that the functor $F^*F_*\mathrm{dR} \otimes_k B$ is coconnective. Since totalizations of coconnective objects commute with filtered colimits, to prove descent for $F^*F_*\mathrm{dR} \otimes_k B$, it suffices to prove descent for $F^*F_*\mathrm{dR}$ (as k is a field, B is a filtered colimit of finite dimensional k-vector spaces). So without loss of generality, we shall assume that $B := k$ throughout the proof.

To this end, since conjugate filtration is increasing and exhaustive, by commuting totalizations of connective objects with filtered colimits, it would be enough to prove that the functor

$$(\mathrm{Sch}_k^{\mathrm{Aff,qsyn}})^{\mathrm{op}} \ni S \mapsto \wedge^i \mathbb{L}_{S/k} \otimes_{S,\varphi_S} S \in D(k) \tag{2.1}$$

satisfies quasi-syntomic descent. We will argue for $i = 1$ and explain how to deduce the $i > 1$ case from it. Let $R \to S$ be a quasisyntomic cover. We will prove descent for this cover, i.e., we will show that there is a natural isomorphism

$$\mathbb{L}_{R/k} \otimes_{R,\varphi_R} R \simeq \mathrm{Tot}(\mathbb{L}_{S^\bullet/k} \otimes_{S^\bullet,\varphi} S^\bullet),$$

where $S^\bullet$ denotes the cosimplicial ring obtained by taking the Cech nerve of $R \to S$. Note that we have a fiber sequence in the category of cosimplicial objects of $D(k)$ of the form

$$\mathbb{L}_{R/k} \otimes_R S^\bullet \otimes_{S^\bullet,\varphi} S^\bullet \to \mathbb{L}_{S^\bullet/k} \otimes_{S^\bullet,\varphi} S^\bullet \to \mathbb{L}_{S^\bullet/R} \otimes_{S^\bullet,\varphi} S^\bullet. \tag{2.2}$$

By faithfully flat descent,

$$\mathrm{Tot}(\mathbb{L}_{R/k} \otimes_R S^\bullet \otimes_{S^\bullet,\varphi} S^\bullet) \simeq \mathrm{Tot}(\mathbb{L}_{R/k} \otimes_{R,\varphi_R} R \otimes_R S^\bullet) \simeq \mathbb{L}_{R/k} \otimes_{R,\varphi_R} R.$$

Therefore, in order to prove descent, we need to show that

$$\mathrm{Tot}(\mathbb{L}_{S^\bullet/R} \otimes_{S^\bullet,\varphi} S^\bullet) \simeq 0. \tag{2.3}$$

Note that for each $[n] \in \Delta$, $S^{[n]}$ appearing in $S^\bullet$ is quasisyntomic, and thus $\mathbb{L}_{S^{[n]}/k}$ has Tor amplitiude in cohomological degrees $[-1, 0]$ as $S^{[n]}$-module. By using the transitivity triangle for the cotangent complex associated to the maps $k \to R \to S^{[n]}$, it follows that each term $\mathbb{L}_{S^{[n]}/R} \otimes_{S^{[n]},\varphi} S^{[n]}$ is concentrated in cohomological degrees $[-2, 0]$. To prove (2.3), it suffices to show that $\mathrm{Tot}(\mathbb{L}_{S^\bullet/R} \otimes_{S^\bullet,\varphi} S^\bullet) \otimes_R S \simeq 0$. Let $S \to T^\bullet$ denote the base change of $R \to S^\bullet$ along $R \to S$. Since $R \to S$ is flat, one can write S as a filtered colimit of finite free R-modules. By commutating totalization of (cohomologically) uniformly bounded below objects with filtered colimits and using base change properties of the cotangent complex, we have

$$\mathrm{Tot}(\mathbb{L}_{S^\bullet/R} \otimes_{S^\bullet,\varphi} S^\bullet) \otimes_R S \simeq \mathrm{Tot}(\mathbb{L}_{T^\bullet/S} \otimes_{T^\bullet,\varphi} T^\bullet).$$

Note that $S \to T^\bullet$ can be viewed as a split augmented cosimplicial S-algebra since it is obtained from Cech nerve of $S \to S \otimes_R S$ which admits a section. By [Mat16, Example 3.11], this implies that

$$\mathrm{Tot}(\mathbb{L}_{T^\bullet/S} \otimes_{T^\bullet,\varphi} T^\bullet) \simeq \mathbb{L}_{S/S} \otimes_{S,\varphi_S} S \simeq 0,$$

as desired. This finishes the proof in the case $i = 1$. For $i > 1$, one argues similarly by using the graded pieces of the finite filtration on $\wedge^i \mathbb{L}_{S^\bullet/k} \otimes_{S^\bullet,\varphi} S^\bullet$ induced from (2.2) (see [BMS19, p. 213]). □

Remark 2.2. The argument for proving the descent statement for (2.1) essentially follows the same idea in [BMS19, Thm. 3.1], with the technical modification of using split cosimplicial objects to avoid talking about "homotopy of cosimplicial objects" in the ∞-categorical set up as in our situation. A flat descent statement for the functor in (2.1) appears in [Pet25, Lem. 5.10].

Let QRSP_k denote the cateory of semi-perfect k-algebras which are quasi-regular over k.[1] They form a basis for the quasisyntomic topology on $\mathrm{Sch}_k^{\mathrm{qsyn}}$. For more discussions on this notion, we refer the readers to [BMS19, §4].

Proposition 2.3. *If $S \in \mathrm{QRSP}_k$, then $F^*F_*\mathrm{dR}(S) \otimes_k B$ is a discrete ring.*

Proof. This follows from the conjugate filtration on (Frobenius twisted) de Rham cohomology (see [Bha12, Proposition 3.5]) and the fact that $\mathbb{L}_{S/k}$ is isomorphic to a flat S-module concentrated in cohomological degree -1. □

Corollary 2.4. *The space of automorphisms $\mathrm{Aut}(F^*F_*\mathrm{dR} \otimes_k B)$ is discrete, and similarly for the space of endomorphisms $\mathrm{End}(F^*F_*\mathrm{dR} \otimes_k B)$.*

Proof. This follows from Proposition 2.1, Proposition 2.3 by considering right extensions and using as in the proof of [LM24, Lemma 3.3]. □

In the proof of Proposition 2.3, we have used the conjugate filtration on p-adic derived de Rham cohomology due to Bhatt. By pulling back along Frobenius, we have an induced filtration on Frobenius twisted de Rham cohomology.

Notation 2.5. Let us denote the Frobenius pullback of the conjugate filtration as

$$\mathrm{Fil}_\bullet^{\mathrm{conj}}(F^*F_*\mathrm{dR} \otimes_k B) := F^*(\mathrm{Fil}_\bullet^{\mathrm{conj}} F_*\mathrm{dR}) \otimes_k B.$$

If there is no risk of confusion, we still refer to the above as the conjugate filtration on $F^*F_*\mathrm{dR} \otimes_k B$.

