SYNTOMIC COHOMOLOGY OF RATIONAL WEIGHTS AND ZETA FUNCTIONS

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Abstract. In this paper, using F-gauges, we introduce the notion of syntomic cohomology of rational weights for varieties over perfect fields of postive characteristic. We use it to deduce expressions of special values of zeta functions at $s = 1/2$, extending certain results of Ramachandran.

Contents

1. Introduction

Let X be a smooth proper variety over a finite field $\mathbb{F}_q$, where q is a power of p. Let $\zeta(X, s)$ denote the zeta function of X. Starting from the ideas of Tate, Beilinson, Lichtenbaum [Lic83], and the work of Milne–Ramachandran [MR13], [MR15], the special values of $\zeta(X, s)$ at integers $s = n \in \mathbb{Z}$ has been related to the syntomic cohomology groups $H^i(X, \mathbb{Z}_p(n))$ of weight n under certain conjectural assumptions.

However, the previous cohomological approaches do not directly help in understanding the behavior of $\zeta(X, s)$ when s is a *rational number*, e.g., $s = 1/2$. This is due to difficulties in making sense of "$\mathbb{Z}(1/2)$" as a motive. As stated in [Ram05], it was suggested by Manin that a certain super singular elliptic curve E might be useful in making sense of $\mathbb{Z}(1/2)$. Using such an E, the behavior of $\zeta(X, s)$ at $s = 1/2$ was explained (unconditionally) in the work of Ramachandran [Ram05], when X is defined over $\mathbb{F}_q$, where q is an even power of p.

In this short paper, we will give an intrinsic definition of syntomic cohomology of rational weights, which will in particular, give a definition of the p-adic Tate twists $\mathbb{Z}_p(1/2)$ without having to choose any elliptic curve. This will be done by using the main theorem in [Mon25], where an *unconditional* formulation and proof of the precise relationship between special values at $s = n$ and syntomic cohomology of weight n has been established. Using our constructions from [Mon25] and the theorem regarding special values of Zeta functions proven by the author in the generality of F-gauges (see [Mon25, Thm. 1.1]), we will also recover and generalize the results of Ramachandran

[Ram05] regarding behavior or $\zeta(X,s)$ at $s=1/2$ that will remove the assumption that q is an even power of p.

Our approach uses the stack theoretic approach to F-gauges proposed in the recent work of Drinfeld [Dri24] and Bhatt–Lurie [Bha23]. This was already used in [Mon25] to conveniently capture geometric notions such as syntomic cohomology and weighted Hodge–Euler characteristic in the context of special values. This stacky approach has recently also been used in several questions related to de Rham cohomology of algebraic varieties in positive characteristic (see e.g., [Mon22b] [Mon22a], [LM24], [Pet23], [Pet25], [LM25]).

Our main theorem is Theorem 3.10. In Section 4, we use our main theorem to recover results of Ramachandran that used Weil–étale cohomology.

Acknowledgements: The author thanks Luc Illusie for pointing out the work of Ramachandran on values of zeta functions at $s=1/2$ and several helpful conversations during a visit to IHES. The author was partially supported by a grant from Purdue University.

2. Syntomic cohomology of rational weights

Let k be a perfect field of characteristic p. Let k^{syn} denote the syntomification stack of k (see [Bha23]). First, we note the following proposition, which gives a description of vector bundles on k^{syn} in terms of linear algebraic data.

Proposition 2.1 ([Bha23, Prop. 4.3.1])**.** *Let k be a perfect field. Then the vector bundles on the stack k^{syn} is equivalent to the data of an isocrystal M over k together with a choice of lattice $M_{W(k)}$ over $W(k)$. Note that the lattice does not need to be preserved by the Frobenius on M.*

The following vector bundles on k^{syn} we construct will play an extremely important role.

Definition 2.2. Let $r,s\in\mathbb{Z}$ be such that $s>0$. Let $M:=W(k)^{\oplus s}$. Let $K:=W(k)[1/p]$. Define a map $K^s\to K^s$ given by

$$(x_1,\dots,x_s)\to(\varphi_K(x_2),\dots,\varphi_K(x_s),p^{-r}\varphi_K(x_1)).$$

By Proposition 2.1, the above data determines a canonical vector bundle of rank s on k^{syn}, which we denote by $\mathcal{O}(r/\!\!/s)$.