Therefore we may view $F^*F_*\mathrm{dR} \otimes_k B$ as a filtered $\mathrm{Fil}_0^{\mathrm{conj}} = \mathcal{O} \otimes_k B$-algebra.

Proposition 2.6. *Any endomorphism $\sigma \in \mathrm{End}(F^*F_*\mathrm{dR} \otimes_k B)$ must preserve the conjugate filtration. Any automorphism of $\mathcal{O} \otimes_k B$ is identity, therefore any automorphism of $F^*F_*\mathrm{dR} \otimes_k B$ must be linear over $\mathcal{O} \otimes_k B$.*

Proof. The first sentence follows from the same proof of [LM24, Lemma 5.3]. For the second sentence: any automorphism of $\mathcal{O} \otimes_k B$ induces an automorphism of $B[x]$ by applying to the algebra $k[x]$. By functoriality, this would necessarily be determined by an automorphism of $\mathbb{A}_B^1$ as a ring scheme, hence must be identity. Since for any k-algebra R, the algebra $R \otimes_k B$ is generated by subalgebras of the form $f(k[x]) \otimes_k B$ for varying maps $f\colon k[x] \to R$, we conclude again by functoriality that the induced automorphism on $R \otimes_k B$ is identity. □

[1]From now on we will call them *qrsp over k*.

3. AUGMENTED $\mathbb{G}_a^{\mathrm{perf}}$-MODULES AND TRANSMUTATION

Below, we fix a k-algebra B and introduce some definitions that will be required.

Definition 3.1. We denote $\mathbb{G}_a^{\mathrm{perf}} := \mathrm{Spec}(\mathbb{F}_p[x^{1/p^\infty}])$ the ring scheme representing the functor sending any $\mathbb{F}_p$-algebra A to its inverse limit perfection $A^\flat := \lim_{a \mapsto a^p} A$. By base change to $\mathrm{Spec}(B)$, we again obtain a ring scheme $\mathbb{G}_{a,B}^{\mathrm{perf}}$. When there is no risk of confusion, we will omit the subscript B and still denote the above by $\mathbb{G}_a^{\mathrm{perf}}$. Its underlying scheme structure is given by $\mathrm{Spec}(B[x^{1/p^\infty}])$.

Definition 3.2 ([Mon22, Definition 2.2.5])**.** A $\mathbb{G}_a^{\mathrm{perf}}$-module is a $\mathbb{G}_a^{\mathrm{perf}}$-module object in the category Aff_B of affine schemes over $\mathrm{Spec}(B)$.

Remark 3.3 ([Mon22, Remark 2.2.8])**.** Let us view $\mathbb{G}_a^{\mathrm{perf}}$ as a monoid scheme, then the category of affine schemes over $\mathrm{Spec}(B)$ together with an action of this monoid is anti-equivalent to $\mathbb{N}[1/p]$-graded B-algebras. The category of abelian group objects in the former category is then anti-equivalent to $\mathbb{N}[1/p]$-graded bi-commutative Hopf-B-algebras. Finally the category of $\mathbb{G}_a^{\mathrm{perf}}$-modules is anti-equivalent to $\mathbb{N}[1/p]$-graded bi-commutative Hopf B-algebras which satisfy a compatibility (analogous to the diagram in [Mon22, Remark 2.1.8]) between summation on $\mathbb{G}_a^{\mathrm{perf}}$ and the graded Hopf-B-algebra structure. As a consequence of this compatibility, one can show that the degree 0 component, which is a priori a random Hopf-B-algebra, is actually given by B (see [Mon22, Proposition 2.2.9]).

Definition 3.4 (Augmented $\mathbb{G}_a^{\mathrm{perf}}$-module)**.** An augmented $\mathbb{G}_a^{\mathrm{perf}}$-module is a $\mathbb{G}_a^{\mathrm{perf}}$-module M together with a map $d\colon M \to \mathbb{G}_a^{\mathrm{perf}}$ of $\mathbb{G}_a^{\mathrm{perf}}$-modules.

Remark 3.5. The above definition appeared in c.f. [Mon22, Definition 2.2.14] where it was called pointed $\mathbb{G}_a^{\mathrm{perf}}$-module.

We recall the following construction, which is a generalization of [Mon22, § 3.4] to an arbirary base ring B of characteristic p. This construction was called "unwinding" in [Mon22], which is called "semiperfect transmutation" or simply "transmutation" here, following the more recent terminology in [Bha23]. Morally speaking, these constructions allow one to build a cohomology theory out of a *single* (!) highly structured object, namely an augmented $\mathbb{G}_a^{\mathrm{perf}}$-module (as in [Mon22]), or a ring stack (as in [Bha23], [LM24]).

Construction 3.6 (Semiperfect transmutation)**.** Let $\mathscr{P}I$ denote the category of pairs (S, I) where S is a perfect k-algebra and I is an ideal of S. Let $d\colon M \to \mathbb{G}_a^{\mathrm{perf}}$ be an augmented $\mathbb{G}_a^{\mathrm{perf}}$-module over B. Below, out of the data (M, d), we shall construct a functor denoted as

$$\mathrm{Tm}(d)\colon \mathscr{P}I \to \mathrm{Alg}_B.$$

Let $(S, I) \in \mathscr{P}I$, we first construct an affine scheme $\mathrm{Sec}_{(S,I)}(d)$ over $\mathrm{Spec}(S \otimes_k B)$ that roughly speaking, is the moduli of *sections of* $d\colon M \to \mathbb{G}_a^{\mathrm{perf}}$ *over the ideal* I. More precisely, note that we have a functor

$$\mathrm{Aff}_{S\otimes_k B}^{\mathrm{op}} \to \mathrm{Sets}$$

defined by

$$\mathrm{Aff}_{S\otimes_k B}^{\mathrm{op}} \ni \mathrm{Spec}(A) \mapsto \mathrm{Eq}\left(\mathrm{Hom}_S(I, M(A)) \underset{*}{\overset{d}{\rightrightarrows}} \mathrm{Hom}_S(I, A^\flat)\right).$$

Recall that $A^\flat := \mathbb{G}_a^{\mathrm{perf}}(A)$. Let us explain the above definition: Applying the limit perfection functor to the natural maps $S \to S \otimes_k B \to A$, we get a composite map $S \to A^\flat$. Therefore we get a natural S-module structure on the $A^\flat$-module $M(A)$. The map $*$ denotes the constant map to the element $* \in \mathrm{Hom}_S(I, A^\flat)$ given by the composition $I \subseteq S \to A^\flat$. In other words, we are looking at the set of dashed arrows that makes the diagram

$$\begin{array}{ccc} & & I \\ & \swarrow & \downarrow{\scriptstyle *} \\ M(A) & \xrightarrow{d} & A^\flat \end{array}$$

commutative. By the adjoint functor theorem, the functor $\mathrm{Aff}^{\mathrm{op}}_{S\otimes_k B} \to \mathrm{Sets}$ is representable by an affine scheme over $\mathrm{Spec}\,(S \otimes_k B)$ which we define to be $\mathrm{Sec}_{(S,I)}(d)$. The functor $\mathrm{Tm}(d)$ is defined by

$$\mathscr{P}I \ni (S, I) \mapsto \Gamma(\mathrm{Sec}_{(S,I)}(d), \mathcal{O}).$$

By definition, this construction is covariantly functorial in the pair (S, I) and contravariantly functorial in the augmented $\mathbb{G}_a^{\mathrm{perf}}$-module (M, d). Hence the above construction produces a functor

$$\mathrm{Tm}\colon \left(\mathbb{G}_a^{\mathrm{perf}} - \mathrm{Mod}_*\right)^{\mathrm{op}} \to \mathrm{Fun}(\mathscr{P}I, \mathrm{Alg}_B),$$

by the assignment $d \mapsto \mathrm{Tm}(d)$, where the source denotes the opposite category of the category of augmented $\mathbb{G}_a^{\mathrm{perf}}$-modules.

Example 3.7. The category $\mathscr{P}I$ contains the category Perf_k of perfect k-algebras via $S \mapsto (S, (0))$. The functor of points of $\mathrm{Sec}_{(S,(0))}(d)$ is given by the singleton, therefore we have $\mathrm{Tm}(d)(S, (0)) \simeq S \otimes_k B$ for any augmented $\mathbb{G}_a^{\mathrm{perf}}$-module (M, d).

Example 3.8. The category $\mathscr{P}I$ has a particular object, which will be of interest to us later, given by $(S, I) := (k[x^{1/p^\infty}], x)$. We will calculate the value of $\mathrm{Tm}(d)$ evaluated at this object. Since the ideal I is free rank 1 as an S-module, the scheme $\mathrm{Sec}_{(S,I)}(d)$ represents the functor $\mathrm{Aff}^{\mathrm{op}}_{S\otimes_k B} \ni \mathrm{Spec}(A) \mapsto \mathrm{Eq}\left(M(A) \underset{*}{\overset{d}{\rightrightarrows}} A^\flat\right)$. Unwinding definitions, this amounts to finding dashed arrows of algebras making the diagram below commutative:

$$\begin{array}{ccc} B[x^{1/p^\infty}] & \xrightarrow{\simeq} & S \otimes_k B \\ \big\downarrow{\scriptstyle d^*} & & \big\downarrow \\ \mathcal{O}(M) & \dashrightarrow & A. \end{array}$$

Therefore we see that $\mathrm{Sec}_{(S,I)}(d) \simeq M$ with structure map to $\mathrm{Spec}(S \otimes_k B)$ given by d, and hence we have an identification

$$\begin{array}{ccc} \mathrm{Tm}(d)(k[x^{1/p^\infty}], (0)) & \xrightarrow{\simeq} & \mathcal{O}(\mathbb{G}_a^{\mathrm{perf}}) \\ {\scriptstyle \mathrm{Tm}(d)(x\mapsto x)}\big\downarrow & & \big\downarrow{\scriptstyle d^*} \\ \mathrm{Tm}(d)(k[x^{1/p^\infty}], (x)) & \xrightarrow{\simeq} & \mathcal{O}(M) \end{array}$$

compatible with the identification in Example 3.7.

Remark 3.9. Using Example 3.8 and the functoriality of Construction 3.6 applied to the map of pairs $F\colon (k[x^{1/p^\infty}], x) \to (k[x^{1/p^\infty}], x)$ which sends $x \mapsto x^p$, we see that for any $(M, d) \in \mathbb{G}_a^{\mathrm{perf}} - \mathrm{Mod}_*$, there is a natural map $\mathcal{O}(M) \xrightarrow{\text{“Frob”}} \mathcal{O}(M)$ which fits into a natural commutative diagram of B-algebras:

$$\begin{array}{ccc} \mathcal{O}(M) & \xrightarrow{\text{“Frob”}} & \mathcal{O}(M) \\ \big\uparrow & & \big\uparrow \\ B[x^{1/p^\infty}] & \xrightarrow{x\mapsto x^p} & B[x^{1/p^\infty}]. \end{array}$$

Remark 3.10. Let (M, d) be an augmented $\mathbb{G}_{a,B}^{\mathrm{perf}}$-module. Unraveling definitions, we see that the identifications in Example 3.7 and Example 3.8 make the following diagram commute:

$$\begin{array}{ccccc} \mathcal{O}(M) & \xrightarrow{\simeq} & \mathrm{Tm}(d)(k[x^{1/p^\infty}], (x)) & \xrightarrow{\mathrm{Tm}(d)(x\mapsto x\otimes x)} & \mathrm{Tm}(d)(k[x^{1/p^\infty}] \otimes_k k[x^{1/p^\infty}], (1 \otimes x)) \\ \big\downarrow{\scriptstyle \mathrm{act}^*} & & & & \big\uparrow \\ \mathcal{O}(\mathbb{G}_{a,B}^{\mathrm{perf}} \times_{\mathrm{Spec}(B)} M) & \xrightarrow{\simeq} (k[x^{1/p^\infty}] \otimes_k B) \otimes_B \mathcal{O}(M) & \xrightarrow{\simeq} & & \mathrm{Tm}(d)(k[x^{1/p^\infty}], (0)) \otimes_B \mathrm{Tm}(d)(k[x^{1/p^\infty}], (x)). \end{array}$$

Example 3.11. The category $(\mathbb{G}_a^{\mathrm{perf}} - \mathrm{Mod}_*)^{\mathrm{op}}$ has initial object given by $\mathbb{G}_a^{\mathrm{perf}} \xrightarrow{\mathrm{id}} \mathbb{G}_a^{\mathrm{perf}}$. Since $d\colon M(A) \to A^\flat$ in this case is an isomorphism, the functor of $\mathrm{Sec}_{(S,I)}(d)$ is given by singleton. Therefore we have $\mathrm{Tm}(\mathrm{id})(S, I) = S \otimes_k B$.

Remark 3.12 (Base change)**.** The transmutation process also satisfies two base change properties: let $S \to T$ be a map of perfect k-algebras such that $I \otimes_S T \simeq I \cdot T$, then for any ideal $I \subset S$ and $d \in (\mathbb{G}_a^{\mathrm{perf}} - \mathrm{Mod}_*)^{\mathrm{op}}$, we have $\mathrm{Tm}(d)(T, I \cdot T) \simeq \mathrm{Tm}(d)(S, I) \otimes_S T$. Let $B \to B'$ be a map of k-algebras, then for any $d \in (\mathbb{G}_{a,B}^{\mathrm{perf}} - \mathrm{Mod}_*)^{\mathrm{op}}$ by base change we obtain a $d' \in (\mathbb{G}_{a,B'}^{\mathrm{perf}} - \mathrm{Mod}_*)^{\mathrm{op}}$, and we have a natural equivalence $\mathrm{Tm}(d') \simeq \mathrm{Tm}(d) \otimes_B B'$ as functors $\mathscr{P}I \to \mathrm{Alg}_{B'}$.