Remark 2.3. When $s=1$, the vector bundles $\mathcal{O}(r/\!\!/s)$ is naturally isomorphic to the Breuil–Kisin twist $\mathcal{O}(r)$, as one can check by reducing to the case $r=1$.

The above vector bundles allow us to make sense of syntomic cohomology of rational weights. Let $D_{\mathrm{perf}}(k^{\mathrm{syn}})$ denote the derived category of perfect complexes on the stack k^{syn}.

Definition 2.4 (Syntomic cohomology of rational weights)**.** Let $\mathscr{M}\in D_{\mathrm{perf}}(k^{\mathrm{syn}})$. Define

$$R\Gamma_{\mathrm{syn}}(\mathscr{M},\mathbb{Z}_p(r/\!\!/s)):=R\Gamma(k^{\mathrm{syn}},\mathscr{M}\otimes\mathcal{O}(r/\!\!/s)).$$

When $\gcd(s,r)=1$, one may rewrite the above as $R\Gamma_{\mathrm{syn}}(\mathscr{M},\mathbb{Z}_p(r/s))$.

When $\mathscr{M}$ arises from a smooth proper scheme X (via derived pushforward of the structure sheaf along the map $X^{\mathrm{syn}} \to k^{\mathrm{syn}}$), we may rewrite the above as $R\Gamma_{\mathrm{syn}}(X, \mathbb{Z}_p(r/\!\!/s))$ or, when $\gcd(s, r) = 1$, as $R\Gamma_{\mathrm{syn}}(X, \mathbb{Z}_p(r/s))$. In this situation, we denote $\mathscr{M}$ by $\mathcal{H}_{\mathrm{syn}}(X)$.

3. Special values of Zeta functions at $s = 1/2$

Let $\mathscr{M} \in D_{\mathrm{perf}}(k^{\mathrm{syn}})$, where $k = \mathbb{F}_q$ for $q = p^r$. As in [Mon25, Notation 3.1], with $\mathscr{M}$, one can functorially associate a complex $\mathscr{M}_K$ of K-vector spaces, such that each of the cohomology groups $H^i(\mathscr{M}_K)$ is equipped a Frobenius semilinear operator F. In [Mon25, Def. 3.2], the zeta function $Z(\mathscr{M}, t)$ was defined to be

$$\prod_{i \geq 0} \det(1 - tF^r | H^i(\mathscr{M}_K))^{(-1)^{i+1}}.$$

One sets $\zeta(\mathscr{M}, s) := Z(\mathscr{M}, q^{-s})$.

Note that $\mathcal{O}(r/\!\!/s)_K$ is naturally an isocrystal over K, and will be simply written as $\mathcal{O}(r/\!\!/s)$ when the context is clear. For an isocrystal V, we will let $V(r/\!\!/s)$

For an isocrystal V over $\mathbb{F}_{q^a}$, with the Frobenius semilinear operator denoted by φ, let $\chi_V(X)$ denote the characteristic polynomial of the linear operator φ^a. Let $\overline{P}$ denote the reciprocal of any polynomial P, i.e., $\overline{P}(t) = t^{\deg P} P(1/t)$. We note the following lemma.

Lemma 3.1. *Let $q = p^a$ and $r, s \in \mathbb{Z}$, $s > 0$, and $d = \gcd(a, s)$. Let V be an isocrystal over $\mathbb{F}_q$ of dimension n as a vector space over $K := W(\mathbb{F}_q)[1/p]$. Let V' denote the isocrystal over $\mathbb{F}_{q^{s/d}}$ obtained by base changing V. In this set up, we have*

$$\overline{\chi_{V(r/\!\!/s)}}(X^{-1}) = \overline{\chi_{V'}}(X^{-s/d} q^{-r/d})^d.$$

Proof. Note that $\chi_{V(r/\!\!/s)}(X)$ denotes the characteristic polynomial of the linear operator φ^a acting on $V \otimes \mathcal{O}(r/\!\!/s)$, which is given by $\varphi_V^a \otimes \varphi^a_{\mathcal{O}(r/\!\!/s)}$. Let $(\mu_j)_{1 \leq j \leq s}$ denote the roots of the polynomial $T^s - p^{-r}$. Then the eigenvalues of $\varphi^a_{\mathcal{O}(r/\!\!/s)}$ is given by the set $(\mu_j^a)_{1 \leq j \leq s}$. Let $(\lambda_i)_{1 \leq i \leq n}$ denote the eigenvalues of φ_V^a. Then the set of eigenvalues of φ^a is given by $(\lambda_i \mu_j^a)_{1 \leq i \leq n, 1 \leq j \leq s}$. Therefore,

$$\chi_{V(r/\!\!/s)}(X) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq s}} (X - \lambda_i \mu_j^a) = \prod_{1 \leq i \leq n} \lambda_i^s \left((X/\lambda_i)^{s/d} - q^{-r/d} \right)^d = \prod_{1 \leq i \leq n} \left(X^{s/d} - q^{-r/d} \lambda_i^{s/d} \right)^d,$$

where the second equality uses the fact that roots of unity form a cyclic group. Now one has

(3.2)
$$\prod_{1 \leq i \leq n} \left(X^{s/d} - q^{-r/d} \lambda_i^{s/d} \right)^d = q^{-rn} \prod_{1 \leq i \leq n} \left(\frac{X^{s/d}}{q^{-r/d}} - \lambda_i^{s/d} \right)^d = q^{-rn} \chi_{V'}(X^{s/d} q^{r/d})^d.$$

Using the notion of reciprocal polynomials, one gets

$$\overline{\chi_{V(r/\!\!/s)}}(X^{-1}) = \overline{\chi_{V'}}(X^{-s/d} q^{-r/d})^d, \tag{3.3}$$

as desired. ∎

Proposition 3.4 (Fractional zeta transformation)**.** *Let $\mathscr{M} \in D_{\text{perf}}(\mathbb{F}_q^{\text{syn}})$, where $q = p^a$. Let $r, s \in \mathbb{Z}$, $s > 0$, and $d = \gcd(a, s)$. Let $\mathscr{M}' \in D_{\text{perf}}(\mathbb{F}^{\text{syn}}_{q^{s/d}})$ denote the pullback of $\mathscr{M}$. Then we have*

$$\zeta\left(\mathscr{M} \otimes \mathcal{O}(r/\!\!/s), z\right) = \zeta\left(\mathscr{M}', \frac{sz + r}{d}\right)^d.$$

Proof. In the situation of the above lemma, by setting $X := q^z$ we get

$$\overline{\chi_{V(r/\!\!/s)}}(q^{-z}) = \overline{\chi_{V'}}(q^{-(sz+r)/d})^d.$$

Therefore, the statement of the proposition follows from the definition of $\zeta(\mathscr{M}, s)$. $\square$

Now we observe that

$$\mathcal{O}\left\{1/\!\!/2\right\}_K \otimes \mathcal{O}\left\{-1/\!\!/2\right\}_K \simeq \mathcal{O}_K^4, \tag{3.5}$$

where $\mathcal{O}$ denotes the structure sheaf of k^{syn}. In the set up of Proposition 3.4, by twisting appropriately, this gives us

$$\zeta(\mathscr{M}, z)^4 = \zeta\left(\mathscr{M}\left\{1/\!\!/2\right\}', \frac{2z - 1}{d}\right)^d, \tag{3.6}$$

where $d = \gcd(a, 2)$. Therefore, if $2 \mid a$, we have

$$\text{ord}_{z=1/2}\zeta(M, z) = 1/2 \cdot \text{ord}_{z=0}(\mathscr{M}\left\{1/\!\!/2\right\}). \tag{3.7}$$

On the other hand, if $2 \nmid a$, then

$$\text{ord}_{z=1/2}\zeta(\mathscr{M}, z) = 1/4 \cdot \text{ord}_{z=0}\zeta(\mathscr{M}\left\{1/\!\!/2\right\}'), \tag{3.8}$$

where $\mathscr{M}\left\{1/\!\!/2\right\}'$ denotes the base change of $\mathscr{M}\left\{1/\!\!/2\right\}$ to $\mathbb{F}_{q^2}^{\text{syn}}$. Now $\text{ord}_{z=0}\zeta(\mathscr{M}\left\{1/\!\!/2\right\}', z)$ is the same as $\text{ord}_{t=1}Z(\mathscr{M}\left\{1/\!\!/2\right\}', t)$.

Let us note the following lemma, where we call a number λ to be a *semisimple eigenvalue* of an operator T if exponent of $(X - \lambda)$ in the characteristic polynomial of T is the same as $\dim \text{Ker}(T - \lambda)$ (which can be zero).