Definition 3.13. Note that there is a functor $\mathrm{QRSP}_k \to \mathscr{P}I$ defined by

$$\mathrm{QRSP}_k \ni S \mapsto (S^\flat, \mathrm{Ker}(S^\flat \twoheadrightarrow S)).$$

This induces a functor also denoted as

$$\mathrm{Tm}\colon \left(\mathbb{G}_a^{\mathrm{perf}} - \mathrm{Mod}_*\right)^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B).$$

Example 3.14. Consider the quasi-ideal $\mathbb{G}_a^{\mathrm{perf}} \xrightarrow{\mathrm{id}} \mathbb{G}_a^{\mathrm{perf}}$. Then by Example 3.11, we have a natural equivalence of functors $\mathrm{Tm}(\mathrm{id}) \simeq (-)^\flat \otimes_k B\colon \mathrm{QRSP}_k \to \mathrm{Alg}_B$. Let us denote this functor by $\mathfrak{G}$.

Since $\mathbb{G}_a^{\mathrm{perf}} \xrightarrow{\mathrm{id}} \mathbb{G}_a^{\mathrm{perf}}$ is the initial object in $(\mathbb{G}_a^{\mathrm{perf}} - \mathrm{Mod}_*)^{\mathrm{op}}$. By the above example, it follows that Definition 3.13 naturally lifts to a functor

$$\mathrm{Tm}\colon \left(\mathbb{G}_a^{\mathrm{perf}} - \mathrm{Mod}_*\right)^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)_{\mathfrak{G}/}, \tag{3.1}$$

where the latter denotes the under category associated with $\mathfrak{G}$.

4. Nilpotent quasi-ideals in $\mathbb{G}_a^{\mathrm{perf}}$

In this section, we restrict attention to a special class of augmeted $\mathbb{G}_a^{\mathrm{perf}}$-modules, which we call nilpotent quasi-ideals in $\mathbb{G}_a^{\mathrm{perf}}$, for which the transmutation construction will be particularly well-behaved. The notion of a quasi-ideal is due to Drinfeld [Dri24].

Definition 4.1. We define a full subcategory $\mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{G}/}$ of $\mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)_{\mathfrak{G}/}$ spanned by functors F that satisfies the three conditions below.

(1) The map $\mathfrak{G}(S) \to F(S)$ is an isomorphism for every perfect ring S.
(2) The natural map $F\left(\frac{k[x^{1/p^\infty}]}{x}\right) \otimes_{F(k)} F(S) \to F\left(\frac{S[x^{1/p^\infty}]}{x}\right)$ is an isomorphism for every perfect ring S.
(3) The natural map $F\left(\frac{S[x^{1/p^\infty}]}{x}\right) \otimes_{F(S)} F\left(\frac{S[x^{1/p^\infty}]}{x}\right) \to F\left(\frac{S[x^{1/p^\infty}]}{x} \otimes_S \frac{S[x^{1/p^\infty}]}{x}\right)$ is an isomorphism for every perfect ring S.

It is worth pointing out that the functor $\mathfrak{G}$ itself does not satisfy either of the conditions (2) and (3) above: Indeed we have $\mathfrak{G}(S[x^{1/p^\infty}]/(x)) = S[\![x^{1/p^\infty}]\!] \otimes_k B$ for any perfect k-algebra S, so only a "topological" version of (2) and (3) holds true. Below, we shall single out a certain full subcategory of $\mathbb{G}_a^{\mathrm{perf}} - \mathrm{Mod}_*$, whose objects $d\colon M \to \mathbb{G}_a^{\mathrm{perf}}$ have transmutations $\mathrm{Tm}(d)$ satisfying the three conditions above. This leads us to the notion of *nilpotent* quasi-ideals, where a certain nilpotency condition is used to discretise the "topological" issue noted earlier.

Definition 4.2 (Nilpotent quasi-ideals)**.** A quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$ is an augmented $\mathbb{G}_a^{\mathrm{perf}}$-module $d : M \to \mathbb{G}_a^{\mathrm{perf}}$, such that $d(y)x = d(x)y$ for all (scheme theoretic) points x, y of M.

A nilpotent quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$ is a quasi-ideal $d\colon M \to \mathbb{G}_a^{\mathrm{perf}}$ such that the image of x under the map $d^*\colon B[x^{1/p^\infty}] \to \Gamma(M, \mathcal{O})$ induced by d is nilpotent. We denote the corresponding full subcategory by $\mathcal{N}\mathrm{QID} - \mathbb{G}_a^{\mathrm{perf}} \subset \mathbb{G}_a^{\mathrm{perf}} - \mathrm{Mod}_*$.

Remark 4.3. By using Example 3.8, Remark 3.10 and diagrams before and inside [Mon22, Definition 3.2.10], we see that if $\mathrm{Tm}(d)$ satisfies the conditions in Definition 4.1, then the augmented module $d\colon M \to \mathbb{G}_a^{\mathrm{perf}}$ is necessarily a quasi-ideal.

Remark 4.4. In the above definition, since d is a map of $\mathbb{G}_a^{\text{perf}}$-modules, the map $d^*\colon B[x^{1/p^\infty}] \to \Gamma(M, \mathcal{O})$ is a map of $\mathbb{N}[1/p]$-graded rings. Note that 1 has degree zero and $d^*(x)$ has degree 1, it follows that $d^*(x)$ is nilpotent if and only if $1 - d^*(x)$ is a unit.

Proposition 4.5. *The functor from* (3.1) *factors to give a functor*

$$\mathrm{Tm}\colon \left(\mathcal{N}\mathrm{QID} - \mathbb{G}_a^{\text{perf}}\right)^{\text{op}} \to \mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{G}/}.$$

Proof. This amounts to checking that the three conditions in Definition 4.1 are satisfied. The first one follows from Example 3.7, and the latter two follows from the proof of [Mon22, Prop. 3.4.16, 3.4.17]. □

Remark 4.6. Let us sketch the idea of showing that transmutation of a nilpotent quasi-ideal satisfies the second condition of Definition 4.1: Recall that the transmutation is defined via a moduli interpretation (see Construction 3.6), so we need to show that the two corresponding moduli functors are naturally isomorphic. The only subtlety is the domains of these moduli functors, assume that $d^*(x^N) = 0$, then we just need to observe the following identification of categories

$$\{S[\![x^{1/p^\infty}]\!]\text{-algebras} \mid x^N = 0\} \cong \{S[x^{1/p^\infty}]\text{-algebras} \mid x^N = 0\},$$

which gives the identification of domains of these moduli functors. Using the same logic, one checks that for $(M, d) \in \mathbb{G}_a^{\text{perf}} - \mathrm{Mod}_*$ such that $d^*(x^N) = 0$ for some $N \in \mathbb{N}$. Then for $(S, I) = (k[\![x^{1/p^\infty}]\!], (x)) \in \mathscr{P}I$, the analog of Example 3.8 still holds true, namely $\mathrm{Sec}_{(S,I)}(d) \cong M$ as schemes over $\mathbb{G}_a^{\text{perf}}$.