Lemma 3.9 (Scaling semisimplicity)**.** *If $\pm q^{1/2}$ are semisimple eigenvalues for an isocrystal V over $\mathbb{F}_q$, then 1 is a semisimple eigenvalue for $V\left\{1/\!\!/2\right\}$.*

Proof. Follows directly by considering tensor products and the fact that the eigenvalues of $\mathcal{O}\left\{1/\!\!/2\right\}_K$ are semisimple and lie in the set $\left\{-q^{-1/2}, q^{-1/2}\right\}$. $\square$

Combining the above discussions, we obtain the following theorem.

Theorem 3.10. *Let $\mathscr{M} \in D_{\text{perf}}(\mathbb{F}_q^{\text{syn}})$, where $q = p^a$. Suppose that*

(1) $2 \mid a$ and $\pm q^{1/2}$ are semisimple eigenvalues of F^a acting on $H^i(\mathscr{M}_K)$ for all i. Then

$$\text{ord}_{z=\frac{1}{2}}\zeta(\mathscr{M}, z) = \frac{1}{2}\left(\sum_{j\geq 0}(-1)^j j \dim H^j(\mathscr{M}, \mathbb{Q}_p(1/2))\right).$$

(2) $2 \nmid a$ *and* $\pm q^{1/2}, \pm i q^{1/2}$ *are semisimple eigenvalues of* F^a *acting on* $H^i(\mathscr{M}_K)$ *for all* i. *Then*

$$\mathrm{ord}_{z=\frac{1}{2}} \zeta(\mathscr{M}, z) = \frac{1}{4}\left(\sum_{j \geq 0} (-1)^j j \dim H^j(\mathscr{M}_{\mathbb{F}_{q^2}}, \mathbb{Q}_p(1/2))\right).$$

Proof. Combine the previous discussion, (3.5), with [Mon25, Prop. 3.9] □

Now we are ready to formulate our result for varieties.

Corollary 3.11. *Let* X *be a smooth proper variety over* $\mathbb{F}_q$, *where* $q = p^a$.

(1) If $2 \mid a$, *then*

$$\mathrm{ord}_{z=\frac{1}{2}} \zeta(X, z) = \frac{1}{2}\left(\sum_{j \geq 0} (-1)^j j \dim H^j(X, \mathbb{Q}_p(1/2))\right).$$

(2) if $2 \nmid a$, *then*

$$\mathrm{ord}_{z=\frac{1}{2}} \zeta(X, z) = \frac{1}{4}\left(\sum_{j \geq 0} (-1)^j j \dim H^j(X_{\mathbb{F}_{q^2}}, \mathbb{Q}_p(1/2))\right).$$

Proof. This will follow from Theorem 3.10 by taking $\mathscr{M} := \mathcal{H}_{\mathrm{syn}}(X)$ once we prove the semisimplicity hypothesis (cf. [Mon25, Rmk. 2.3]). By the Riemann Hypothesis part of the Weil conjectures (Deligne's theorem), $\pm q^{1/2}, \pm i q^{1/2}$ can only possibly be eigenvalues of $H^1_{\mathrm{crys}}(X)[1/p]$. By replacing X with the Albanese of X, it suffices to verify the claim when X is an abelian variety, where it follows from a theorem of Tate [Tat66]. This finishes the proof. □

We will now also give an expression for the special value of $\zeta(\mathscr{M}, z)$ at $z = 1/2$. By (3.6), one can relate the special value of $\zeta(\mathscr{M}, z)$ at $z = 1/2$, with the special value of $\zeta(\mathscr{M}'\{1/2\}, z)$ at $z = 0$ in a manner that will be made more explicit later. In what follows, we will work with a $\mathscr{N} \in D_{\mathrm{perf}}(k^{\mathrm{syn}})$. Our first goal is to relate the weighted Hodge–Euler characteristic (as defined in [Mon25, Def. 6.3]) $\chi(\mathscr{N}, 0)$ to that of $\chi(\mathscr{N}\{1/2\}, 0)$. By construction ([Mon25, Def. 6.3]), this is defined in terms of the "Hodge realization" obtained via pulling back along the closed immersion $u : B\mathbb{G}_m \to k^{\mathrm{syn}}$.

Lemma 3.12. *Let* k *be a perfect field. Let* $s \geq 1$. *Then* $\mathcal{O}\{1/s\}^{\mathrm{Hodge}} := u^*\mathcal{O}\{1/s\} \in \mathrm{Vect}(B\mathbb{G}_m)$ *identified as a graded vector space is isomorphic to* $k^{\oplus(s-1)} \oplus k(-1)$, *where* $k^{\oplus(s-1)}$ *has weight* 0 *and* $k(-1)$ *has weight* -1.