Construction 4.7. Consider the quasi-ideal in $\mathbb{G}_{a,k}^{\text{perf}}$

$$\alpha^\flat := \mathrm{Ker}(\mathbb{G}_a^{\text{perf}} \to \mathbb{G}_a).$$

Now let $F \in \mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{G}/}$, we may apply $\mathrm{Spec}(F(-))$ to the map $k[x^{1/p^\infty}] \twoheadrightarrow k[x^{1/p^\infty}]/(x) \cong \mathcal{O}(\alpha^\flat)$. By the conditions in Definition 4.1, the result is a quasi-ideal in $\mathbb{G}_a^{\text{perf}}$. Next we claim that the resulting quasi-ideal is nilpotent: Using Remark 4.4 it is equivalent to checking the image of $1 - x$ being a unit in $F(k[x^{1/p^\infty}]/(x))$. Note that the map $k[x^{1/p^\infty}] \twoheadrightarrow k[x^{1/p^\infty}]/(x)$ factors through the x-adic completion of the source, which is perfect and in which $1 - x$ is already a unit, so our claim follows from the first condition in Definition 4.1. Since $\mathrm{Spec}(-)$ is contravariant, the above "restriction to $\alpha^\flat$" process defines a functor

$$\mathrm{Res}\colon \mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{G}/} \to \left(\mathcal{N}\mathrm{QID} - \mathbb{G}_a^{\text{perf}}\right)^{\text{op}}.$$

Proposition 4.8 (Full faithfulness of transmutation)**.** *Let B be a k-algebra fixed as before. The functors*

$$\mathrm{Tm}\colon \left(\mathcal{N}\mathrm{QID} - \mathbb{G}_a^{\text{perf}}\right)^{\text{op}} \rightleftarrows \mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{G}/} \colon \mathrm{Res}$$

are adjoint to each other and the left adjoint Tm *is fully faithful.*

Proof. Using Proposition 4.5 and Remark 4.6, we first get an equivalence $\mathrm{id} \xrightarrow{\sim} \mathrm{Res} \circ \mathrm{Tm}$, which is the desired unit. Next we need to construct a counit and check compatibility conditions. The key idea is to use the universal property of transmutation construction to produce a natural transformation $\mathrm{Tm}(\mathrm{Res}\, F) \to F$ for any $F \in \mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{G}/}$. For details, see the proof of [Mon22, Prop. 3.2.18, Prop. 3.4.20]. Note that the unit map $\mathrm{id} \xrightarrow{\sim} \mathrm{Res} \circ \mathrm{Tm}$ is an equivalence, we see that the left adjoint Tm is fully faithful. □

Construction 4.9 (Pullback functors)**.** For any ring scheme R, one can define a notion of augmented R-modules as in Definition 3.4. For a map of ring schemes $u\colon R_1 \to R_2$, and an augmented R_2-module $d\colon M \to R_2$, the scheme theoretic pullback $M \times_{R_2} R_1 \to R_1$ is naturally an augmented R_1-module, which we denote by u^*d.

Proposition 4.10. *Let $u\colon \mathbb{G}_a^{\text{perf}} \to \mathbb{G}_a$. Then the pullback functor from augmented $\mathbb{G}_a$-modules to augmented $\mathbb{G}_a^{\text{perf}}$-modules is fully faithful.*

Proof. See [Mon22, Prop. 2.2.17]. □

Let us give some examples of nilpotent quasi-ideals and their corresponding transmutations.

Example 4.11. The functor $\mathrm{id} \otimes \mathrm{B}\colon \mathrm{QRSP}_k \to \mathrm{Alg}_B$ that sends $S \mapsto S \otimes_k B$ is isomorphic to $\mathrm{Tm}(\alpha^\flat \to \mathbb{G}_a^{\mathrm{perf}})$.

Example 4.12 (de Rham cohomology, see [Mon22, Prop. 4.0.3])**.** Consider the quasi-ideal in $\mathbb{G}_a$ given by $W[F] \to \mathbb{G}_a$, and let us pullback this quasi-ideal along the map $u\colon \mathbb{G}_a^{\mathrm{perf}} \to \mathbb{G}_a$ of ring schemes. Then the derived de Rham cohomology, viewed as an object $\mathrm{dR} \in \mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{G}/}$ is equivalent to $\mathrm{Tm}\left(u^*(W[F] \to \mathbb{G}_a)\right)$.

Construction 4.13 (Frobenius pushforwards)**.** Let $\varphi\colon \mathbb{G}_a^{\mathrm{perf}} \to \mathbb{G}_a^{\mathrm{perf}}$ be the relative Frobenius map over B. Given a $\mathbb{G}_a^{\mathrm{perf}}$-module M, one can consider the tensor product $M^{(1/p)} := M \otimes_{\mathbb{G}_a^{\mathrm{perf}},\varphi} \mathbb{G}_a^{\mathrm{perf}}$, which is representable by an affine scheme and is naturally a $\mathbb{G}_a^{\mathrm{perf}}$-module. Now given an augmented $\mathbb{G}_a^{\mathrm{perf}}$-module $d\colon M \to \mathbb{G}_a^{\mathrm{perf}}$, we may compose the maps

$$M^{(1/p)} \to \mathbb{G}_a^{\mathrm{perf}\,(1/p)} \xrightarrow{\varphi} \mathbb{G}_a^{\mathrm{perf}}$$

to obtain another augmented $\mathbb{G}_a^{\mathrm{perf}}$-module that we denote by $\varphi_* d\colon M^{(1/p)} \to \mathbb{G}_a^{\mathrm{perf}}$.