Proof. By realizing $\mathcal{O}\{1/s\}$ as an isocrystal together with a lattice, one observes that the associated Gauge is concentrated in the interval $[-1, 0]$. Therefore, $\mathcal{O}\{1/s\}$ has Hodge–Tate weights in degrees ≤ 0 , or in order words, $u^*\mathcal{O}\{1/s\}$ as a graded vector space is concentrated in degrees ≤ 0. Let $u^*\mathcal{O}\{1/s\} = \oplus_{i \leq 0} V^i$. Note that $\wedge^s \mathcal{O}\{1/s\}$ is a line bundle, and thus $\wedge^s u^*\mathcal{O}\{1/s\}$ is a 1-dimensional vector space concentrated in a single weight $w \in \mathbb{Z}$. Since $\mathcal{O}\{1/s\}_{W(k)[1/p]}$ has slope $-1/s$, it follows that $w/s = -1/s$, implying that $w = -1$. It follows that $\sum_{i \leq 0} i \dim V_i = w = -1$. This implies that

$\dim V_i = 0$ for $i < -1$, and $\dim V_{-1} = 1$. Since $\sum_{i \leq 0} \dim V_i = s$, one must have $\dim V_0 = (s-1)$. This gives the desired claim. □

Lemma 3.13. *Let $\mathscr{N}$ be an effective (see* [Bha23, Def. 3.4.8]*) object of $D_{\mathrm{perf}}(k^{\mathrm{syn}})$. Then the weighted Hodge–Euler characteristic satisfies the following formula:*

$$\chi(\mathscr{N}\{1/s\}, 0) = \sum_{j \in \mathbb{Z}} (-1)^{j-1} h^{0,j-1}(\mathscr{N}).$$

Proof. By definition, $\chi(\mathscr{N}\{1/s\}, 0) = \sum_{\substack{i,j \in \mathbb{Z} \\ i \leq 0}} (-1)^{i+j}(-i) h^{i,j}(\mathscr{N}\{1/s\})$. By Lemma 3.12, we have $\mathscr{N}\{1/s\}^{\mathrm{Hodge}} \simeq (\mathscr{N}^{\mathrm{Hodge}})^{\oplus(s-1)} \oplus \mathscr{N}^{\mathrm{Hodge}}(-1)$. Therefore,

$$h^{i,j}(\mathscr{N}\{1/s\}) = (s-1)h^{i,j}(\mathscr{N}) + h^{i+1,j-1}(\mathscr{N}).$$

Now since $\mathscr{N}$ is additionally assumed to be effective, the Hodge–Tate weights of $\mathscr{N}$ are in degrees ≥ 0. Therefore, $h^{i,j}(\mathscr{N}) = 0$ for $i < 0$. Therefore,

$$\chi(\mathscr{N}\{1/s\}, 0) = \sum_{j \in \mathbb{Z}} (-1)^{j-1} h^{0,j-1}(\mathscr{N}),$$

as desired. □

Proposition 3.14. *Let X be a smooth proper variety over $\mathbb{F}_q$. Multiplication by the canonical class $\sigma \in H^1(\mathbb{F}_q^{\mathrm{syn}}, \mathcal{O}) \simeq \mathrm{Hom}(\hat{\mathbb{Z}}, \mathbb{Z}_p)$ gives a complex*

$$\ldots \to H^i(X, \mathbb{Z}_p(1/2)) \to H^{i+1}(X, \mathbb{Z}_p(1/2)) \to H^{i+2}(X, \mathbb{Z}_p(1/2)) \to \ldots,$$

whose cohomology groups are finite abelian groups.