On the other hand, given an object $\mathfrak{G} \to F$ of $\mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)_{\mathfrak{G}/}$, we set

$$F^{(1/p)}(S) := F(\varphi_{k,*}S).$$

There is a natural transformation of functors $\mathfrak{G}^{(1/p)} \to F^{(1/p)}$. For any $S \in \mathrm{QRSP}_k$, the k-linear Frobenius map $S^\flat \to \varphi_{k,*}S^\flat$ induces a map $S^\flat \otimes_k B \to \varphi_{k,*}S^\flat \otimes_k B$ that gives a natural transformation $\mathfrak{G} \xrightarrow{\mathrm{Frob}} \mathfrak{G}^{(1/p)}$. The composition $\mathfrak{G} \xrightarrow{\mathrm{Frob}} \mathfrak{G}^{(1/p)} \to F^{(1/p)}$ is naturally an object of $\mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)_{\mathfrak{G}/}$, which will be denoted as $\mathrm{Frob}_*(\mathfrak{G} \to F)$. By construction, transmutation is compatible with the two operations we discussed. Namely, we have

$$\mathrm{Tm}(\varphi_* d) \simeq \mathrm{Frob}_*(\mathrm{Tm}(d)).$$

By the same logic of Remark 3.9, using the equivalence $\mathrm{id} \xrightarrow{\sim} \mathrm{Res} \circ \mathrm{Tm}$ from Proposition 4.8, for any $(M, d) \in \mathcal{N}\mathrm{QID} - \mathbb{G}_a^{\mathrm{perf}}$, we get a map $\varphi_M\colon \varphi_* d \to d$ in $\mathcal{N}\mathrm{QID} - \mathbb{G}_a^{\mathrm{perf}}$, which corresponds to the map $\mathrm{Tm}(d) \to \mathrm{Frob}_*(\mathrm{Tm}(d))$ given by applying functoriality to the relative Frobenius of the input $S \to \varphi_* S$.

In general the map φ_M from above is not transparent, but in the case of $\alpha^\flat$ it is easy to determine.

Lemma 4.14. *The map $\varphi_{\alpha^\flat}$ is given by the "B-linear Frobenius":*

$$\alpha^{\flat,(1/p)} \cong \mathrm{Spec}(B[x^{1/p^\infty}]/(x)) \xrightarrow{x^{1/p^{i+1}} \mapsto x^{1/p^i}} \mathrm{Spec}(B[x^{1/p^\infty}]/(x)) \cong \alpha^\flat.$$

Note that the element $x^i \in \mathcal{O}(\alpha^{\flat,(1/p)})$ has degree i/p, therefore the relative Frobenius is indeed a graded map.

Proof. This follows from the commutative diagram from Remark 3.9. □

Remark 4.15. Note that the category of augmented $\mathbb{G}_a^{\mathrm{perf}}$-modules have pullbacks. In fact, the forgetfull functor $\mathbb{G}_a^{\mathrm{perf}} - \mathrm{Mod}_* \to \mathbb{G}_a^{\mathrm{perf}} - \mathrm{Mod}$ that sends $(M \to \mathbb{G}_a^{\mathrm{perf}}) \mapsto M$ preserves pullbacks. Furthermore, the category $\mathcal{N}\mathrm{QID} - \mathbb{G}_a^{\mathrm{perf}}$ is closed under pullbacks and the (contravariant) transmutation functor

$$\mathrm{Tm}\colon \left(\mathcal{N}\mathrm{QID} - \mathbb{G}_a^{\mathrm{perf}}\right)^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{G}/}$$

takes pullbacks to pushouts.

Example 4.16. The global sections of the underlying $\mathbb{G}_a^{\mathrm{perf}}$-module $u^*W[F]$ gives a $\mathbb{N}[1/p]$-graded B-Hopf algebra, whose underlying $\mathbb{N}[1/p]$-graded B-algebra is explicitly described as

$$\frac{B[x_0^{1/p^\infty}, x_1, \ldots, x_i, \ldots]}{x_i^p},$$

where $\deg x_i = p^i$ for all $i \in \mathbb{N}$. The global sections of $\alpha^\flat$ can be explicitly described as $B[x_0^{1/p^\infty}]/x_0$, where $\deg x_0 = 1$. Note that there is no map $u^*W[F] \dashrightarrow \alpha^\flat$ of augmented $\mathbb{G}_a^{\mathrm{perf}}$-modules.

Remark 4.17. The 0-th conjugate filtration (Notation 2.5) gives rise to a map $\mathrm{id} \otimes_k B \to F_* \mathrm{dR} \otimes_k B$ in $\mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{G}/}$. By Example 4.11, the source is $\mathrm{Tm}(\alpha^\flat)$. By Example 4.12 and Construction 4.13, the target is $\mathrm{Tm}(u^* W[F])^{(1/p)})$. Then according to Proposition 4.8, the 0-th conjugate filtration must be given, via transmutation, by a map $(u^*W[F])^{(1/p)} \to \alpha^\flat$ of nilpotent quasi-ideals in $\mathbb{G}_a^{\mathrm{perf}}$.

There is unique such map because the augmentation $d\colon \alpha^\flat \hookrightarrow \mathbb{G}_a^{\mathrm{perf}}$ is a closed immersion. Below let us explicate this map. Note that the $\mathbb{N}[1/p]$-graded B-algebra underlying $(u^*W[F])^{(1/p)}$ can be described as

$$\frac{B[x_0^{1/p^\infty}, x_1, \ldots, x_i, \ldots]}{x_i^p},$$

where $\deg x_i = p^{i-1}$ for all $i \in \mathbb{N}$. Then the map must be given by

$$\mathcal{O}(\alpha^\flat) \cong \frac{B[x_0^{1/p^\infty}]}{x_0} \to \frac{B[x_0^{1/p^\infty}, x_1, \ldots, x_i, \ldots]}{x_i^p},\ x_0^i \mapsto x_0^{pi},$$

as it has to be a map of schemes over $\mathbb{G}_a^{\mathrm{perf}}$.

Now we describe an augmented $\mathbb{G}_a^{\mathrm{perf}}$-module that will be used in the next proposition.

Construction 4.18. Let $W[F]^{(p)}$ denote the $\mathbb{G}_a$-module obtained from the $\mathbb{G}_a$-module $W[F]$ via restriction of scalars along (the B-linear) Frobenius map $\mathbb{G}_a \to \mathbb{G}_a$. Consider the augmented $\mathbb{G}_a$-module $W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a$. Pulling back along $u\colon \mathbb{G}_a^{\mathrm{perf}} \to \mathbb{G}_a$ and applying the Frobenius pushforward Construction 4.13, we obtain an augmented $\mathbb{G}_a^{\mathrm{perf}}$-module $\varphi_* u^*(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a)$. By construction, it is a nilpotent quasi-ideal in $\mathbb{G}_a^{\mathrm{perf}}$.

For later purpose, let us explicate the outcome of the above construction:

Remark 4.19. Similar to Remark 3.3, a $\mathbb{G}_a$-module is the same as an $\mathbb{N}$-graded bi-commutative Hopf B-algebra satisfying certain extra compatibility (see [Mon22, Remark 2.2.8]). Under this identification, the $\mathbb{G}_a$-module $W[F]^{(p)}$ over B is given by $B[x_0, x_1, \ldots]/(x_i^p)$, with grading $\deg x_i = p^{i+1}$. Its comultiplication is determined using the Witt vector summation, for instance the comultiplication sends

$$x_0 \mapsto x_0 \otimes 1 + 1 \otimes x_0, \text{ and } x_1 \mapsto x_1 \otimes 1 + 1 \otimes x_1 - \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} x_0^i \otimes x_0^{p-i}, \ldots.$$

The augmentation of $W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a$ is simply given by $d : B[t] \to B[x_0, x_1, \ldots]/(x_i^p)$ sending $t \mapsto 0$.