Proof. By [Mon25, Prop. 2.20], the discussion in [Mon25, Def. 6.4] and Rmk. 4.8 in loc. site, it suffices to prove that 1 is a semisimple eigenvalue for the isocrystals $H^i_{\mathrm{crys}}(X)[1/p]\{1/2\}$. By the same argument in the proof of Corollary 3.16 we know that $\pm q^{1/2}$ are semisimple eigenvalues of $H^i_{\mathrm{crys}}(X)[1/p]$. Therefore, the claim follows from Lemma 3.9. □

Let $K^\bullet$ denote the complex in Proposition 3.14 (where $H^0(X, \mathbb{Z}_p(1/2))$ is in degree 0). Let $\mathcal{H}_{\mathrm{syn}}(X)$ denote the derived pushforward along the structure sheaf of the map $X^{\mathrm{syn}} \to \mathbb{F}_q^{\mathrm{syn}}$. Let $\mu_{\mathrm{syn}}(\mathcal{H}_{\mathrm{syn}}(X)\{1/2\}, 0)$ denote the stable Bockstein characteristic as defined in [Mon25, Def. 5.2]. Then one has the following.

Lemma 3.15. $\mu_{\mathrm{syn}}(\mathcal{H}_{\mathrm{syn}}(X)\{1/2\}, 0) = \prod_i |H^i(K^\bullet)|^{(-1)^i}$.

Proof. This follows from the proof of [Mon25, Prop. 5.1], as the Bockstein characteristic is well-defined for $r = 1$ in the notation of loc. cit. □

Corollary 3.16. *Let X be a smooth proper variety over $\mathbb{F}_q$, where $q = p^a$. Let $\chi^\times(X, 1/2) := \prod_i |H^i(K^\bullet)|^{(-1)^i}$. Let $\rho_X := \mathrm{ord}_{s=1/2}\zeta(X, s)$. Let*

$$c_X := \lim_{s \to 1/2} \frac{\zeta(X, s)}{(1 - q^{1/2-s})^{\rho_X}}$$

(1) If $2 \mid a$, then

$$|c_X|_p^2 = \frac{1}{\chi^\times(X, 1/2) \cdot q^{\chi(X, \mathcal{O}_X)}}.$$

(2) if $2 \nmid a$, then

$$|c_X|_p^4 = \frac{|2|_p^{\rho_{X'}}}{\chi^\times(X', 1/2) q^{\chi(X', \mathcal{O}_{X'})}},$$

where X' denotes $X_{\mathbb{F}_{q^2}}$.

Proof. By Lemma 3.13, we have $\mu_{\mathrm{syn}}(\mathcal{H}_{\mathrm{syn}}(X)\{1/2\}, 0) = \chi(X, \mathcal{O}_X)$ (and similarly for X'). Therefore, the result follows from Lemma 3.15 and [Mon25, Thm. 1.1]. ☐

4. Weil–étale cohomology

In this section, we use our results in the previous section to deduce the results of Ramachandran [Ram05] involving Weil–étale cohomology of certain elliptic curves. To this end, we choose an elliptic curve E over $\mathbb{F}_{p^{2f}}$ such that the set of eigenvalues of Frobenius on $H^1_{\mathrm{crys}}(E)[1/p]$ is given by p^f, p^f. Let $k := \mathbb{F}_{p^{2f}}$ and $K := W(k)[1/p]$.

Lemma 4.1. *For any elliptic curve E over k such that the set of eigenvalues of Frobenius on $H^1_{\mathrm{crys}}(E)[1/p]$ is given by p^f, p^f, the isocrystal $H^1_{\mathrm{crys}}(E)[1/p]$ is isomorphic to $\mathcal{O}\{-1/2\}_K$.*

Proof. By Honda–Tate theory for simple abelian varieties (see [Tat66]), all such elliptic curves E are isogenous. Therefore, it suffices to handle the case when E arises as base change of an elliptic curve E' defined over $\mathbb{F}_{p^2}$ satisfying the eigenvalue conditions. Thus, by base change, we reduce to the case when $f = 1$. Let $V = H^1_{\mathrm{crys}}(E')[1/p]$. Writing the Frobenius on the isycrystal V by F, we have $F^2 = p$ by semisimplicity of F^2. Let $e_1 \in V$ be a nonzero vector. Let $e_2 := F(e_1)$. Then e_1 and e_2 can not be linearly dependent since by Dieudonné–Manin classification, as the isocrystal V is of rank 2 and slope 1/2, it can not have non-trivial subobjects. Thus, e_1, e_2 form a basis of V such that $F(e_1) = e_2$ and $F(e_2) = pe_1$. Permuting the basis, we see that the isocrystal V is isomorphic to $\mathcal{O}\{-1/2\}_K$. ☐

Definition 4.2. For the chosen elliptic curve E over k, let $\mathcal{M}(E)$ denote the Dieudonné F-gauge associated to E, which is a vector bundle on k^{syn} of rank 2 with Hodge–Tate weights in $\{0, 1\}$ (see [Mon24]). Since $H^2_{\mathrm{crys}}(BE) \simeq H^1_{\mathrm{crys}}(E)$ (see [Mon21]), by the above lemma, it follows that $\mathcal{M}(E)_K \simeq \mathcal{O}\{-1/2\}_K$.