Next we consider the effect of the functor u^* to augmented $\mathbb{G}_a$-module $W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a$. The underlying $\mathbb{G}_a^{\mathrm{perf}}$-module of $u^*(W[F]^{(p)})$ is given by push out along $u : B[t] \to B[x^{1/p^\infty}]$ that sends $t \mapsto x$

$$B[x_0, x_1, \ldots]/(x_i^p) \otimes_{d, B[t], u} B[x^{1/p^\infty}] = B[x^{1/p^\infty}, x_0, x_1, \ldots]/(x, x_i^p),$$

with grading $\deg x^{1/p^i} = 1/p^i$ and $\deg x_i = p^{i+1}$. The comultiplication sends $x \mapsto x \otimes 1 + 1 \otimes x$, and respects the above comultiplication formula on the x_i's.

Lastly after we apply the functor φ_*, the underlying Hopf B-algebra of $\varphi_* u^*(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a)$ is the same as above, but the grading is divided by p, so we have $\deg x^{1/p^i} = 1/p^{i+1}$ and $\deg x_i = p^i$. The augmentation is given by

$$B[x^{1/p^\infty}] \to B[x^{1/p^\infty}, x_0, x_1, \ldots]/(x, x_i^p), \text{ where } x^{1/p^i} \mapsto x^{1/p^{i-1}}.$$

5. Automorphisms of $F^* F_* \mathrm{dR}$ and decomposition

Now we can state the main observation involving transumtation that is relevant for the results of our paper.

Proposition 5.1. *Let k be a perfect field and B be a k-algebra. Then we have an equivalence in* $\mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{G}/}$

$$F^* F_* \mathrm{dR} \otimes_k B \simeq \mathrm{Tm}(\varphi_* u^*(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a)).$$

Proof. By Remark 3.12, we can assume $B := k$. Let $d\colon u^*(W[F] \to \mathbb{G}_a)$; by Example 4.12, it follows that $\mathrm{Tm}(d) \simeq \mathrm{dR}$. Let $S \in \mathrm{QRSP}_k$. By Remark 4.17, the k-linear map $S \to \varphi_{k,*}\mathrm{dR}(S)$ is induced by applying transmutation to the map $(u^*W[F])^{(1/p)} \to \alpha^\flat$ of augmented $\mathbb{G}_a^{\mathrm{perf}}$-modules. By Example 4.11 and Lemma 4.14, the k-linear Frobenius map $S \to \varphi_{k,*}S$ is induced by applying transmutation to the "B-linear Frobenius" $(\alpha^\flat)^{(1/p)} \to \alpha^\flat$. By Remark 4.15, it follows that

$$F^*F_*\mathrm{dR} \simeq \mathrm{Tm}((u^*W[F])^{(1/p)} \times_{\alpha^\flat} (\alpha^\flat)^{(1/p)}).$$

The proposition now follows from observing that

$$(u^*W[F])^{(1/p)} \times_{\alpha^\flat} (\alpha^\flat)^{(1/p)} \simeq \varphi_* u^*(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a), \tag{5.1}$$

which we explain below.

Let P denote the augmented $\mathbb{G}_a^{\mathrm{perf}}$-module defined as the pullback

$$P := u^*W[F] \times_{u^*\alpha_p} \alpha^\flat.$$

Since $(u^*\alpha_p)^{(1/p)} \simeq \alpha^\flat$, it follows that the left hand side in (5.1) is naturally isomorphic to φ_*P. By commuting products and pullbacks, we also have

$$P \simeq u^*(W[F] \times_{\alpha_p} (0)),$$

where (0) denotes the quasi-ideal in $\mathbb{G}_a$ given by $0 \to \mathbb{G}_a$. Now the Verschiebung operator V induces an exact sequence of $\mathbb{G}_a$-modules

$$0 \to W[F]^{(p)} \xrightarrow{V} W[F] \to \alpha_p \to 0,$$

which promotes to an isomorphism

$$W[F] \times_{\alpha_p} (0) \simeq (W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a)$$

of augmented $\mathbb{G}_a$-modules. This finishes the proof. □

Proposition 5.2. *Let k be a perfect field and B be a k-algebra. Then $\varphi_* u^*(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a)$ has endomorphism monoid given by $(B, \cdot)$. In particular, its automorphism group is $B^\times$. Moreover, denote the underlying $\mathbb{G}_a^{\mathrm{perf}}$-module by M, the action of $b \in B$ sends $x_0 \in \mathcal{O}(W[F]^{(p)}) \subset \mathcal{O}(M)$ to $b \cdot x_0$.*

Proof. Endomorphisms of $\varphi_* u^*(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a)$ are equivalent to endomorphisms of $u^*(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a)$. Further, by full faithfulness of u^*, it is equivalent to endomorphisms of $(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a)$. This is equivalent to endomorphisms of $W[F]^{(p)}$ as a $\mathbb{G}_a$-module, which is further equivalent to endomorphisms of $W[F]$ as a graded group scheme. By graded Cartier duality, see [Mon22, Ex. 2.4.9 and Prop. 2.4.10], the latter is equivalent to endomorphisms of $\mathbb{G}_a$ as a graded group scheme (with coordinate given by the functional $(x_0)^\vee$). Our proposition follows from the observation that endomorphism of the graded group scheme is given by the monoid $(B, \cdot)$, where $b \in B$ sends $(x_0)^\vee \mapsto b \cdot (x_0)^\vee$. □

Proof of Theorem 1.3. By Proposition 2.1, it is equivalent to considering endomorphisms automorphisms of the left Kan extension $F^*F_*\mathrm{dR} \otimes_k B \in \mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)$ (by restricting the target category from quasisyntomic rings to QRSP_k). By Proposition 2.6, it is further equivalent to considering automorphisms of $F^*F_*\mathrm{dR} \otimes_k B \in \mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{G}/}$. By Proposition 5.1, the functor $F^*F_*\mathrm{dR} \otimes_k B$ is the transmutation of the nilpotent quasi-ideal $\varphi_* u^*(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a)$. By Proposition 4.8, we are reduced to computing endomorphisms automorphisms of $\varphi_* u^*(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a)$, which is the content of Proposition 5.2. □

Remark 5.3. Following the above reasoning, it is possible to determine the whole endomorphism monoid of $F^*F_*\mathrm{dR} \otimes_k B$, let us indicate the necessary steps and leave the fun of computing all endomorphisms to the interested readers. First of all, from Proposition 2.6, there is a natural map

$$\mathrm{End}(F^*F_*\mathrm{dR} \otimes_k B) \to \mathrm{End}(\mathcal{O} \otimes_k B)$$

given by restricting to the 0-th conjugate filtration. Moreover the proof there can be generalized to help computing the later monoid, it should be a submonoid of $\mathrm{Frob}^{\mathbb{N}}$ depending on the cardinality of k. The above map admits a section, given by applying various powers of Frobenius to the input algebra. Lastly one needs to compute the fiber above Frob^i, this is the same as computing the following homomorphism set

$$\mathrm{Hom}((\mathrm{Frob}^i)^*(F^*F_*\mathrm{dR}\otimes_k B), F^*F_*\mathrm{dR}\otimes_k B)$$

in the category $\mathrm{Fun}(\mathrm{QRSP}_k, \mathrm{Alg}_B)^{\otimes}_{\mathfrak{S}/}$. Finally, similar to Proposition 5.1, one can get a nilpotent quasi-ideal corresponding to the source and then use the proof of Proposition 5.2 to compute the above homomorphism in the category of nilpotent quasi-ideals.