Now, we are ready to prove the following proposition, recovering Weil–étale cohomology in terms of syntomic cohomology of rational weights. In what follows below, we will use the fact that [Ram05, Thm. 14, (b)] the Weil-étale cohomology groups $H^*_W(X, E)$ lives in finitely many degrees and are finitely generated. We will also use Dieudonné theory in the language of prismatic F-gauges as in [Mon24] (see also [Mon21] and [MM25]).

Proposition 4.3. *Let $k = \mathbb{F}_{p^{2f}}$. Let X be a smooth proper variety over k. Let E be an elliptic curve over k fixed as before. Then we have*

$$H^{j+1}(X, \mathbb{Q}_p(1/2)) \simeq H^j_W(X, E) \otimes_{\mathbb{Z}} \mathbb{Q}_p.$$

Proof. Using the short exact sequences $0 \to E[p^k] \to E \xrightarrow{p^k} E \to 0$ in the Weil–étale topology, one arrives at the natural isomorphisms

$$H^j_W(X,E)\otimes_{\mathbb{Z}}\mathbb{Z}_p \simeq H^{j+1}(X, T_p(E)). \tag{4.4}$$

Indeed, we have

$$\varprojlim_n R\Gamma_W(X,E)\otimes^L \mathbb{Z}/p^n\mathbb{Z} \simeq R\Gamma(X, T_p(E))[1].$$

Since $R\Gamma_W(X,E)$ is a perfect complex of $\mathbb{Z}$-modules, the left hand side above is naturally isomorphic to $R\Gamma_W(X,E)\otimes^L_{\mathbb{Z}}\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module, by taking cohomology, one arrives at (4.4).

Further, by the expression of cohomology in terms of Dieudonné F-gauge (see [Mon24, Prop. 3.5.16]), we have

$$R\Gamma(X, T_p(E)) \simeq R\Gamma(X^{\mathrm{syn}}, \mathcal{M}(E^\vee)\{1\}).$$

Since E is an elliptic curve $E\simeq E^\vee$ and $\mathcal{M}(E^\vee)_K \simeq \mathcal{M}(E)_K \simeq \mathcal{O}\{-1/2\}_K$. Therefore, we obtain

$$R\Gamma(X, T_p(E))[1/p] \simeq R\Gamma(X, \mathbb{Q}_p(1/2)).$$

This gives the desired claim. □

Proposition 4.5. *Let X be a smooth proper variety over $\mathbb{F}_{p^{2f}}$. Then*

$$\mathrm{ord}_{z=\frac{1}{2}}\zeta(X,z) = -1/2\left(\sum_{j\geq 0}(-1)^j j\cdot \mathrm{rank} H^j_W(X,E)\right).$$

Proof. Note that by [Mon25, Prop. 3.5], we have

$$\sum_{j\geq 0}(-1)^j \dim H^j(X, \mathbb{Q}_p(1/2)) = 0.$$

Therefore,

$$\sum_{j\geq 0}(-1)^j j \dim H^j(X,\mathbb{Q}_p(1/2)) = \sum_{j\geq 0}(-1)^j (j-1)\dim H^j(X,\mathbb{Q}_p(1/2)).$$

By Proposition 4.3, the right hand side is equal to

$$-\left(\sum_{j\geq 0}(-1)^j j\cdot \mathrm{rank} H^j_W(X,E)\right).$$

Therefore, the result follows from Corollary 3.16. □

Lemma 4.6. *Let E be the elliptic curve fixed as before. Let $u: B\mathbb{G}_m \to k^{\mathrm{syn}}$. Let $\mathcal{M}(E)^*$ denote the dual of $\mathcal{M}(E)$ as a vector bundle on k^{syn}. Then*

$$u^*\mathcal{M}(E)^* \simeq k \oplus k(-1),$$

where $k(-1)$ has weight -1.