By Theorem 1.3, similar to the discussion before [LM24, Theorem 5.4], we obtain a functorial action of $\mathbb{G}_m$ on $F^*F_*\mathrm{dR}\colon (\mathrm{Sch}_k^{\mathrm{sm}})^{\mathrm{op}} \to \mathrm{CAlg}(k)$.

Theorem 5.4 (c.f. [LM24, Theorem 5.4])**.** *The functorial $\mathbb{G}_m$-action preserves the conjugate filtration defined in Notation 2.5, and $\mathrm{gr}_i^{\mathrm{conj}} F^*F_*\,\mathrm{dR}[i]$ is a $\mathbb{G}_m$-representation purely of weight i.*

Proof. The preservation of conjugate filtration and the triviality of the action on $\mathrm{Fil}_0^{\mathrm{conj}}$ were proved in Proposition 2.6. Following the same proof strategy of [LM24, Theorem 5.4], it would suffice to explicate the $\mathbb{G}_m$-action on $\mathrm{R}\Gamma(\mathbb{A}^1_k, F^*F_*\mathrm{dR})$ and exhibit a nonzero weight 1 element in $\mathrm{H}^1(\mathbb{A}^1_k, F^*F_*\mathrm{dR})$.

Recall that the $\mathbb{G}_m$-action on $F^*F_*\mathrm{dR}\otimes_k B$(smooth algebras) comes, via descent Proposition 2.1, from the $\mathbb{G}_m$-action on $F^*F_*\mathrm{dR}\otimes_k B$(qrsp algebras). The action on $F^*F_*\mathrm{dR}\otimes_k B$(qrsp algebras) then, due to Proposition 5.1, comes from Definition 3.13. Therefore we arrive at the following explicit cosimplicial presentation in a $\mathbb{G}_m$-equivariant fashion:

$$F^*F_*\mathrm{dR}(k[x]) = \lim_{[n]\in\Delta} F^*F_*\mathrm{dR}(k[x^{1/p^\infty}]^{\otimes_{k[x]}(n+1)}) = \lim_{[n]\in\Delta} \mathrm{Tm}(\varphi_*u^*(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a))(k[x_0^{1/p^\infty}, \dots, x_n^{1/p^\infty}]/(x_i - x_j)).$$

For simplicity, let us denote $G(-) \coloneqq \mathrm{Tm}(\varphi_*u^*(W[F]^{(p)} \xrightarrow{0} \mathbb{G}_a))(-)$. We are interested in degree 1 cohomology, to this end, let us explicate the first three terms in a convenient presentation together with the coface maps: They are given by

$$G\Big(k[x^{1/p^\infty}]\Big) \rightrightarrows G\Big(k[x^{1/p^\infty}]\otimes_k k[y^{1/p^\infty}]/(y)\Big) \Rrightarrow G\Big(k[x^{1/p^\infty}]\otimes_k k[y^{1/p^\infty}]/(y)\otimes_k k[z^{1/p^\infty}]/(z)\Big),$$

the first two arrows are induced by $x\mapsto x$ and $x\mapsto x+y$ respectively; whereas the latter three arrows are induced by $(x,y)\mapsto(x,y)$, $(x,y)\mapsto(x,y+z)$, and $(x,y)\mapsto(x+y,z)$ respectively.

Combining Example 3.8 and Example 3.11, we see that for any augmented $\mathbb{G}_a^{\mathrm{perf}}$-module $M \xrightarrow{d} \mathbb{G}_a^{\mathrm{perf}}$, applying $\mathrm{Tm}(d)(-)$ to $k[x^{1/p^\infty}] \to k[x^{1/p^\infty}]/(x)$ gives $d^*\colon k[x^{1/p^\infty}] \to \mathcal{O}(M)$. According to Proposition 4.5, the functor $G(-)$ satisfies the axioms listed in Definition 4.1. Therefore, using the computation made in Remark 4.19, the above presentation becomes:

$$k[x^{1/p^\infty}] \rightrightarrows k[x^{1/p^\infty}]\otimes_k \tfrac{k[y^{1/p^\infty}, y_i; i\in\mathbb{N}]}{(y, {y_i}^p)} \Rrightarrow k[x^{1/p^\infty}]\otimes_k \tfrac{k[y^{1/p^\infty}, y_i; i\in\mathbb{N}]}{(y, {y_i}^p)} \otimes_k \tfrac{k[z^{1/p^\infty}, z_i; i\in\mathbb{N}]}{(z, {z_i}^p)},$$

the first two arrows are given by $x\mapsto x$ and $x\mapsto x+y$ respectively; whereas the latter three arrows are given by $\mathrm{id}\otimes 1$, $\mathrm{id}\otimes$ comultiplication, and $(x\mapsto x+y)\otimes(y, y_i\mapsto z, z_i)$ respectively. In Remark 4.19 we have seen that the comultiplication sends $y_0\mapsto(y_0+z_0)$, hence we see that y_0 defines a nonzero cohomology class. By the last part of Proposition 5.2, we see that y_0 has weight 1. Note that $\mathrm{H}^1(F^*F_*\mathrm{dR}(k[x]))$ is a free rank 1 module over $k[x]$, we see that it is indecomposable, hence the whole H^1 has weight 1. The rest follows from Künneth formula consideration for polynomial algebras, and the fact that $\mathrm{Fil}_\bullet^{\mathrm{conj}}(F^*F_*\mathrm{dR})$ is obtained by left Kan extension from polynomial algebras. □

Since any $\mathbb{G}_m$-representation decomposes canonically according to $\mathbb{G}_m$-weights, we get the following:

Corollary 5.5 (c.f. [Pet25, Theorem 1.2])**.** *Let X be a smooth k-scheme, then the Frobenius twisted de Rham complex is functorially formal:*

$$F^*F_*\mathrm{dR}_X \cong \bigoplus_{i\geq 0} F^*\Omega^i_{X^{(1)}/k}[-i].$$

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