Proof. Using [Bha23, Rmk. 5.3.14] the statement follows from the Hodge filtration on $H^1_{\mathrm{dR}}(E)$, whose associated graded is given by $H^1(E,\mathcal{O})\oplus H^0(E,\Omega^1_A)\simeq k\oplus k(1)$, and taking duals. □

Note that since $E \simeq E^\vee$, by [Mon24, Prop. 3.4.9] we have $\mathcal{M}(E)^* \simeq \mathcal{M}(E)\{1\}$. By Lemma 4.1, we have $\mathcal{M}(E)^*_K \simeq \mathcal{O}\{1/2\}$. We also have the following lemma, where for an F-gauge $\mathcal{F}$ over k, we use $R\Gamma(X^{\mathrm{syn}}, \mathcal{F})$ to denote $R\Gamma(k^{\mathrm{syn}}, \mathcal{H}_{\mathrm{syn}}(X) \otimes F)$.

Lemma 4.7. *Let E be an elliptic curve fixed as before. Let X be a smooth proper variety over $\mathbb{F}_{p^{2f}}$. Multiplication by the canonical class $\sigma \in H^1(\mathbb{F}^{\mathrm{syn}}_{p^{2f}}, \mathcal{O}) \simeq H^1_{\acute{e}t}(\mathrm{Spec}\,\mathbb{F}_q, \mathbb{Z}_p)$ gives a complex*

$$\dots \to H^i(X^{\mathrm{syn}}, \mathcal{M}(E)^*) \to H^{i+1}(X^{\mathrm{syn}}, \mathcal{M}(E)^*) \to \dots$$

that we denote by $K^\bullet_E$. Then cohomology groups of $K^\bullet_E$ are finite abelian groups.

Proof. By Lemma 4.1, the same proof as Proposition 3.14 applies. □

Definition 4.8. Let E be fixed as before. We denote $\chi^\times(X,E) := \prod_i |H^i(K^\bullet_E)|^{(-1)^i}$.

Proposition 4.9. *Let X be a smooth proper variety over $\mathbb{F}_{p^{2f}}$. In the notations of Corollary 3.16, we have*

$$|c_X|_p^2 = \frac{1}{\chi^\times(X,E) \cdot q^{\chi(X,\mathcal{O}_X)}}.$$

Proof. Same argument as Corollary 3.16 applies. Indeed, setting $\mathscr{N} := \mathcal{H}_{\mathrm{syn}}(X) \otimes (\mathcal{M}(E)^*)$, by Lemma 4.6, we have $\chi(\mathscr{N}, 0) \simeq \chi(X, \mathcal{O}_X)$ (similar to the proof of Lemma 3.13) and the claim follows from applying [Mon25, Thm. 1.1] to $\mathscr{N}$. □

Remark 4.10. Note that as a consequence of the above proposition, $\chi^\times(X,E)$ is independent of the chosen elliptic curve E over $\mathbb{F}_{p^{2f}}$ as long as the set of eigenvalues of Frobenius on $H^1_{\mathrm{crys}}(E)[1/p]$ is given by p^f, p^f.

Lemma 4.11. *Let $\theta \in H^1_W(\mathrm{Spec}\,\mathbb{F}_q, \mathbb{Z}) \simeq \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ correspond to the identity. Multiplication by θ defines a complex*

$$\dots \to H^0_W(X,E) \to H^1_W(X,E) \to H^2_W(X,E) \to \dots$$

denoted by $K^\bullet_{W,E}$. Then cohomology groups of $K^\bullet_{W,E}$ are finite abelian groups.

Proof. As $H^i_W(X,E)$ is a finitely generated abelian group, it suffices to prove that the cohomology groups vanish after applying $(\cdot) \otimes_{\mathbb{Z}} \mathbb{Q}_p$. This follows from Proposition 4.3 and Lemma 4.7. □

Proposition 4.12. *In the above set up, $v_p(\prod H^i(K^\bullet_{W,E})^{(-1)^i}) = -v_p(\chi^\times(X,E))$.*

Proof. Since $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module this follows from structure theorem of finitely generated abelian groups and (4.4). □

Now we can recover the expression due to Ramachandran (after taking p-adic norms).

Corollary 4.13. *Let X be a smooth proper variety over $\mathbb{F}_{p^{2f}}$. In the notations of Corollary 3.16, we have*

$$|c_X|_p^2 = \left|\frac{q^{\chi(X,\mathcal{O}_X)}}{\prod H^i(K^\bullet_{W,E})^{(-1)^i}}\right|_p.$$

Proof. Follows from Proposition 4.9 and Proposition 4.12. □

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